High Energy Asymptotics for N-body Scattering Matrices with Arbitrary Channels

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Abstract

In this paper, we study the high energy asymptotics in weak sense of scattering matrices associated to arbitrary scattering channels for generalized N-body Schrödinger operators. In the case where the cluster decomposition corresponding to the incoming and outgoing channel is the same, we obtain the leading term of the high energy asymptotics under the condition that the eigenfunctions ψ_α, ψ_β associated to the outgoing and incoming channel satisfy: $\psi_\alpha \in L^{2,\varepsilon_\alpha}, \psi_\beta \in L^{2,\varepsilon_\beta}$ with $\varepsilon_\alpha + \varepsilon_\beta \geq 1$. When the cluster decompositions corresponding to the incoming and outgoing channel are different, we prove that if the potentials are smooth and rapidly decreasing, the scattering matrices are of the order $O(\lambda^{-\infty})$ as the energy λ tends to infinity.

RESUME. Dans ce travail, nous étudions l'asymptotiuqe à haute énergie au sens faible pour les matrices de diffusion associées à des canaux de diffusion quelconques pour l'opérateur de Schrödinger à N-corps généralisé. Au cas où la partition en amas pour les canaux entrant et sortant est la même, on donne le terme principal de l'asymptotique à haute énergie sous certaine hypothèse sur les fonctions propres. Quand les décompositions en amas pour les canaux entrant et sortant sont différentes, on prouve que les matrices de diffusions sont de l'ordre de $O(\lambda^{-\infty})$ si les potentiels sont réguliers et à décroissance rapide.

MOTS-CLES: Opérateurs de Schrödinger, Problème à N-corps, Asymptotique à haute énergie, Matrices de la diffusion.

1 Introduction

This work is a continuation of [25] in which the author studied the high energy asymptotics for free channel − free channel scattering matrix and proved the uniqueness of inverse scattering problems at high energies for generalized N-body Schrödinger operators . In this work, we shall study the high energy asymptotics for N-body scattering matrix with arbitrary scattering channels. For two-body Schrödinger operators, the high energy asymptotics of various scattering quantities are now well understood and there exists a large litterature on these subjects including more complicated cases where coupling constants are present. See, for example, [6, 8, 12, 15, 16, 20, 26, 27, 29]. For N-body systems, the problem is more complicated. Let us just mention here that the high energy asymptotics in N-body problems are already appeared in the book [12] and that in [2, 3, 6], the finiteness of total cross-section with initial two-cluster channel is proved and upper bounds in high energy case are given. In [10, 23, 24], the high energy asymptotics for total cross-sections are established in three-body and general N-body scattering theory, respectively. In [11, 17], the semiclassical asymptotics of total cross-sections with initial two-cluster channel are obtained. In [4, 9, 19], the authors studied the regularity or singularity of scattering amplitudes for scattering matrices where one of the scattering channels is a two-cluster channel with non-threshold energy. In the case where none of the scattering channels is a two-cluster one with non-threshold energy, less is known. Apart from the result of [25] mentioned above, we can only quote [28] in which Yafaev established representation formula for scattering matrices with arbitrary scattering channels and proved their weak continuity in energy and a recent work of Novikov ([13]) in which he studied the inverse scattering of 3-body problems by using Faddeev's method and assumptions. Since as far as the author knows, a pointwise definition of scattering amplitudes with arbitrary scattering channels is unkown, we content ourselves with the high energy asymptotics of scattering matrices in weak sense, which already reveals fruitful as shown in [25].

Let us now introduce some notations. Let Δ be the Laplacian on the Euclidean space $\mathbf{X} = \mathbf{R}^d, d \geq 2$. Let $\mathcal{A}$ be the set of all possible cluster decompositions of an N-body system labelled by $\{1, 2, \cdots, N\}$. For $a, b \in \mathcal{A}$, we write $b \subset a$ if the cluster decomposition b is a refinement of a. The generalized N-body Schrödinger operator to be studied in this work is of the form:

$$P = -\Delta + \sum_{a \in \mathcal{A}} V_a(x^a).$$

Here $x^a = \pi^a x$ with π^a the orthogonal projection from $\mathbf{X}$ onto some subspace $\mathbf{X}^a$ associated to the cluster decomposition $a \in \mathcal{A}$. The physical N-body Schrödinger operators can always be put into the above form by appropriate change of coordinates. We shall not recall the conventions on the geometrical structure for the configuration of generalized N-body systems and refer to, for example, [22, 24, 25] for more details.

For each $a \in \mathcal{A}$, we denote by $\mathbf{X}_a$ the orthogonal complement (with respect to the Euclidean structure on $\mathbf{X}$) of $\mathbf{X}^a$ in $\mathbf{X}$: $\mathbf{X} = \mathbf{X}^a \oplus \mathbf{X}_a$. Accordingly, a generic point $x \in \mathbf{X}$ can be decomposed as: $x = x^a + x_a$. Sometimes, we also write it as $x = (x^a, x_a)$. Denote $-\Delta^a$ ($-\Delta_a$, resp.) the Laplacian in x^a-variables (x_a- variables, resp.) and $D^a = -i\partial/\partial x^a$, $D_a = -i\partial/\partial x_a$. Put

$$P^a = -\Delta^a + \sum_{b \subseteq a} V_b(x^b), P_a = P^a - \Delta_a,$$

$$I_a(x) = \sum_{b \not\subseteq a} V_b(x^b).$$

Let $\mathcal{T}$ denote the set of thresholds and eigenvalues of P:

$$\mathcal{T} = \cup_a \sigma_{pp}(P^a).$$

Let $\mathbf{S}^a$, $\mathbf{S}_a$ denote the unit sphere in $\mathbf{X}^a$ and $\mathbf{X}_a$, respectively. Put

$$\Sigma_a = \mathbf{S}_a \setminus \cup_{b \not\subseteq a} \mathbf{X}_b. \tag{1.1}$$

Due to the geometry of an N-body system, one can check that $\Sigma_a = \mathbf{S}_a$ if $\#a = 2$ ($\#a$ being the number of clusters in a). The norm and the scalar product in $L^2(\mathbf{X}_a)$, (in $L^2(\mathbf{S}_a)$, respectively), will be denoted by $\|\cdot\|_a$ and $< \cdot, \cdot >_a$ (by $|\cdot|_a, (\cdot, \cdot)_a$, respectively), while those in $L^2(\mathbf{X})$ will be denoted by $\|\cdot\|$ and $< \cdot, \cdot >$.

Let a be a non-trivial cluster decomposition (i.e., $a \in \mathcal{A}$ with the number of clusters $\#a \geq 2$). A scattering channel α stands for a collection of data: $\alpha = (a, E_\alpha, \varphi_\alpha)$, where $E_\alpha \in \sigma_{pp}(P^a)$ and φ_α is an associated normalized eigenfunction:

$$P^a \varphi_\alpha = E_\alpha \varphi_\alpha, \ \|\varphi_\alpha\| = 1.$$

When $a = a_{\min}$ (i.e., $\#a = N$), one uses the convention that $P^a = 0$, $P_a = -\Delta$ and in this case, the only scattering channel is the free one: $\alpha = (a_{\min}, 0, 1)$. We shall say that α is a scattering channel with non-threshold energy, if

$$E_\alpha \in \sigma_{pp}(P^a) \setminus \cup_{b \subset a} \sigma_{pp}(P^b).$$

Let $\mathcal{J}_\alpha : L^2(\mathbf{X}_a) \to L^2(\mathbf{X})$ the channel identification:

$$(\mathcal{J}_\alpha f)(x) = \varphi_\alpha(x^a) f(x_a).$$

Assume that $\forall a \in \mathcal{A}$, V_a satisfies for some $R > 0$,

$$|(y \cdot \nabla_y)^k V_a(y)| \leq C \langle y \rangle^{-\rho}, \quad |y| > R, \tag{1.2}$$

for $k = 0, 1, 2$ and for some $\rho > 1$ and $(y \cdot \nabla_y)^k V_a$, $k = 0, 1, 2$, is relatively compact with respect to $-\Delta^a$ in $L^2(\mathbf{X}^a)$. Under the assumption (1.2), it is well known that the channel wave opeartors

$$W_\pm^\alpha = s - \lim_{t \to \pm\infty} U(t)^* U_a(t) \mathcal{J}_\alpha$$

exist for any scattering channel α and are complete ([18]). Here $U(t)$ and $U_a(t)$ are unitary groups generated by P and P_a, respectively.

Now let $\alpha = (a, E_\alpha, \psi_\alpha)$ and $\beta = (b, E_\beta, \psi_\beta)$ be two given scattering channels. Let

$$S_{\alpha\beta} = W_+^{\beta *} W_-^\alpha$$

be the scattering operator from an initial channel α to a final channel β. Let $S_{\alpha\beta}(\lambda)$ be the corresponding scattering matrices. The purpose of this work is to study the asymptotics of $(S_{\alpha\beta}(\lambda)u_a, u_b)_b$ as $\lambda \to \infty$, for any $u_c \in C_0^\infty(\Sigma_c)$, $c = a, b$. Remark that the choice of the support of test functions allows to avoid the singularities of scattering amplitude and the result obtained in this work shows that $(S_{\alpha\beta}(\lambda)u_a, u_b)_b$ should have a different behavior as $\lambda \to \infty$, if we just take $u_c \in C^\infty(\mathbf{S}_c)$, $c = a, b$. Let $T_{\alpha\beta}(\lambda) = i(S_{\alpha\beta}(\lambda) - \delta_{\alpha\beta})/(2\pi)$. Under appropriate assumptions, we prove in this paper that there exists some $\eta > 0$ such that if $a = b$, one has for any $\varphi_a, \varphi_a' \in C_0^\infty(\Sigma_a)$

$$(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_a')_a - (\mathcal{F}_\beta(\lambda) I_a \mathcal{F}_\alpha(\lambda)^* \varphi_a, \varphi_a')_a = O(\lambda^{-1/2-\eta}), \quad \lambda \to \infty. \tag{1.3}$$

If $a \neq b$, one has for any $\varphi_c \in C_0^\infty(\Sigma_c)$, $c = a, b$,

$$(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b = O(\lambda^{-1/2-\eta}), \quad \lambda \to \infty. \tag{1.4}$$

When potentials are smooth and decay rapidly, we prove in the case $a \neq b$ that $(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b = O(\lambda^{-\infty})$. See Theorem 3.6 and Theorem 4.1 for more precisions.

The plan of this work is as follows. In Section 2, we establish spectral representation formula for scattering matrices with arbitrary scattering channels. Recall that the scattering matrices of N-body systems have already been studied in [28]. But it is not clear to the author how to obtain high energy asymptotics from the representation given in [28]. Our study of high energy asymptotics is based on the high energy microlocal resolvent estimates obtained in [22] and their generalizations given in the following Subsection 2.1. Therefore we need to represent scattering matrices in terms of microlocalizations appeared in these results. In Section 3, we study the high energy asymptotics of scattering matrices for bounded potentials and prove (1.3) and (1.4). Some technical difficulty arises when we want to control the commutors of Δ with various cut-offs in the high energy regime. To overcome this, we make the following assumption on the scattering channels: $\psi_\alpha \in L^{2,\varepsilon_\alpha}(\mathbf{X}^a)$ and $\psi_\beta \in L^{2,\varepsilon_\beta}(\mathbf{X}^b)$ with $\varepsilon_\alpha + \varepsilon_\beta \geq 1$. This condition is always satisfied if one of the channels is a non-threshold channel or is the free channel. In the later case, $\mathbf{X}^a = \{0\}$. (1.4) suggests that when potentials are bounded, the probability for particles to transit from one cluster to another during the scattering is small at high energies. When potentials are of sufficiently short range (i.e., $V_a \in \mathcal{S}(\mathbf{X}^a)$), we prove in Section 4 that this probability is of the order $O(\lambda^{-\infty})$ as $\lambda \to \infty$.

