# On the Magnetic Stark Resonances in Two Dimensional Case

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**Abstract**

In this work, we study the resonances of Schrödinger operator with homogeneous magnetic and electric fields: $P = (D_x - \vec{A}(x))^2 + \vec{e}.x + V(x)$ in the case the strength of the magnetic field tends to $\infty$.

## 1 Introduction

The hydrogen atom in a strong homogeneous magnetic field is a model of "quantum chaos". See for example [HRW], [DBG1] and the references cited therein. In particular, it is said that in this problem, there exist resonances (see for example [HRW], [DBG2]). This work is an attempt to understand this phenomenom of resonances from a mathematical point of view. In fact, we shall only study a much simplified model: a two dimensional Schrödinger operator with homogeneous magnetic and electric fields

$$P = (D_x - \vec{A}(x))^2 + \vec{e}.x + V(x), x \in \mathbf{R}^2 \tag{1.1}$$

Here $\vec{A}(x) = 1/2\vec{B} \times x$, $V$ is assumed to be a real smooth function tending to zero at the infinity, $\vec{B}$ and $\vec{e}$ are the constant magnetic and electric field, respectively. For the Schrödinger operator (1.1) in three dimensional case, there are works of some physists [CLLPT], in which they considered the case where the homogeneous magnetic and electric fields are parallel and studied the structure of diamagnetic Stark spectrum for $P$ in the regime $|\vec{e}|/|\vec{B}| \to 0$. The structure of the spectrum of magnetic Schrödinger opertors in two dimensional case is completely different from that in three dimensional case. In particular, in two dimensional case, one can imagine that the magnetic field points to the third imaginary direction and the magnetic field is thus always perpendicular to the electric field. So in some sense, our work may be regarded as a complement to the work [CLLPT].

By an appropriate gauge transformation and change of coordinates, we can write the operator $P$ in the form:

$$P(B,\beta) = (D_x - By)^2 + D_y^2 + \beta x + V(x,y), (x,y) \in \mathbf{R}^2, \tag{1.2}$$

where $B$ and $\beta$ are proportionnal to the strength of the homogeneous magnetic and electric field. We are interested in the resonances of $P(B,\beta)$ in the case $\beta/B \to 0$, which we call magnetic Stark resonances. Typically, there are two situations: The first one is that the magnetic field is fixed: $B = 1$ and the strength of the electric field tends to 0, i.e., $\beta \to 0$. The second one is that the electric field is fixed and the strength of the magnetic field tends to the infinity: $\beta = 1, B \to \infty$. In the first case, the operator $P(1,\beta)$ is to be considered as a perturbation of the operator $P(1,0)$. The spectrum of $P(1,0)$ consists of the Landau levels, which are the essential spectrum of $P(1,0)$, and the discrete eigenvalues outside Landau levels. In this case, it is natural to hope that as soon as a weak homogeneous electric field switches on, the discrete eigenvalues of $P(1,0)$ will be turned into the resonances of $P(1,\beta)$. This is very similar to usual Stark resonances, on which there is now a large litterature (see for example [S1], [S2], [W1], [W2] and also references therein). We shall not study this case in details. In fact, one can show that the eigenfunctions of $P(1,0)$ associated to disrete eigenvalues outside Landau levels decay exponentially and by the method developped in the sequel of this paper, we can easily prove the existence and the stability of resonances. This paper is devoted to studying the resonances of magnetic Stark Hamiltonian in strong magnetic field: $B \to \infty, \beta = 1$. Before stating the results of this work, let us first give the definition of resonances.

Remark that having a constant double characteristic on $\{\xi = By\}$, the magnetic Schrödinger operator $(D_x - By)^2 + D_y^2$ is unstable under arbitrary analytical deformations: the domain of the distorted operator usually depends on the distortion, once the imaginary part of the deformation parameter becomes non zero. This causes difficulties in locating the essential spectrum of distorted operator. If one uses sympletic changes of coordinates ([HS2]), one can write $P(B,1)$ as

$$P(B) = B(D_y^2 + y^2) + x - \frac{1}{4B^2} \tag{1.3}$$

$$+V(x - B^{-1/2}D_y - \frac{1}{2B^2}, B^{-1}D_x + B^{-1/2}y),$$

where we denoted by $a(x, D_x), x \in \mathbf{R}^n$ the Weyl pseudodifferential operator ($\Psi DO$, in short) with symbol $a(x,\xi)$. (Throughout this paper, we shall use Weyl quantization formula to define $\Psi DOs$). In the form (1.3), one may try to deform $x$ and $y$ variables individually . But then any non linear deformation requires an analyticity of $V$ over the whole complex space $\mathbf{C}^2$, which is unrealistic. For these reasons, we do not see for the moment how to apply the theory of Helffer-Sjöstrand [HS1] or other method of analytical distortions to study the magnetic-Stark resonances in two dimensional case. In the following, we shall just use complex dilation to define resonances of $P(B)$. Note that one can equally use complex translation in $x$ variable to study this problem.

Assume that $V$ extends to a holomorphic function in the region $\{z \in \mathbf{C}^2; |\Re z| \le \delta|\Im z| + R\}$ for some $\delta, R > 0$ and satisfies there:

$$|V(z)| \le C < \Re z >^{-\epsilon'}, \text{ for some } \epsilon' > 0. \tag{1.4}$$

Since we shall utilize the magnetic field to reduce the number of variables, we regard $y$ as a dual variable of $x$ and therefore it is more convenient to work with the operator $P(B)$ given by (1.3). In this configuration, consider the dilation $(x,y) \to (e^\theta x, e^{-\theta}y)$. The operator (1.3) becomes now

$$P(B,\theta) = B(e^{-2\theta}D_y^2 + e^{2\theta}y^2) + e^\theta x - \frac{1}{4B^2} \tag{1