2 Some Preliminaries

In this Section, we establish the spectral representation formula for scattering matrices with arbitrary scattering channels which is adapted to our study of high energy asymptotics. The main difference from the free channel case already treated in [25] is that the microlocal resolvent estimates obtained in [22] are not sufficient to the present situation and we often need a localization in intra-cluster momentum space. Intuitively, the presence of scattering channel means that the intra-cluster energy is fixed. If the potentials are bounded, this would imply that the intra-cluster kinetic energy is finite. So we can always insert a localization in intra-cluster momentum space. We begin with justifying this intuition and establishing some results on microlocal resolvent estimates needed in the spectral representation of scattering matrices with arbitrary scattering channels.

2.1 Resolvent Estimates

Let P be a generalized N-body Schrödinger operator : $P = -\Delta + \Sigma_{a\in\mathcal{A}} V_a(x^a)$. We write formally $V_a^0(x^a) = V_a(x^a)$ and $V_a^j(x^a) = (x^a\cdot\nabla^a)V_a^{j-1}(x^a)$, for $j = 1, 2, \cdots$. To obtain microlocal resolvent estimates in the free channel region, we need only assume that V_a^j is $-\Delta^a$-compact for $0 \le j \le 3$. To establish microlocal resolvent estimates with intercluster microlocalizations, we need stronger assumptions on potentials. In this Section, we assume that the potentials satisfy the following conditions: $\forall a \in \mathcal{A}, \forall 0 \le j \le 3, V_a^j(\cdot)$ is relatively compact in $L^2(\mathbf{X}^a)$ with respect to $-\Delta^a$ and there exist $\varepsilon_0 > 0$ and $R > 0$ such that

$$|\partial_{x^a}^\alpha V_a(x^a)| \le C\langle x^a\rangle^{-\varepsilon_0-|\alpha|}, \text{ for } |x^a| > R \text{ and } |\alpha| \le \max\{3, d/2+1\}. \tag{2.1}$$

Let us indicate that different from the main body of the work, potentials can be long range in this Subsection.

Under the assumption (2.1), we can apply Theorem 2.8 in [22] with $n = 3$ and obtain that for any bounded symbols $p_{\pm,a}(x,\xi_a)$, $a\in\mathcal{A}$, with supp $p_{\pm,a} \subset \{(x,\xi_a); \pm x_a\cdot\xi_a \ge -(1-\varepsilon)|x_a||\xi_a|\}\cap\{x; \forall b\in\mathcal{A}, b\not\subseteq a, |x^b| \ge \varepsilon|x|\}$, $\varepsilon > 0$, one has: $\exists\lambda_0 > 0$ depending on the support of $p_{\pm,a}$ such that for any $s\in]1/2, 2[$,

$$\|\langle x\rangle^{-s}R(\lambda \pm i0)p_{\pm,a}(x, D_a)\langle x\rangle^{s-1}\| \le C\lambda^{-1/2}. \tag{2.2}$$

As a consequence of (2.2), if p_c, $c = a, b\in\mathcal{A}$, is supported in $\{(x,\xi_c); |x_c\cdot\xi_c| \le (1-\varepsilon)|x_c||\xi_c|\}\cap\{x; \forall d\in\mathcal{A}, d\not\subseteq c, |x^d| \ge \varepsilon|x|\}$,

$$\|\langle x\rangle^{-1/2}p_b(x, D_b)R(\lambda\pm i0)p_a(x, D_a)\langle x\rangle^{-1/2}\| \le C\lambda^{-1/2}, \tag{2.3}$$

for $\lambda \ge \lambda_0$. If we have a symbol q_c with *larger* support in $\{(x,\xi_c); |x^c| \ge \delta|x|, |x\cdot\xi_c| \le (1-\varepsilon)|x||\xi_c|\}$ (this is the case for microlocalizations appeared in the representation formula for scattering matrices with *arbitrary* scattering channels), the above results can not be directly used. In fact, we can only apply these estimates to $g_c(x)q_c$ where $g_c(x)$ is an additional cut-off function supported in $\{x; |x^c| \le \varepsilon'|x|\}$ for some $\varepsilon' > 0$ so small that on the support of $g_c(x)q_c(x,\xi_c)$, one has:

$$|x_c\cdot\xi_c| \le (1-\varepsilon_1)|x_c||\xi_c|\}\cap\{x; \forall d\in\mathcal{A}, d\not\subseteq c, |x^d| \ge \varepsilon_1|x|$$

for some $\varepsilon_1 > 0$. Remark that if $\mathcal{F}_\alpha(\lambda)$ is the spectral representation for the sub-Hamiltonian P^a with scattering channel α (see (2.9) for the definition), one has

$$\eta_1(D_a)\eta_2(P^a)\mathcal{F}_\alpha(\lambda) = \mathcal{F}_\alpha(\lambda),$$

for any $\eta_1\in C_0^\infty$ which is equal to 1 for near $\{|\xi_a|^2 = \lambda - E_\alpha\}$ and for any η_2 which is equal to 1 near E_α. So to study the scattering matrices, we just need microlocal resolvent estimates with microlocalizations of the form $q_a(x, D_a)\eta_1(D_a)\eta_2(P^a)$. For this reason, we prove the following results which allow microlocalizations with larger supports.

Proposition 2.1 *For $c = a, b\in\mathcal{A}$, let $q_{\pm,c}$ be bounded symbols supported in $\{\pm x\cdot\xi_c \ge -(1-\varepsilon)\lambda^{1/2}|x|\}\cap\{x; |x^c| \ge d|x|\}$ for some $d > 0, \varepsilon > 0$ and*

$$|\partial_x^\alpha\partial_\xi^\beta q_{\pm,c}(x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}, \text{ uniformly in } \lambda. \tag{2.4}$$

Put $Q_{\pm,c} = q_{\pm,c}(x, D)\eta(P^c)$ where η is any smooth function with compact support on $\mathbf{R}$. Assume the conditions (2.1). Then there exists $\lambda_0 > 0$ such that for $s\in]1/2, 2[$,

$$\|\langle x\rangle^{-s}R(\lambda\pm i0)Q_{\pm,a}\langle x\rangle^{s-1}\| \le C_s\lambda^{-1/2}, \quad \forall\lambda \ge \lambda_0. \tag{2.5}$$

Let p_a be a bounded symbol as in (2.3). Let $Q_c = q_c(x,D)\eta(P^c)$ where q_c is bounded symbol supported in $\{|x\cdot\xi_c| \le (1-\varepsilon)\lambda^{1/2}|x|\} \cap \{x; |x^c| \ge d|x|\}$. Then one has:

$$\|\langle x\rangle^{-1/2} p_a(x,D) R(\lambda \pm i0) Q_b \langle x\rangle^{-1/2}\| \le C_s \lambda^{-1/2}, \tag{2.6}$$

$$\|\langle x\rangle^{-1/2} Q_a R(\lambda \pm i0) Q_b \langle x\rangle^{-1/2}\| \le C_s \lambda^{-1/2}, \tag{2.7}$$

for all $\lambda \ge \lambda_0$.

Proof. The point of the proof is to show that we can obtain from $\eta(P^a)$ a localization by $\eta'(-\Delta^a)$. Then we can apply the known results of [22] to $q_{\pm,a}(x, D_a)\eta'(-\Delta^a)$ for λ large enough. We just give the details for the proof of (2.5). (2.6) and (2.7) can be derived from (2.2) and (2.5) by an argument of interpolation. We shall use an induction on $\#a$ the number of clusters in the cluster decomposition $a \in \mathcal{A}$ beginning from $\#a = N$. When $\#a = N$, $x_a = x$. The result is proved in [7, 22]. When $\#a = N-1$, $P^a = -\Delta^a + V_a(x^a) = -\Delta^a + O(\langle x^a\rangle^{-\varepsilon_0})$. Here $O(\langle x^a\rangle^{-\varepsilon_0})$ is a term which can be estimated as

$$\|\langle x^a\rangle^{\varepsilon_0} O(\langle x^a\rangle^{-\varepsilon_0})(-\Delta^a + i)^{-1}\| \le C.$$

Note that $(-\Delta^a + i)^{-1}$ can be obtained from $\eta(P^a)$ because η is of compact support. Let $\chi(x)$ be a cut-off function which is equal to 1 on the support of $q_{\pm,a}$ and is supported in a set of the form $|x^a| \ge d'|x|$, $0 < d' < d$ such that $|\partial_x^\alpha \chi(x)| = O(\langle x\rangle^{-|\alpha|})$. On the support of χ, we have: $P^a = -\Delta^a + O(\langle x\rangle^{-\varepsilon_0})$. Since by the assumption (2.1), we can commute V_a with $-\Delta^a$ at least twice outside some compact set in x^a and each commutation gives an additional decay of the order $O(\langle x^a\rangle^{-1})$, one can prove by the method of functional calculus used, for example, in Appendix of [22] that

$$\chi(x)\eta(P^a) = \eta(-\Delta^a)\chi(x) + \sum_{j=1}^{N} \eta_j(-\Delta^a) W_j(x) + R_1,$$

where $N \in \mathbf{N}$ with $N\varepsilon_0 > 2$, $\eta_j \in C_0^\infty(\mathbf{R})$ with supp $\eta_j \subseteq \eta$, $W_j(x) = O(\langle x\rangle^{-j\varepsilon_0})$ and $R_1 = O(\langle x\rangle^{-2-\varepsilon_0})$. Since $\eta \in C_0^\infty$, one has on the support of $q_{+,a}(x,\xi)\eta(|\xi^a|^2)$:

$$x\cdot\xi = x\cdot\xi_a + x\cdot\xi^a \ge -(1-\varepsilon)\lambda^{1/2}|x| - M|x| \ge -(1-\varepsilon/2)\lambda^{1/2}|x|,$$

for $\lambda > \lambda_0$ if we take $\lambda_0 = (2M/\varepsilon)^2$. So we can apply Theorem 2.1 in [22] to $q_{+,a}(x,D)\eta(-\Delta^a)$ and obtain that for $1/2 < s < 2$,

$$\|\langle x\rangle^{-s} R(\lambda + i0) q_{+,a}(x,D)\eta(-\Delta^a)\langle x\rangle^{s-1}\| \le C_s \lambda^{-1/2}, \quad \lambda > \lambda_0.$$

Similar estimate holds for the microlocalization by $q_{+,a}(x,D)\eta_j(-\Delta^a)$, $j = 1, \cdots, N$. Since $s - 1 - (2+\varepsilon_0) < -1/2$, the term related to R_1 is bounded by $O(\lambda^{-1/2})$. (2.5) is proved for a with $\#a = N-1$.

Suppose now (2.5) is true for any a with $\#a > k$ ($k \ge 2$). When $\#a = k$, we introduce a partition of unity on $\mathbf{X}^a$:

$$\chi_0(x^a) + \sum_{c\in\mathcal{A}_a} \chi_c(x^a) = 1 \text{ on } \mathbf{X}^a,$$

where $\mathcal{A}_a = \{c \in \mathcal{A}; c \subset a, \#c < N\}$ and supp $\chi_0 \subset \{x^a; \forall c \in \mathcal{A}_a, |x^c| > \delta|x^a|\}$, supp $\chi_c \subset \{x^a; \forall d \in \mathcal{A}_a, d \not\subseteq c, |x^d| > \delta|x^a|\}$. By the geometrical assumptions on the configuration of generalized N-body systems, such a partition exists at least for $\delta > 0$ sufficiently small. On the support of $\chi_0(x^a)$, we have:

$$P^a = -\Delta^a + \sum_{c\subseteq a} V_c(x^c) = -\Delta^a + O(\langle x^a\rangle^{-\varepsilon_0}),$$

where $O(\langle x^a\rangle^{-\varepsilon_0})$ has the same meaning as before. Since we can write for $c \subset a$

$$P^a = -\Delta^a + \sum_{d\subseteq c} V_d(x^d) + \sum_{d\in\mathcal{A}_a, d\not\subseteq c} V_d(x^d) = P^c - \Delta_{x_c^a} + \sum_{d\not\subseteq c} V_d(x^d),$$

one has on supp χ_c, $P^a = P^c - \Delta_{x_c^a} + O(\langle x^a\rangle^{-\varepsilon_0})$. Since supp $q_{\pm,a} \subset \{|x^a| \geq \delta|x|\}$, by functional calculus, one obtains for some $N > 2/\varepsilon_0$:

$$Q_{\pm,a} = q_{\pm,a}(x, D)\sum_{j=0}^{N}\{\eta_{j,0}(-\Delta^a)\chi_{j,0}(x^a) + \sum_{c\in\mathcal{A}_a}\eta_{j,c}(P^c - \Delta_{x_c^a})\chi_{j,c}(x^a)\} + R_2,$$

where $\eta_{j,0}$ and $\eta_{j,c}$ are smooth functions with support contained in supp η and $\chi_{j,c}(x^a) = O(\langle x^a\rangle^{-j\varepsilon_0}) = O(\langle x\rangle^{-j\varepsilon_0})$ on the support of $q_{\pm,a}$ and $R_2 = O(\langle x\rangle^{-2-\varepsilon_0})$. By the arguments used above, (2.5) is true if we replace $Q_{\pm,a}$ by $q_{\pm,a}(x, D)\eta_{j0}(-\Delta^a)$.

To treat other terms, assume that supp $\eta \subset]-M, M[$ and $P^c \geq -M_c$ in the sense of selfadjoint operators. Put $M_1 = \max\{M_c; c \in \mathcal{A}_a\}$. Take $\eta_1, \eta_2 \in C_0^\infty(\mathbf{R})$ such that supp $\eta_1 \subset]-(M + M_1 + 1), M + M_1 + 1[$ and $\eta_1 = 1$ on $[-(M + M_1), M + M_1]$; supp $\eta_2 \subset]-(M + 1), M + 1[$ and $\eta_2 = 1$ on $[-M, M]$. Then, since supp $\eta_{j,c} \subseteq$ supp η,

$$\eta_{j,c}(P^c - \Delta_{x_c^a}) = \eta_1(-\Delta_{x_c^a})\eta_2(P^c)\eta_{jc}(P^c - \Delta_{x_c^a}).$$

Notice that for $c \subset a$, $x^a = x^c + x_c^a$ and $\xi_c = \xi_a + \xi_c^a$. The support of $q'_{\mp,c} \equiv q_{\pm,a}(x,\xi)\eta_1(|\xi_c^a|^2)\chi_c(x^a)$ is contained in

$$\{x; \forall d \not\subseteq c, |x^d| \geq c|x|\} \cap \{(x,\xi); \mp x\cdot\xi_c \leq \pm(1 - \varepsilon/2)\lambda^{1/2}|x|\}$$

for $\lambda > \lambda_0$ if we take $\lambda_0 >> 1$. Let $g_1(s) + g_2(s) \equiv 1$ on $\mathbf{R}$ be a partition of unity on $\mathbf{R}$ such that $g_1(s) = 1$ for $s < 1 + \delta$; 0 for $s > 1 + 2\delta$, $\delta > 0$. On the support of $q'_{+,c}g_1(|x|/\langle x_c\rangle)$, one has: $|x| \leq (1 + 2\delta)|x_c|$ and $x_c \cdot \xi_c = x \cdot \xi_c \geq -(1 - \varepsilon/2)(1 + 2\delta)\lambda^{1/2}|x_c|\} \geq -(1 - \varepsilon/4)\lambda^{1/2}|x_c|\}$ for $\delta << \varepsilon$. We can then apply Theorem 2.9 in [22] to estimate the term corresponding to this piece. On the suppot of $g_2(|x|/\langle x_c\rangle)$, one has $|x| \geq (1 + \delta)|x_c|$, which implies $|x^c|^2 \geq \delta|x|^2$. Let $q''_{\pm,c} = q'_{\pm,c}g_2(|x|/\langle x_c\rangle)$. $q''_{\pm,c}$ has the same support properties as $q_{\pm,a}$ (with a replaced by c). Since $c \subset a$, $\#c > \#a = k$. We can then apply the induction assumption to $\tilde{Q}_{\pm,c} = q''_{\pm,c}(x, D)\eta_2(P^c)$ to prove that (2.5) is true with $Q_{\pm,a}$ replaced by $\tilde{Q}_{\pm,c}$. Finally the term related to R_2 satifies also (2.5), because $R_2\langle x\rangle^{s-1} = O(\langle x\rangle^{-1})$. (2.5) is proved by induction. □

The following result is not needed in this work. We formulate it just for the sake of completeness.

Proposition 2.2 *Let $q_a(x, \xi_a)$ be bounded symbol (satisfying (2.4)) supported in $\{(x, \xi_a); |x^a| \geq c_0|x|, |\xi_a| \leq (1 + \varepsilon')\sqrt{\lambda}\}$ with $0 < \varepsilon' = \varepsilon'(c_0)$ small enough. Then (2.5), (2.6) and (2.7) are true with $Q_{\pm,c}, Q_c$ replaced by $Q'_c = q_c(x, D_c)\eta(P^c)$, $c = a, b \in \mathcal{A}$. Here η is of compact support.*

Proof. As in the proof of Proposition 2.1, we can reduce the problem to the operators of the form $Q''_c = q_c(x, D_c)\eta'(-\Delta^c)$, where η' is of compact support. On the support of $q_c(x, \xi_c)\eta'(|\xi^c|^2)$, we have $|x_c| \leq \sqrt{1 - c_0^2}|x|$ and $|\xi^c| \leq M$ and therefore

$$|x \cdot \xi| \leq |x^c \cdot \xi^c| + |x_c \cdot \xi_c| \leq (M + (1 - c_0^2)^{1/2}(1 + \varepsilon')\sqrt{\lambda})|x|.$$

For $\varepsilon' > 0$ with $(1 - c_0^2)^{1/2}(1 + \varepsilon') < 1$, we can choose λ_0 large enough so that for $\lambda > \lambda_0$, the support of the symbol of Q''_c is contained in$\{|x \cdot \xi| \leq (1 - \varepsilon)\sqrt{\lambda}|x|\}$, $\varepsilon > 0$. We can then apply the results of [22] to Q''_c. □

2.2 Representation of Scattering Matrices

From now on, we assume that the potentials are short range and that the condition (2.1) is satisfied with $\varepsilon_0 = \rho > 1$. Let α and β be two arbitrary scattering channels. Denote $S_{\alpha\beta} = W_\beta^{+*}W_\alpha^-$: $L^2(\mathbf{X}_a) \to L^2(\mathbf{X}_b)$ the scattering operator associated with the incoming channel α and the outgoing channel β. We want to study the spectral representation of the scattering matrices for

$$T_{\alpha\beta} = \frac{i}{2\pi}\{S_{\alpha\beta} - \delta_{\alpha\beta}\}.$$

Let $\mathbf{I}_\beta =]E_\beta, +\infty[$. Let $F_\beta : L^2(\mathbf{X}_b) \to \mathbf{H}_\beta \equiv L^2(\mathbf{I}_\beta; L^2(\mathbf{S}_b))$ be defined by:

$$(F_\beta f)(\lambda, \theta) = c_\beta(\lambda) \int e^{-i\sqrt{(\lambda - E_\beta)}\,\theta\cdot x_b} f(x_b) dx_b, \quad (\lambda, \theta) \in \mathbf{I}_\beta \times \mathbf{S}_b, \tag{2.8}$$

where

$$c_\beta(\lambda) = (2\pi)^{-n_b/2}(\lambda - E_\beta)^{(n_b-2)/4},$$

with $n_b = \dim \mathbf{X}_b$. We can verify that $\|F_\beta f\|_{\mathbf{H}_\beta} = \|f\|_b$. Put $\mathcal{F}_\beta = F_\beta \mathcal{J}_\beta^*$. Then $\mathcal{F}_\beta P_b \mathcal{F}_\beta^*$ acts as the multiplication by λ in $\mathbf{H}_\beta$. By the Sobolev's lemma, F_β defines a family of maps, $F_\beta(\lambda), \lambda \in \mathbf{I}_\beta$, from $L^{2,s}(\mathbf{X}_b)$, $s > 1/2$, to $L^2(\mathbf{S}_b)$:

$$(F_\beta(\lambda) f)(\theta) = (F_\beta f)(\lambda, \theta),$$

Here $L^{2,s}$ is the weighted L^2 space $L^{2,s}(\mathbf{X}_b) = L^2(\mathbf{X}_b, \langle x_b\rangle^{2s} dx_b)$. The spectral representation for the sub-Hamiltonian P_b with scattering channel β is now defined by

$$\mathcal{F}_\beta(\lambda) = F_\beta(\lambda)\mathcal{J}_\beta^*. \tag{2.9}$$

One has $\mathcal{F}_\beta(\lambda) P_b \mathcal{F}(\lambda)^* = \lambda$ in the sense of non-bounded operators in $\mathbf{H}_\beta$. Similarly, we can construct a spectral representation $\mathcal{F}_\alpha$ for the sub-Hamiltonian P_a with scattering channel α.

Remark 2.1 The spectral representation given above (equations (2.8) and (2.9)) is actually only valid in the case $n_b = \dim \mathbf{X}_b \geq 2$. If $n_b = 1$, $\mathbf{S}_b$ is just two points: $\mathbf{S}_b = \{-1, 1\}$. In this case, $L^2(\mathbf{S}_b)$ should be understood as the space of two by two matrices. In order to avoid complications of notations, we always assume in the following without explicit mention that $n_b \geq 2$ for any $b \in \mathcal{A}$ with $\#b \geq 2$.

Denote now $\mathbf{I}_{\alpha\beta} =]\max\{E_\alpha, E_\beta\}, +\infty[$. Then $F_\beta T_{\alpha\beta} F_\alpha^*$ can be represented by a family of operators $\{T_{\alpha\beta}(\lambda) = F_\beta(\lambda) T_{\alpha\beta} F_\alpha(\lambda)^*; \lambda \in \mathbf{I}_{\alpha\beta}\}$ mapping $L^2(\mathbf{S}_b)$ to $L^2(\mathbf{S}_a)$. To give more precisions on $T_{\alpha\beta}(\lambda)$, we introduce appropriate cut-offs to avoid bad directions in momentum space.

Let $\mathbf{Y}_a = \mathbf{X}_a \setminus \cup_{b \not\subseteq a} \mathbf{X}_b$ and $\Sigma_a = \mathbf{Y}_a \cap \mathbf{S}_a$. Let $\chi_a(\xi_a)$ be of compact support with its conic support contained in $\mathbf{Y}_a$. Let $\chi_b(\xi_b)$ be chosen in a similar way. Instead of looking for spectral representations for $T_{\alpha\beta}$, we consider the operator $\chi_b(D_b) T_{\alpha\beta} \chi_a(D_a)$. Take $j \in C_0^\infty(\mathbf{R})$ with $j(t) = 0$ if $t < 1/2$ and $j(t) = 1$ if $t \geq 1$. For $a \in \mathcal{A}$, put:

$$j_a(x) = \prod_{c \not\subseteq a} j(\frac{|x^c|}{\delta |x|}),$$

and

$$J_a(x) = j_a(x) j(|x|) + (1 - j(|x|)). \tag{2.10}$$

Similarly, we introduce the cut-off function $J_b(\cdot)$. One can check that for $\delta > 0$ small enough, $J_a(x)$ is equal to 1 for x in a conic neighbourhood of $\mathrm{supp}\chi_a(\cdot)$. Here $\mathbf{X}_a$ is considered as a subspace of $\mathbf{X}$. Consequently, one has:

$$|\hat{x} \cdot \hat{\xi_c}| \leq 1 - \varepsilon, \varepsilon > 0, \tag{2.11}$$

for (x, ξ_c) in the support of $\nabla J_c(x) \chi_c(\xi_c)$, $c = a, b$. Here $\hat{x} = x/|x|$ and $\hat{\xi_c} = \xi_c/|\xi_c|$.

Assume the condition (2.1) for some $\varepsilon_0 = \rho > 1$. For any $f_c \in \mathcal{S}(\mathbf{X}_c)$ with $c = a$ or b, we denote: $f_b(\lambda, \theta) = (F_\beta f_b)(\lambda, \theta)$ and $f_a(\lambda, \theta') = (F_\alpha f_a)(\lambda, \theta')$. Take $\chi_c \in C_0^\infty(\mathbf{Y}_c \setminus 0)$ such that $\chi_c(\xi_c)\hat{f}_c(\xi_c) = \hat{f}_c(\xi_c)$ for $c = a, b$. By a formal computation, we can check (see [24] in the case β is a two-cluster scattering channel with non-threshold energy) that

$$< T_{\alpha\beta} f_a, f_b >_b = \int_{\mathbf{I}_{\alpha\beta}} (\tilde{T}_{\alpha\beta}(\lambda) f_a(\lambda, .), f_b(\lambda, .))_b d\lambda, \tag{2.12}$$

where

$$\tilde{T}_{\alpha\beta}(\lambda) = \lim_{\varepsilon \to 0_+} \mathcal{F}_\beta(\lambda)\{\chi_b(D_b) J_b(x) - Q_b^* R(\lambda + i\varepsilon)\} Q_a \mathcal{F}_\alpha(\lambda)^*, \text{ in } \lambda \in \mathbf{I}_{\alpha\beta}. \tag{2.13}$$

Here Q_c is defined by

$$Q_c = \{I_c(x)J_c(x) + [-\Delta, J_c]\}\chi_c(D_c). \tag{2.14}$$

Remarks 2.2 (a). General theory for the representation of scattering matrices only says that $T_{\alpha\beta}(\lambda)$ is defined about everywhere in λ. To study the high energy asymptotics, we shall prove that $\exists\lambda_0 > 0$ such that $\tilde{T}_{\alpha\beta}(\lambda)$ defines a bounded operator from $L^2(\mathbf{S}_a)$ to $L^2(\mathbf{S}_b)$ for any $\lambda > \lambda_0$ and is weakly continuous in λ. (See also [28]). Then we can identify $(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b$, $\varphi_c \in C_0^\infty(\Sigma_c)$, with $(\tilde{T}_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b$ and study the high energy asymptotics of $(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b$.

(b). For technical reasons, we use the representation (2.13) only in the case $b \not\subset a$. In the case $b \subset a$, we have $a \not\subset b$ and we can show that (2.12) is still true with $\tilde{T}_{\alpha\beta}(\lambda)$ now given by

$$\tilde{T}_{\alpha\beta}(\lambda) = \lim_{\varepsilon\to 0_+} \mathcal{F}_\beta(\lambda)Q_b^*\{J_a(x)\chi_a(D_a) - R(\lambda - i\varepsilon)Q_a\}\mathcal{F}_\alpha(\lambda)^*. \tag{2.15}$$

Here Q_c is still defined by (2.14). In fact (2.13) is deduced from time-dependent expression for $S_{\alpha\beta} - \delta_{\alpha\beta} = W_\beta^{+*}\{W_\alpha^- - W_\alpha^+\}$. (2.15) can be deduced by the same method, but making use of the identity $S_{\alpha\beta} - \delta_{\alpha\beta} = \{W_\beta^{+*} - W_\beta^{-*}\}W_\alpha^-$.

To prove that $\tilde{T}_{\alpha\beta}(\lambda)$ is bounded, we first check the structure of Q_c. By the assumption (2.1) and the choice of J_c, we have:

$$Q_c = O(\langle x\rangle^{-\rho}) + [-\Delta, J_c]\chi_c(D_c) = O(\langle x\rangle^{-\rho'}) - 2\nabla J_c\chi_c(D_c)\cdot\nabla, \tag{2.16}$$

for some $\rho' > 1$ and $c = a, b$. Since χ_c is of compact support, $\chi_c(D_c)\nabla$ is bounded on the range of $\mathcal{F}_\alpha(\lambda)$ or $\mathcal{F}_\beta(\lambda)$ according to $c = a$ or b. The presence of ∇ is not harmful if we just study the scattering matrices locally in λ. But it causes some serious difficulties, if one is interested in the high energy behaviour of scattering matrices, because then ∇ acting on the range of $\mathcal{F}_\alpha(\lambda)$ will give a contribution of the order $O(\lambda^{1/2})$. This is why we need to introduce an additional condition on scattering channels in next Section.

The following result is useful in this work.

Lemma 2.3 *Let α, β be two arbitrary scattering channels with $b \not\subseteq a$. With the above notations, one has*

$$\||\nabla J_a|^{1/2}\chi_a(D_a)\mathcal{F}_\alpha(\lambda)^*\|_{\mathcal{L}(L^2(\mathbf{S}_a);L^2(\mathbf{X}))} \le C\lambda^{-1/4}, \tag{2.17}$$

$$\||\nabla J_a|^{1/2}\mathcal{F}_\beta(\lambda)^*\|_{\mathcal{L}(L^2(\mathbf{S}_b);L^2(\mathbf{X}))} \le C\lambda^{-1/4}, \tag{2.18}$$

for $\lambda \ge 1$.

Proof. Notice first that $\nabla J_a = O(|x|^{-1})$ and by (2.11), $\nabla J_a(x)\chi_a(\xi_a)$ is supported in $\{|x\cdot\xi_a| \le (1-\varepsilon)|x||\xi_a|\}$. Introduce a partition of unity on $\mathbf{R}$: $g_1(s) + g_2(s) = 1, \forall s \in \mathbf{R}$, with $g_1(s) = 1$ for $s \le 1+\delta$; 0 for $s \ge 1+2\delta, \delta > 0$. On the support of $p_a(x,\xi_a) \equiv g_1(|x|/\langle x_a\rangle)|\nabla J_a(x)|^{1/2}\chi_a(\xi_a)$, we have $|x| \le (1+2\delta)|x_a|$ and

$$|x\cdot\xi_a| \le (1-\varepsilon)|x||\xi_a| \le (1-\varepsilon')|x_a||\xi_a|, \text{ for some } \varepsilon' > 0,$$

if $\delta > 0$ is chosen sufficiently small. So we can apply the results of [22] to the microlocalization by $p_a(x, D_a)$. Since $\nabla J_a(x) = O(\langle x\rangle^{-1})$ and

$$\mathcal{F}_\alpha(\lambda)^*\mathcal{F}_\alpha(\lambda) = \frac{1}{2i\pi}((-\Delta_a + E_\alpha - \lambda - i0)^{-1} - (-\Delta_a + E_\alpha - \lambda + i0)^{-1}), \tag{2.19}$$

we obtain from (2.3) for the free resolvent that

$$\|p_a(x, D_a)\mathcal{F}_\alpha(\lambda)^*\|_{\mathcal{L}(L^2(\mathbf{S}_a);L^2(\mathbf{X}))} \le C\lambda^{-1/4}.$$

To prove (2.17), it is sufficient to prove

$$\|\langle x\rangle^{-1/2}g_2(|x|/\langle x_a\rangle)\mathcal{F}_\alpha(\lambda)^*\|_{\mathcal{L}(L^2(\mathbf{S}_a);L^2(\mathbf{X}))} \le C\lambda^{-1/4}. \tag{2.20}$$

It is known (see [1]) that there exists $C > 0$ such that

$$\frac{1}{R}\int_{|x_a|\le R} |F_\alpha(1)^*\varphi|^2 dx_a \le C|\varphi|_a^2,$$

for any $\varphi \in L^2(\mathbf{S}_a)$ and any $R > 1$. By a suitable change of scale in x_a-variables, we obtain,

$$\frac{1}{R}\int_{|x_a|\le R} |F_\alpha(\lambda)^*\varphi|^2 dx_a \le C\lambda^{-1/2}|\varphi|_a^2,$$

for any $\varphi \in L^2(\mathbf{S}_a)$ and any $R > 1, \lambda > 1$. Now we first integrate

$$|\langle x\rangle^{-1/2} g_2(|x|/\langle x_a\rangle)\mathcal{F}_\alpha(\lambda)^*\varphi|^2$$

on $\mathbf{X}_a$. Taking notice that $|x_a| \le C|x^a|$ for x in the support of $g_2(|x|/\langle x_a\rangle)$ and that $\mathcal{F}_\alpha(\lambda)^*\varphi = \psi_\alpha(x^a)\otimes F_\alpha(\lambda)^*\varphi$, we see the integral over $\mathbf{X}_a$ is bounded by

$$M|\psi_\alpha(x^a)|^2\frac{1}{|x^a|}\int_{|x_a|\le C|x^a|} |F_\alpha(\lambda)^*\varphi|^2 dx_a \le M'\lambda^{-1/2}|\psi_\alpha(x^a)|^2|\varphi|_a^2,$$

for any $\varphi \in L^2(\mathbf{S}_a)$. Since $\psi_\alpha \in L^2(\mathbf{X}^a)$, this proves (2.20) and therefore, (2.17).

To prove (2.18), we remark that if $b \not\subset a$, $|x^b| \ge C|x|$ for x in supp J_a for some $C > 0$. One just needs to repeat the arguments used in the proof of (2.20). □

Lemma 2.4 *Let $J'_c, c = a, b$ be bounded cut-off function which is equal to 1 on supp ∇J_c and has the same support properties as ∇J_c (in particular, (2.11) still holds on the support of $J'_c(x)\chi_c(\xi_c)$ with a possibly smaller $\varepsilon > 0$). Let g_1, g_2 be the same as in the proof of Lemma 2.3. Put $g_{k,c}(x) = g_k(|x|/\langle x_c\rangle), c = a, b \in \mathcal{A}, k = 1, 2$. Define*

$$L_{1,c} = \langle x\rangle^{-1/2} g_{1,c}(x)J'_c(x)\chi_c(D_c), L_{2,c} = \langle x\rangle^{-1/2} g_{2,c}(x)J'_c(x)\chi_c(D_c)\eta(P^c),$$

where η_c is a smooth function with compact support. Then under the assumptions of Proposition 2.1, there exists $\lambda_0 > 0$ such that the following results hold for $\lambda > \lambda_0$.

$$\|\langle x\rangle^{-s}R(\lambda \pm i0)L_{k,c}\langle x\rangle^{s-1/2}\| \le C_s\lambda^{-1/2}, s \in]1/2, 2[, k = 1, 2, c = a, b. \tag{2.21}$$

$$\|L_{j,b}R(\lambda \pm i0)L_{k,a}\| \le C\lambda^{-1/2}, j, k = 1, 2. \tag{2.22}$$

Proof. Notice that $\langle x\rangle^{1/2}L_{j,c}$ is bounded. By the choice of J'_c and χ_c, we can apply (2.2) to prove (2.21) for $k = 1$ and (2.5) to prove (2.21) for $k = 2$. (2.22) can be derived from (2.3), (2.6) and (2.7). □

Now we can give a meaning to the representation formula (2.12).

Theorem 2.5 *Assume the condition (2.1) with $\varepsilon_0 = \rho > 1$. Let α, β be two arbitrary scattering channels. For any conic sets $\Gamma_c \subset Y_c$, $c = a, b$, there exists $\lambda_0 = \lambda(\Gamma_a, \Gamma_b) > 0$ such that the representation formula (2.12) is true for any $f_c \in \mathcal{S}(\mathbf{X}_c)$ with $\hat{f}_c \in C_0^\infty(\Gamma_c \setminus \{0\})$ and $f_c(\lambda; \cdot) = 0$ if $\lambda < \lambda_0$, where $\tilde{T}_{\alpha\beta}(\lambda)$ is given by (2.13) if $b \not\subset a$ and by (2.15) if $a \not\subset b$. Q_c in (2.14) is defined with χ_c a bounded smooth function with compact support in Γ_c such that $\chi_c(\xi_c) = 1$ for ξ_c) near supp $\hat{f}_c$ and J_c a bounded smooth cut-off function defined by (2.10) with $\delta > 0$ small enough so that (2.11) is true. In addition, $\tilde{T}_{\alpha\beta}(\lambda)$ is a bounded operator from $L^2(\mathbf{S}_a)$ to $L^2(\mathbf{S}_b)$ for all $\lambda > \lambda_0$ and $\lambda \to \tilde{T}_{\alpha\beta}(\lambda)$ is strongly continuous in λ.*

Proof. We only consider the case $b \not\subset a$. The other case can be treated similarly. Let $\rho' > 1$ be given by (2.16). It is known that for $s = \rho'/2 > 1/2$,

$$\|\langle x\rangle^{-s}R(\lambda \pm i0)\langle x\rangle^{-s}\| \le C\lambda^{-1/2},$$

and by (2.19)

$$\|\langle x\rangle^{-s}\mathcal{F}_\alpha(\lambda)^*\| \le C\lambda^{-1/4}.$$

So $\mathcal{F}_\beta(\lambda)O(\langle x\rangle^{-\rho'})R(\lambda\pm i0)O(\langle x\rangle^{-\rho'})\mathcal{F}_\alpha(\lambda)^*$ is bounded with the norm of the order $O(\lambda^{-1})$ as $\lambda\to\infty$.

Remark that $\mathcal{F}_\beta(\lambda)g_b(D_b)h_b(P^b)=\mathcal{F}_\beta(\lambda)$ for any bounded functions g_b,h_b such that $g_b(\xi_b)=1$ for ξ_b near $\{|\xi_b|^2=\lambda-E_\beta\}$ and $h_b(t)=1$ for t near E_β. Similar properties are also true for $\mathcal{F}_\alpha(\lambda)$. With this remark, we can decompose:

$$(\nabla J_a)\cdot\nabla\cdot\chi_a(D_a)\mathcal{F}_\alpha(\lambda)^*=(\tilde{L}_{1,a}+\tilde{L}_{2,a})M_a\mathcal{F}_\alpha(\lambda)^*.$$

Here

$$\tilde{L}_{1,a}\equiv\langle x\rangle^{1/2}\nabla J_a\cdot\nabla\cdot\chi_a(D_a)g_{1,a}\eta_a(P^a),$$

$$\tilde{L}_{2,a}\equiv\langle x\rangle^{1/2}\nabla J_a\cdot\nabla\cdot\chi_a(D_a)g_{2,a}\eta_a(P^a),$$

and $M_a\equiv\langle x\rangle^{-1/2}J'_a\chi'_a(D_a)$ with J'_a, (respectively, χ'_a), equal to 1 on supp ∇J_a , (respectively, supp χ_a), and 0 outside a sufficiently small neighbourhood and $\eta_a\in C_0^\infty$ with $\eta_a(E_\alpha)=1$. "$\equiv$" means here equality modulo a term $O(\langle x\rangle^{-s})$ for some $s>1/2$. This decomposition is true, because $\eta_a(P^a)$ commutes with functions of D_a and the commutator of $\eta_a(P^a)$ with various cut-off functions gives rise to terms of the order $O(|x|^{-1})$. The latter fact can be proved as in Appendix in [22]. Similarly, we can decompose $(\nabla J_b)\cdot\nabla\cdot\chi_b(D_b)\mathcal{F}_\beta(\lambda)^*$ as

$$(\nabla J_b)\cdot\nabla\cdot\chi_b(D_b)\mathcal{F}_\beta(\lambda)^*=(\tilde{L}_{1,b}+\tilde{L}_{2,b})M_b\mathcal{F}_\beta(\lambda)^*.$$

Now we can apply Proposition 2.1, Lemmas 2.3 and 2.4 to $\tilde{L}_{k,c}$ and M_c, $k=1,2$ and $c=a,b$, respectively and conclude that there exists $\lambda_0>0$ such that

$$\lim_{\varepsilon\to 0_+}\mathcal{F}_\beta(\lambda)Q_b^*R(\lambda+i\varepsilon)Q_a\mathcal{F}_\alpha(\lambda)^*$$

exists and is a bounded operator from $L^2(\mathbf{S}_a)$ to $L^2(\mathbf{S}_b)$ for $\lambda\geq\lambda_0$.

To show that $\tilde{T}_{\alpha\beta}(\lambda)$ is bounded for λ sufficiently large, it remains to prove that

$$\mathcal{F}_\beta(\lambda)\chi_b(D_b)J_bQ_a\mathcal{F}_\alpha(\lambda)^*$$

is bounded from $L^2(\mathbf{S}_a)$ to $L^2(\mathbf{S}_b)$. Since $b\not\subset a$, we have either $b=a$ or $b\not\subseteq a$. In the case $b=a$, we can equally apply (2.17) to $\mathcal{F}_\beta$. This proves the boundedness of $\mathcal{F}_\beta(\lambda)\chi_b(D_b)J_bQ_a\mathcal{F}_\alpha(\lambda)^*$ when $a=b$. When $b\not\subseteq a$, the desired result follows from (2.17) and (2.18). This proves that $\tilde{T}_{\alpha\beta}(\lambda)$ is well defined as bounded operator from $L^2(\mathbf{S}_a)$ to $L^2(\mathbf{S}_b)$ for $\lambda>\lambda_0$. The strong continuity of $\tilde{T}_{\alpha\beta}(\lambda)$ can be proved as in [25] in the case $\alpha=\beta$ is the free channel. See also [28] where the weak continuity of $T_{\alpha\beta}(\lambda)$ is proved. The details are omitted. □

Theorem 2.5 shows that $(T_{\alpha\beta}(\lambda)f_a(\lambda,\cdot),f_b(\lambda,\cdot))_b=(\tilde{T}_{\alpha\beta}(\lambda)f_a(\lambda,\cdot),f_b(\lambda,\cdot))_b$ is pointwisely well defined for all $\lambda>\lambda_0$ and is continuous in λ. Now for any given $\varphi_c\in C_0^\infty(\Sigma_c), c=a,b$, take a λ-dependent cut-off function $\chi_c(\cdot)\in C_0^\infty(\mathbf{Y}_c^*\setminus\{0\})$ such that

$$|\partial_\eta^\gamma\chi_c(\eta)|\leq C_\gamma\lambda^{-|\gamma|/2},\forall\eta\in\mathbf{X}_c^*,\lambda>1\text{ and }\gamma\in\mathbf{N}^{n_a},$$

and that $\mathcal{F}_\alpha(\lambda)^*\varphi_a=\chi_a(D_a)\mathcal{F}_\alpha(\lambda)^*\varphi_a$ (and the similar relation for $\chi_b(D_b)$). Let $J_c(\cdot)$ be constructed as before. Then there exists λ_0 depending on supp φ_a and supp φ_b such that

$$(T_{\alpha\beta}(\lambda)\varphi_a,\varphi_b)_b=(\tilde{T}_{\alpha\beta}(\lambda)\varphi_a,\varphi_b)_b,\tag{2.23}$$

for $\lambda>\lambda_0$, where $\tilde{T}_{\alpha\beta}(\lambda)$ is given by (2.13) if $b\not\subset a$; by (2.15) if $a\not\subset b$. In the next Section, we shall use (2.23) to study the asymptotics of $(T_{\alpha\beta}(\lambda)\varphi_a,\varphi_b)_b$ as $\lambda\to\infty$.

3 High Energy Asymptotics of Scattering Matrices

Even though we have established λ-dependent estimates in Proposition 2.1 and Lemmas 2.3 and 2.4, we have only proved in Theorem 2.5 that $\tilde{T}_{\alpha\beta}(\lambda)$ is bounded from $L^2(\mathbf{S}_a)$ to $L^2(\mathbf{S}_b)$ for each fixed λ. This is sufficient to etablish the representation formula (2.12), because $f_c(\lambda, \theta)$ is of compact support in λ. New difficulties arise when we want to study the high energy asymptotics for $(\tilde{T}_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b$. The first one is methodological. It is well known that the Born approximation is valid only in the case where the potential energy is small compared with the kinetic energy. That is why we shall assume that the potentials are bounded. The case where the potentials present singularities and the Born approximation is not valid will be studied elsewhere. To simplify some technical estimates, we replace in this Section the assumption (2.1) by the following stronger assumption

$$|\partial_y^\gamma V_a(y)| \leq C\langle y\rangle^{-\rho-|\gamma|}, \forall y \in \mathbf{X}^a, \forall a \in \mathcal{A}, \tag{3.1}$$

for some $\rho > 1$ and all γ with $|\gamma| \leq \max\{3, \frac{d}{2}+1\}$. The second one is technical and is related to the representation formula established in Section 2. In fact Q_c (see (2.14)) is a first order differential operator and the method of Section 2 can only lead to an estimate $(\tilde{T}_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b = O(1)$ as $\lambda \to \infty$. This is not satisfactory, because we know in the free channel case that the high energy asymptotics should be of the order $O(\lambda^{-1/2})$. For this reason, we shall introduce a modification in the representation for $T_{\alpha\beta}(\lambda)$ and more essentially, a mild assumption on the decay of eigenfunctions ψ_α, ψ_β.

For $\varphi_c \in C_0^\infty(\Sigma_c)$, let J_c and χ_c be constructed as at the end of Section 2. Put $J_c^\lambda(x) = J_c(x/\sqrt{\lambda}), \lambda > 0$. Then we can check that we still have

$$(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b = (\tilde{T}_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b, \quad \lambda > \lambda_0,$$

where $\tilde{T}_{\alpha\beta}(\lambda)$ is still given by (2.13) if $b \not\subset a$, by (2.15) if $a \not\subset b$ with the only modification that now Q_c is defined by

$$Q_c = \{I_c(x)J_c^\lambda(x) + [-\Delta, J_c^\lambda]\}\chi_c(D_c), \quad \text{for } c = a, b. \tag{3.2}$$

To see why we need an assumption on the eigenfunctions ψ_α, ψ_β, let us first study the leading term in $\tilde{T}_{\alpha\beta}(\lambda)$.

3.1 The Leading Term

Assume without loss that $b \not\subset a$. Otherwise we use the representation (2.15). Let

$$I_1(\lambda) = (\mathcal{F}_\beta(\lambda)\chi_b(D_b)J_b^\lambda Q_a \mathcal{F}_\alpha(\lambda)^*\varphi_a, \varphi_b)_b.$$

By the choice of $\chi_c(\cdot)$, we can write $I_1(\lambda)$ as

$$I_1(\lambda) = < J_b^\lambda\{I_a J_a^\lambda + [-\Delta, J_a^\lambda]\}\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >, \tag{3.3}$$

where $< \cdot, \cdot >$ is the scalar product in $L^2(\mathbf{X})$ and $\varphi_\alpha(\lambda) = \mathcal{F}_\alpha(\lambda)^*\varphi_a, \varphi_\beta(\lambda) = \mathcal{F}_\beta^*(\lambda)\varphi_b$. Since φ_a and φ_b are C^∞, by the method of stationary phase (see also [25]), one has:

$$\begin{aligned} \varphi_\alpha(x;\lambda) &= 2^{-1/2}(2\pi)^{-1/2}\lambda_\alpha^{-1/4}|x_a|^{(1-n_a)/2}\psi_\alpha(x^a) \\ \times \quad &\{e^{i(\sqrt{\lambda_\alpha}|x_a| + \frac{(n_a-1)\pi}{4})}(\varphi_a(\hat{x}_a) + r_{+,\alpha}(x_a,\lambda)) \\ &+ e^{-i(\sqrt{\lambda_\alpha}|x_a| + \frac{(n_a-1)\pi}{4})}(\varphi_a(-\hat{x}_a) + r_{-,\alpha}(x_a,\lambda))\}. \end{aligned} \tag{3.4}$$

Here $\lambda_\alpha = \lambda - E_\alpha$, $n_a = \dim X_a$ is assumed to be ≥ 2 (see Remark 2.1) and $r_{\pm,\alpha}$ are smooth functions having an asymptotic expansion of the form

$$r_{\pm,\alpha}(x_a,\lambda) \sim \sum_{j=1}^{\infty} \lambda^{-j/2}|x_a|^{-j}a_{j,\pm}(x_a), \lambda^{1/2}|x_a| >> 1,$$

where $|a_{j,\pm}(x_a)| \le C_j$ uniformly in x_a and $a_{j,\pm}(x_a) = 0$ for all $j \ge 1$ if $\pm\hat{x}_a$ is not in the support of φ_α. The same result is true for $\varphi_\beta(x,\lambda)$ when we replace α by β and a by b in (3.4).

In the free channel case treated in [25], $\#a = N$, $x_a = x$ and by the choice of J_a, ∇J_a is equal to 0 for $\mp x$ in a conic neighbourhood of the test function φ_a. We have by (3.4)

$$[-\Delta, J_a^\lambda]\varphi_\alpha(\lambda) = O((\lambda|x|)^{-\infty})$$

as $\sqrt{\lambda}|x| \to \infty$. This is no longer true if $\#a < N$. The choice of J_a^λ only gives: $\nabla J_a^\lambda(x) = 0$ if $\pm\hat{x}_a \in$ supp $\varphi_a(\cdot)$ *and* $|x^a| < \varepsilon|x|$ with $\varepsilon > 0$ small enough. In the region $|x^a| \ge \varepsilon|x|$, $\nabla J_a^\lambda(x) \not\equiv 0$ in a conic neighbourhood of supp φ_a and by (3.4), $[-\Delta, J_a^\lambda]\varphi_\alpha(x,\lambda) = O(\lambda^{-1/4}|x_a|^{-(n_a-1)/2})\psi_\alpha(x^a)$. Without additional assumption on $\psi_\alpha(x^a)$, we do not see, for example, how to prove the norm of this term in $L^2(\mathbf{X})$ is of the order $O(\lambda^{-1/4})$. For this reason, we introduce the following assumption on scattering channels:

$$\psi_\alpha \in L^{2,\varepsilon_\alpha}(\mathbf{X}^a), \psi_\beta \in L^{2,\varepsilon_\beta}(\mathbf{X}^b) \quad \text{with } \varepsilon_\alpha + \varepsilon_\beta \ge 1. \tag{3.5}$$

Notice that (3.5) is always satisfied if E_α is not a threshold of P^a or E_β is not a threshold of P^b. Note also that if $\#c = N$, then, $X^c = \{0\}$ is compact. So in the case where one of the scattering channels α, β is the free channel, (3.5) is satisfied with $\varepsilon_\alpha + \varepsilon_\beta = +\infty$. Therefore, if one of the scattering channels is of non-threshold energy or is the free channel, the other can be arbitrary.

Proposition 3.1 *(i). Let $a = b \in \mathcal{A}$. Assume the conditions (3.1) and (3.5). Let $I_1(\lambda)$ be defined by (3.2) with φ_b replaced by $\varphi_a' \in C_0^\infty(\Sigma_a)$. Then,*

$$I_1(\lambda) - (\mathcal{F}_\beta(\lambda) I_a \mathcal{F}_\alpha(\lambda)^* \varphi_a, \varphi_a')_a = O(\lambda^{-1/2-\eta}), \quad \lambda \to \infty. \tag{3.6}$$

(ii). Let $a \ne b$. Assume the condition (3.1). One has for any $\varphi_c \in C_0^\infty(\Sigma_c)$, $c = a, b$,

$$I_1(\lambda) = O(\lambda^{-3/2}), \quad \lambda \to \infty. \tag{3.7}$$

Here $I_a = \sum_{c \not\subseteq a} V_c(x^c)$ and $\eta \ge 0$ is defined by $\eta = \min\{(\varepsilon_\alpha + \varepsilon_\beta - 1)/2, 1/2\}$.

Proof. We first prove (3.6). Let $a = b$. We begin with estimating the term $< [-\Delta, J_a^\lambda]\varphi_\alpha(\lambda), J_a^\lambda\varphi_\beta(\lambda) >$. Let $\chi_{\varepsilon,a}$ be a smooth cut-off function with support in $\{x; |x^a| < 2\varepsilon|x|\}$ and equal to 1 on $\{x; |x^a| < \varepsilon|x|\}$. We have seen that

$$< \chi_{\varepsilon,a}[-\Delta, J_a^\lambda]\varphi_\alpha(\lambda), J_a^\lambda\varphi_\beta(\lambda) >= O(\lambda^{-\infty}), \quad \text{if } \varepsilon > 0 \text{ is small enough.}$$

To treat the term $< (1-\chi_{\varepsilon,a})[-\Delta, J_a^\lambda]\varphi_\alpha(\lambda), J_a^\lambda\varphi_\beta(\lambda) >$ we write J_a^λ as

$$J_a^\lambda = j_0^\lambda + (1 - j_0^\lambda) j_a,$$

with $j_0^\lambda(x) = 1 - j(|x|/\sqrt{\lambda})$ (see (2.10) for the definition of j and j_a). Then

$$\nabla J_a^\lambda = \nabla j_0^\lambda (1 - j_a) + (1 - j_0^\lambda)\nabla j_a.$$

On the support of $(1-\chi_{\varepsilon,a})\nabla J_a^\lambda$, $|x| \ge c\lambda^{1/2}$ and $|x_a| \le C|x^a|$. We deduce from the assumption (3.5) by the arguments used in the proof of Lemma 2.4 that

$$\langle x\rangle^{\varepsilon_\alpha - 1/2}(1-\chi_{\varepsilon,a})\varphi_\alpha(\lambda) = O(\lambda^{-1/4}),$$

and

$$\langle x\rangle^{\varepsilon_\beta - 1/2}(1-\chi_{\varepsilon,a})\varphi_\beta(\lambda) = O(\lambda^{-1/4}) \quad \text{in } L^2(\mathbf{X}).$$

Since $(1-j_0^\lambda)(\nabla j_a)\cdot\nabla\varphi_\alpha(\lambda)$ behaves like $O(\lambda^{\tau/2}|x|^{-\tau})\varphi_\alpha(\lambda)$ for any $\tau \in [0,1]$, we obtain

$$< (1-\chi_{\varepsilon,a})(1-j_0^\lambda)(\nabla j_a)\nabla\varphi_\alpha(\lambda), J_a^\lambda\varphi_\beta(\lambda) >= O(\lambda^{-1/2-(\varepsilon_\alpha+\varepsilon_\beta-1)/2}). \tag{3.8}$$

We can show by the same arguments that

$$< \chi_{\varepsilon,a}(1-(J_a^\lambda)^2) I_a \varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-\infty})$$

and

$$< (1-\chi_{\varepsilon,a})(1-(J_a^\lambda)^2)I_a\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >$$
$$= O(\lambda^{-1/2-(\varepsilon_\alpha+\varepsilon_\beta-1)/2}).$$

This proves (3.6).

Let now $b \not\subseteq a$. We shall use the method of oscillatory integrals to estimate

$$r_1 =< [-\Delta, J_a^\lambda]\varphi_\alpha(\lambda), J_b^\lambda\varphi_\beta(\lambda) > .$$

Let $\theta \in C_0^\infty(\mathbf{X})$ which is equal to 1 for $|x| \le 1$. Put

$$r_1(x;\lambda) = ([-\Delta, J_a^\lambda]\varphi_\alpha(\lambda))\overline{J_a^\lambda\varphi_\beta(\lambda)}.$$

Then we can check for each fixed λ that

$$r_1 = \int_{\mathbf{X}} r_1(x;\lambda)dx = \lim_{R\to\infty}\int \theta(x/R)r_1(x;\lambda)dx.$$

The last integral can be written as

$$c_\alpha(\lambda)c_\beta(\lambda)\int_{\mathbf{S}_a\times\mathbf{S}_b}\varphi_a(\omega_a)\overline{\varphi_b(\omega_b)}\int_{\mathbf{X}} e^{ix\cdot(\sqrt{\lambda_\alpha}\omega_a-\sqrt{\lambda_\beta}\omega_b)}\theta(x/R)q(x;\lambda)dx,$$

where

$$q(x;\lambda) = (e^{-i\sqrt{\lambda_\alpha}x\cdot\omega_a}[-\Delta, J_a^\lambda]e^{i\sqrt{\lambda_\alpha}x\cdot\omega_a}\psi_\alpha)\overline{J_a^\lambda\psi_\beta}.$$

Since $b \not\subseteq a$, one has for $\omega_a \in$ supp φ_a $\omega_a \notin \mathbf{X}_b$. So $\sqrt{\lambda_\alpha}\omega_a - \sqrt{\lambda_\beta}\omega_b \neq 0$ for any $\omega_b \in \mathbf{S}_b$ and $\sqrt{\lambda_\alpha}, \sqrt{\lambda_\beta} \neq 0$. This means that the phase $e^{ix\cdot(\sqrt{\lambda_\alpha}\omega_a-\sqrt{\lambda_\beta}\omega_b)}$ has no critical point on $\mathbf{X}$ and we can apply the method of non-stationary phase. Let

$$L = \frac{(\sqrt{\lambda_\alpha}\omega_a - \sqrt{\lambda_\beta}\omega_b)\cdot D}{|\sqrt{\lambda_\alpha}\omega_a - \sqrt{\lambda_\beta}\omega_b|^2}$$

which satisfies the equation

$$L(e^{ix\cdot(\sqrt{\lambda_\alpha}\omega_a-\sqrt{\lambda_\beta}\omega_b)}) = e^{ix\cdot(\sqrt{\lambda_\alpha}\omega_a-\sqrt{\lambda_\beta}\omega_b)}.$$

The assumptions on potentials allow us to integrate by parts at least thrice and each integration by parts produces a factor of the order $O(\lambda^{-1/2})$. Taking the limit $R\to\infty$ after integration by parts, the reader can check by the arguments used in the proof of Theorem 2.5 that $r_1 = O(\lambda^{-3/2})$.

Similarly, one can prove that $< I_aJ_a^\lambda J_b^\lambda\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-3/2})$. This proves (3.7). □

Remark that the proof of Proposition 3.1 shows that if all potentials are smooth, then we can integrate by parts using the operator L an infinite number of times and deduce that in the case $a \neq b$, $I_1(\lambda) = O(\lambda^{-\infty})$ as $\lambda\to\infty$.

3.2 Remainder Estimate

Let $I_2(\lambda) =< Q_b^*R(\lambda+i0)Q_a\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >$. We want to show that $I_2(\lambda)$ is negligible as $\lambda\to\infty$.

Proposition 3.2 *Under the assumptions (3.1) and (3.5), we have*

$$||I_2(\lambda)|| = O(\lambda^{-1/2-\eta}), \lambda\to\infty, \tag{3.9}$$

where $\eta > 0$ is defined by

$$\eta = \min\{1/2, (\rho+\varepsilon_\alpha-1)/2, (\rho+\varepsilon_\beta-1)/2, (\varepsilon_\alpha+\varepsilon_\beta-1)/2\}.$$

The proof of Proposition 3.2 is technical and will be divided into the following three Lemmas. We first treat the term related to $I_cJ_c^\lambda$.

Lemma 3.3 *Under the assumptions of Proposition 3.2, one has*

$$< \chi_b(D_b) I_b J_b^\lambda R(\lambda + i0) I_a J_a^\lambda \chi_a(D_a)\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-1/2-\eta}), \lambda \to \infty, \tag{3.10}$$

where $\eta > 0$ is defined as in Proposition 3.2.

Proof. We write for $c = a$ or b: $I_c J_c^\lambda = I_c j_c + I_c j_0^\lambda (1 - j_c) = O(\langle x\rangle^{-\rho}) + I_c j_0^\lambda(1 - j_c)$. Introduce a partition of unity $g_{1,c}(x) + g_{2,c}(x) = 1$ on $\mathbf{X}$, where $g_{1,c} = 1$ if $|x^c| \le \delta'|x|$ and 0 outside a slightly larger neighbourhood. If $\delta' > 0$ is small enough, $g_{1,c}(x)(1 - j_c)(x) = 0$ for $x = x^c + x_c$ with $\pm\hat{x}_c$ in the support of φ_c, $c = a, b$. In this case, we can apply (3.4) to show that, for example, $g_{1,a}(1 - j_a)\varphi_\alpha(\lambda) = O(|\lambda x|^{-\infty})$. This shows

$$< \chi_b(D_b) I_b J_b^\lambda R(\lambda + i0) I_a J_a^\lambda \chi_a(D_a)\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >$$

$$= O(\lambda^{-\infty}) + < \chi_b(D_b) g_{2,b} I_b J_b^\lambda R(\lambda + i0) I_a J_a^\lambda g_{2,a} \chi_a(D_a)\varphi_\alpha(\lambda), \varphi_\beta(\lambda) > .$$

According to the assumption (3.5) and Lemma 2.3, one has

$$\|\langle x\rangle^{\varepsilon_\alpha - 1/2} g_{2,a}\varphi_\alpha(\lambda)\| \le C\lambda^{-1/4}.$$

Since j_0^λ is supported in $\{|x| \le C\lambda^{1/2}\}$, it is easy to prove that

$$\|\langle x\rangle^s j_0^\lambda g_{2,a}\varphi_\alpha(\lambda)\| \le C_{s,\varepsilon}\lambda^{-1/4+(s-\varepsilon_\alpha+1/2)/2}, \forall s \ge 0.$$

The same is true if we replace a by b and α by β. Choosing appropriately s, we can apply Proposition 2.1 to obtain the following estimates:

$$< O(\langle x\rangle^{-\rho}) R(\lambda + i0) O(\langle x\rangle^{-\rho})\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-1}),$$

$$< O(\langle x\rangle^{-\rho}) R(\lambda + i0) I_a j_0^\lambda (1 - j_a) g_{2,a}\chi_a(D_a)\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-(\rho+\varepsilon_\alpha)/2}),$$

and

$$< \chi_b(D_b) I_b j_0^\lambda(1 - j_b) g_{2,b} R(\lambda + i0) I_a j_0^\lambda (1 - j_a) g_{2,a}\chi_a(D_a)\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >$$
$$= O(\lambda^{-(\varepsilon_\alpha+\varepsilon_\beta)/2}).$$

Similar results hold if we interchange a with b and α with β. (3.10) follows from the above estimates. □

Lemma 3.4

$$\begin{aligned} &< \chi_b(D_b)[-\Delta, J_b^\lambda] R(\lambda + i0)[-\Delta, J_a^\lambda]\chi_a(D_a)\varphi_\alpha(\lambda), \varphi_\beta(\lambda) > \\ &= O(\lambda^{-1/2-\min\{1/2, (\varepsilon_\alpha+\varepsilon_\beta-1)/2\}}), \end{aligned} \tag{3.11}$$

as $\lambda \to \infty$.

Proof. Since $[\Delta, J_c^\lambda] = 2\nabla J_c^\lambda \cdot \nabla + O(\langle x\rangle^{-2})$, we only check the terms related to ∇J_c^λ. Let $g_{1,c}, g_{2,c}$ be defined as in the proof of Lemma 3.3. By (3.4), we have

$$g_{1,a}\nabla J_a^\lambda \cdot \nabla\varphi_\alpha(\lambda) = O(|\lambda x|^{-\infty}).$$

On the support of $g_{2,c}$, $|x^c| \ge \delta'|x|$ for some $\delta' > 0$. Notice that $\nabla J_c^\lambda = \nabla j_0^\lambda(1 - j_c) + (1 - j_0^\lambda)\nabla j_c = O(\lambda^{-s/2}|x|^{-1+s})$ for any $s \in [0, 1]$, $c = a, b$. Write $\nabla = \nabla^c + \nabla_c$, $c = a, b$. Since $\nabla^a\varphi_\alpha(\lambda) = (\nabla^a\psi_\alpha(x^a))F_\alpha(\lambda)\varphi$ which causes no loss in λ, we concentrate our attention to the term related to ∇_a. Put $B_c = g_{2,c}(\nabla J_c^\lambda) \cdot \nabla_c \cdot \chi_c(D_c)$. Note that ∇_c acting on $\varphi_\alpha(\lambda)$ or $\varphi_\beta(\lambda)$ according to $c = a$ or b gives a loss of order $O(\sqrt{\lambda})$ as $\lambda \to \infty$. But the symbol of B_c is bounded uniformly with respect to λ due to the λ dependent choice of χ_c and J_c^λ. By the calculus of pseudodifferential operators, we can find $B_c'(x, \xi_c)$ a bounded symbol which is equal to 0 outside a sufficiently small neighbourhood of the support of $g_{2,c}(\nabla J_c^\lambda) \cdot \xi_c\chi_c(\xi_c)$ such that

$$B_c = B_c B_c' + R_{1,c}$$

where $R_{1,c}$ is a pseudo-differential operator with symbol of the order $O(\langle x\rangle^{-2})$ uniformly in λ.

Let η_a be a smooth function with compact support which is equal to 1 at E_α. We can decompose $B_a\varphi_\alpha(\lambda)$ as in the proof of Theorem 2.5

$$B_a\varphi_\alpha(\lambda) = (L_aM_a + R_{1,a})\varphi_\alpha(\lambda),$$

where $L_a \equiv B_a\eta_a(P^a)$ and $M_a \equiv B'_a$ and the "$\equiv$ " here means the equality modulo a term of the order $O(\langle x\rangle^{-1/2})$ and having the similar support property as the leading term.

As in the proof of Proposition 3.1, we obtain from Proposition 2.1 and Lemma 2.4 that

$$\begin{aligned}
&|< B_b^*R(\lambda+i0)O(\langle x\rangle^{-2})\chi_a(D_a)g_{2,a}\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >| \\
&\leq O(\lambda^{-1}) + C\|\langle x\rangle^{1/2}L_b^*R(\lambda+i0)\langle x\rangle^{-3/2}\| \\
&\quad\times\|\langle x\rangle^{-1/2}g_{2,a}\varphi_\alpha(\lambda)\|\|\langle x\rangle^{-1/2}M_b\varphi_\beta(\lambda)\| \\
&\leq C'\lambda^{-1},
\end{aligned}$$

and making use of the assumption (3.5), one has

$$\begin{aligned}
&|< B_b^*R(\lambda+i0)B_a\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >| \\
&\leq C\lambda^{-1} + \|\langle x\rangle^{-\varepsilon_\beta+1/2}L_b^*R(\lambda+i0)L_a\langle x\rangle^{-\varepsilon_\alpha+1/2}\|\|\langle x\rangle^{\varepsilon_\alpha-1/2}M_a\varphi_\alpha(\lambda)\| \\
&\quad\times\|\langle x\rangle^{\varepsilon_\beta-1/2}M_b\varphi_\beta(\lambda)\| \\
&\leq C'(\lambda^{-1} + \lambda^{-(\varepsilon_\alpha+\varepsilon_\beta)/2}),
\end{aligned}$$

as $\lambda\to\infty$. In the last estimate, we used the fact that the symbol of L_c is of the order $O(\lambda^{s/2}|x|^{-s})$ for any $s\in[0,1]$ and consequently, by Proposition 2.1,

$$\|\langle x\rangle^{-\varepsilon_\beta+1/2}L_b^*R(\lambda+i0)L_a\langle x\rangle^{-\varepsilon_\alpha+1/2}\| = O(\lambda^{-\min\{1/2,(\varepsilon_\alpha+\varepsilon_\beta-1)/2\}}).$$

Summing up, we have proved:

$$\begin{aligned}
&< \chi_b(D_b)[-\Delta, J_b^\lambda]R(\lambda+i0)[-\Delta, J_a^\lambda]\chi_a(D_a)\varphi_\alpha(\lambda), \varphi_\beta(\lambda) > \\
&= O(\lambda^{-\infty}) + 4 < \{B_b^* + O(\langle x\rangle^{-2})\}R(\lambda+i0)\{B_a + O(\langle x\rangle^{-2})\}\varphi_\alpha(\lambda), \varphi_\beta(\lambda) > \\
&= O(\lambda^{-1/2-\min\{1/2,(\varepsilon_\alpha+\varepsilon_\beta-1)/2\}}), \quad \lambda\to\infty.
\end{aligned}$$

This proves Lemma 3.4. □

Lemma 3.5 *The following estimates hold as $\lambda\to\infty$:*

$$< I_bJ_b^\lambda R(\lambda+i0)[-\Delta, J_a^\lambda]\chi_a(D_a)\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-\min\{1,(\rho+\varepsilon_\alpha)/2,(\varepsilon_\alpha+\varepsilon_\beta)/2\}}), \tag{3.12}$$

$$< \chi_b(D_b)[-\Delta, J_b^\lambda]R(\lambda+i0)I_aJ_a^\lambda\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-\min\{1,(\rho+\varepsilon_\beta)/2,(\varepsilon_\alpha+\varepsilon_\beta)/2\}}). \tag{3.13}$$

Proof. Lemma 3.5 can be proved by combining the methods used in Lemma 3.3 and Lemma 3.4 The details are omitted. □

Proof of Proposition 3.2 It follows immediately from Lemmas 3.3-3.5. □

It follows from Propositions 3.1 and 3.2 that we have proved the following

Theorem 3.6 *Under the assumptions (3.1) and (3.5), the following results hold.*
(i). If $a=b$, one has for any $\varphi_a, \varphi'_a \in C_0^\infty(\Sigma_a)$

$$(T_{\alpha\beta}(\lambda)\varphi_a, \varphi'_a)_a - (\mathcal{F}_\beta(\lambda)I_a\mathcal{F}_\alpha(\lambda)^*\varphi_a, \varphi'_a)_a = O(\lambda^{-1/2-\eta}), \quad \lambda\to\infty. \tag{3.14}$$

(ii). If $a\neq b$, one has for any $\varphi_c\in C_0^\infty(\Sigma_c)$, $c=a,b$,

$$(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b = O(\lambda^{-1/2-\eta}), \quad \lambda\to\infty. \tag{3.15}$$

Here $I_a = \sum_{c\not\subseteq b} V_c(x^c)$ and η is defined as in Proposition 3.2.

Remark that if $\varepsilon_\alpha + \varepsilon_\beta = 1$, we just proved that $(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b = O(\lambda^{-1/2})$. In the case $\varepsilon_\alpha + \varepsilon_\beta > 1$, $\eta > 0$. In this case, we can prove as in [25] that (3.14) really gives the leading term of the high energy asymptotics for $(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b')_b$ in the case $a = b$.

Corollary 3.7 *Let $a = b$ and assume the conditions (2.1) and (3.5) with $\varepsilon_\alpha + \varepsilon_\beta > 1$. Then there exists $\delta > 0$ such that for any $\varphi, \psi \in C_0^\infty(\Sigma_a)$, one has:*

$$(T_{\alpha\beta}(\lambda)\varphi, \psi)_a = \frac{1}{4\pi\sqrt{\lambda}} \int_{\mathbf{X}_a} \frac{(I_{\alpha\beta}(x_a) + I_{\alpha\beta}(-x_a))}{|x_a|^{n_a - 1}} \varphi(\hat{x}_a)\overline{\psi(\hat{x}_a)} dx_a + O(\lambda^{-1/2-\delta}), \tag{3.16}$$

as $\lambda \to \infty$. Here

$$I_{\alpha\beta}(x_a) = \int_{\mathbf{X}^a} I_a(x)\psi_\alpha(x^a)\overline{\psi_\beta(x^a)} dx^a.$$

Proof. The result is proved in [25] in the case $\alpha = \beta$ is the free channel. Making use of the same argument and (3.4), we can derive that

$$\begin{aligned}
&(\mathcal{F}_\beta(\lambda) I_a \mathcal{F}_\alpha(\lambda)^* \varphi, \psi)_a \\
&= \frac{1}{4\pi\sqrt{\lambda}} \int_{\mathbf{X}_a} \frac{(I_{\alpha\beta}(x_a) + I_{\alpha\beta}(-x_a))}{|x_a|^{n_a - 1}} \varphi(\hat{x}_a)\overline{\psi(\hat{x}_a)} dx_a + O(\lambda^{-1/2-\delta}).
\end{aligned}$$

(3.16) is then a consequence of (3.14). □

4 The Case $a \neq b$

In the proof of Proposition 3.1, we have seen that in the case $a \neq b$, the leading term $I_1(\lambda)$ of $(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b$ is in fact an oscillatory integral with non-stationary phase. If everything is smooth with suitable decay, the standard techniques of oscillatory integrals show that $I_1(\lambda) = O(\lambda^{-\infty})$. The remainder of $(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b$ is more difficult to study. In this paper, we content ourselves with the following

Theorem 4.1 *Assume that $V_a \in \mathcal{S}(\mathbf{X}^a)$ for all $a \in \mathcal{A}$. Let $\alpha = (a, E_\alpha, \psi_\alpha)$, $\beta = (b, E_\beta, \psi_\beta)$ be two scattering channels with $a \neq b$ and ψ_α, ψ_β rapidly decreasing in x^a, x^b, respectively. Then one has for any $\varphi_c \in C_0^\infty(\Sigma_c)$ with $c = a, b$,*

$$(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b = O(\lambda^{-\infty}), \ \text{as } \lambda \to \infty. \tag{4.1}$$

The proof of Theorem 4.1 is based on the following resolvent estimate in weighted Sobolev space which is due to Ito [9].

Proposition 4.2 *Put $P(\lambda, \omega) = e^{i\sqrt{\lambda_\alpha} x_a \cdot \omega}(P - \lambda)e^{-i\sqrt{\lambda_\alpha} x_a \cdot \omega}, \omega \in \mathbf{S}_a$. Under the assumptions of Theorem 4.1, for any $k \in \mathbf{N}$ and $s > k + 1/2$, there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, the limits $(P(\lambda, \omega) \pm i0)^{-1} = \lim_{\varepsilon \to 0_+}(P(\lambda, \omega) \pm i\varepsilon)^{-1}$ exist in the norm of bounded operators from $H^{k,s}$ to $H^{k,-s}$ and*

$$\sup_{\omega \in \mathbf{S}_a} \|(P(\lambda, \omega) \pm i0)^{-1}\|_{k,s} \leq C_{k,s}\lambda^{-1/2}, \quad \lambda \geq \lambda_0. \tag{4.2}$$

Here $H^{k,s}$ is the Sobolev space of order k on $\mathbf{X}$ with weight $\langle x \rangle^{2s}$ and $\|\cdot\|_{k,s}$ is the norm of bounded operators from $H^{k,s}$ to $H^{k,-s}$.

Proposition 4.2 follows from Theorem 4.2 in [9] by repeating the proof of Proposition 3.1 in [9] in three-body case. Clearly, the results of Proposition 4.2 are also true with a replaced by b and α by β.

Proof of Theorem 4.1. We represent $(T_{\alpha\beta}(\lambda)\varphi_a, \varphi_b)_b$ as in Section 3. But this time we take $J_c^\lambda(x) = J_c(x/\lambda^{1/8})$. Since everything is smooth now, we can use the operator L introduced in Section 3.1 to integrate by parts an infinite number of times and obtain that $< J_b^\lambda\{I_a J_a^\lambda + [-\Delta, J_a^\lambda]\}\varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-\infty})$. It remains to prove that

$$< Q_b^* R(\lambda+i0) Q_a \varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-\infty}),$$

where Q_c is given by (3.2).

Assume without loss that $b \not\subseteq a$. Since ψ_α and ψ_β are rapidly decreasing and supp $\nabla J_c^\lambda \subseteq \{|x| \geq C\lambda^{1/8}\}$, by introducing a partition of unity $g_{1,c}(x) + g_{2,c}(x) = 1$ on $\mathbf{X}$ as in Section 3 and applying (3.4), one obtains

$$[-\Delta, J_a^\lambda]\varphi_\alpha(\lambda) = O(|\lambda x|^{-\infty}), \quad [-\Delta, J_b^\lambda]\varphi_\beta(\lambda) = O(|\lambda x|^{-\infty}).$$

This gives

$$< Q_b^* R(\lambda+i0) Q_a \varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-\infty}) + < I_b J_b^\lambda R(\lambda+i0) I_a J_a^\lambda \varphi_\alpha(\lambda), \varphi_\beta(\lambda) > .$$

Since for each fixed λ, $I_a J_a^\lambda \psi_\alpha$ is rapidly decreasing in x, we can exchange the order of integrations and write

$$R(\lambda+i0) I_a J_a^\lambda \varphi_\alpha(\lambda) = \int_{\mathbf{S}_a} e^{i\sqrt{\lambda_\alpha} x\cdot\omega_a} \varphi_a(\omega_a) r(x, \omega_a, \lambda) d\omega_a.$$

Here $r(x, \omega_a, \lambda) = (P(\lambda, \omega_a) - i0)^{-1} I_a J_a^\lambda \psi_\alpha$. Proposition 4.2 shows that $r(x, \omega, \lambda)$ is smooth in x and

$$\|\langle x\rangle^{-s} \partial_x^\gamma r(\cdot, \omega_a, \lambda)\| \leq C_{\gamma,s} \lambda^{-1/2} \|\langle x\rangle^s I_a J_a^\lambda \psi_\alpha\| \leq C' \lambda^{-1/2+(|s|+d+1)/8}$$

for any $s > |\gamma| + 1/2$. In the last estimate, we used the fact that $I_a J_a^\lambda \psi_\alpha = O(\langle x\rangle^{-\infty}) + O(1) j_0(x/\lambda^{1/8})$ with $j_0(x) = 1 - j(|x|)$. See (2.10) for the choice of j. Since $b \not\subseteq a$,

$$\begin{aligned} &< Q_b^* R(\lambda+i0) Q_a \varphi_\alpha(\lambda), \varphi_\beta(\lambda) > \\ &= \int_{\mathbf{S}_a}\int_{\mathbf{S}_b}\int_{\mathbf{X}} e^{i\Phi(x;\lambda,\omega_a,\omega_b)} \varphi_a(\omega_a) \overline{\varphi_b(\omega_b)} r(x, \omega_a, \lambda) \overline{I_b J_b^\lambda \psi_\beta} dx d\omega_a d\omega_b \end{aligned}$$

is an oscillatory integral with the non-degenerate phase

$$\Phi(x; \lambda, \omega_a, \omega_b) = x \cdot (\sqrt{\lambda_\alpha}\omega_a - \sqrt{\lambda_\beta}\omega_b)$$

for ω_a in the support of φ_a and ω_b in the support of φ_b. See the proof of Proposition 3.1. We can again use the operator L introduced in Section 3.1 to first integrate by parts with respect to x an infinite number of times. Since the support of $j_0(\cdot/\lambda^{1/8})$ is contained in $\{|x| \leq C\lambda^{1/8}\}$, each integration by parts allows us to obtain a decrease of the order $O(\lambda^{-1/4})$. This shows $< Q_b^* R(\lambda+i0) Q_a \varphi_\alpha(\lambda), \varphi_\beta(\lambda) >= O(\lambda^{-\infty})$. Theorem 4.1 is proved. □

References

[1] S. Agmon and L. Hörmander, Asymptotic properties of solutions of differential equations with simple characteristics. J. Analyse Math., 30(1976), 1-38.

[2] W.O. Amrein, D.B. Pearson and K.B. Sinha, Bounds on the total scattering cross section for N-body systems, Nuovo Cimento, 52 A (1979), 115-131.

[3] W.O. Amrein and K.B. Sinha, On three-body scattering cross sections, J. Phys. A: Math. Gen., 15(1982), 1567-1586.

[4] A. Bommier, Régularité et prolongement méromorphe de la matrice de diffusion pour les problèmes à N-corps à longue portée. Thèse de Doctorat, Ecole Polytechnique, 1993.

[5] H. Cycon, R. Froese, W. Kirsch and B. Simon, Schrödinger Operators, Texts and Monographs in Physics, Springer Verlag, 1987.

[6] V. Enss and B. Simon, Finite total cross-sections in nonrelativistic quantum mechanics, Commun. in Math. Phys., 76(1980), 177-209.

[7] C. Gérard, H. Isozaki and E. Skibsted, Commutator algebra and resolvent estimates, preprint 1993.

[8] W. Hunziker, Potential scattering at high energies, Helv. Phys. Acta 36(1963), 838-856.

[9] H. Isozaki, Structure of S-matrices for three body Schrödinger operators, Comm. Math. Phys., 146(1992), 241-258.

[10] H.T. Ito, High energy behavior of total scattering cross sections for 3-body quantum systems, preprint 1992.

[11] H.T. Ito and H. Tamura, Semi-classical asymptotics for total scattering cross sections of 3-body systems, J. Math. Kyoto Univ. 32(1992), 533-555.

[12] L.D.. Landau and E. M. Lifchitz, Quantum Mechanics, Nonrelativistic Theory, Pergmon Press, Oxford, 1965.

[13] R. Novikov, Le problème de scattering inverse pour les systèmes à trois corps, exposé au Séminaire des E.D.P., Univ. de Nantes, Mai 1994.

[14] M. Reed and B. Simon, Methods of Modern Mathematical Physics, IV. Analysis of Operators, Academic Press, New York, 1978.

[15] D. Robert, Asymptotique de la phase de diffusion à haute énergie pour des perturbations du second ordre du Laplacien, Ann. Sci. Ecole Norm. Sup. 25(1992), 107-134.

[16] D. Robert and H. Tamura, Semiclassical estimates for resolvents and asymptotics for total scattering cross sections, Ann. Inst. H. Poincaré, 46(1987), 415-442.

[17] D. Robert and X.P. Wang, Pointwise semiclassical asymptotics for total cross sections in N-body problems, preprint 1992, Univ. Nantes, to appear in " Spectral and Scattering Theory", M. Ikawa ed., Marcel Dekker.

[18] I.M. Sigal and A. Soffer, The N-particle scattering problem: asymptotic completeness for short range systems, Ann. of Math. 126(1987), 35-108.

[19] E. Skibsted, Smoothness of N-body scattering amplitudes, Rev. Math. Phys., 4(4) 1992, 619-658.

[20] A.V. Sobolev and D.R. Yafaev, On the quasiclassical limit of total scattering cross-section in non-relativistic quantum mechanics, Ann. Inst. H. Poincaré, 44(1986), 195-210.

[21] X.P. Wang, On the three-body long-range scattering problems, Letters in Math. Phys., 25(1992), 267-276. Detailed version: Propagation estimates and asymptotic completeness in three-body long-range scattering, J. of Funct. Analysis, Vol. 125(1) 1994, 1-36.

[22] X.P. Wang, Microlocal resolvent estimates of N-body Schrödinger operators, J. of the Fac. Sc., Univ. Tokyo, Sect. IA, Mathematics, Vol. 40 (2) (1993), 337-385.

[23] X.P. Wang, Sections efficaces dans le problème à N-corps, Exposé No. IX au Séminaire EDP à l'Ecole Polytechnique, Palaiseau, Décembre 1992.

[24] X.P. Wang, Total cross sections in N-body problems: Finiteness and high energy asymptotics, Commun. Math. Phys., 156(1993), 333-354.

[25] X.P. Wang, On the uniqueness of inverse scattering for N-body systems, Inverse Problems, 10(1994), 765-784.

[26] D.R. Yafaev, The eikonal approximation for the Schrödinger equation, Proc. of the Steklov Inst. Math., 179(2) (1989), 251-266.

[27] D.R. Yafaev, Quasiclassical asymptotics of the scattering cross-section for the Schrödinger equation, Math. USSR Izv., 32(1) (1989), 141-165.

[28] D.R. Yafaev, Resolvent estimates and scattering matrix for N-particle hamiltonians, preprint Univ. Rennes 1, September 1993.

[29] K. Yajima, The quasiclassical limit of scattering amplitude $-$ L^2 approach for short range potentials$-$, Japan J. Math., 13(1987), 77-126.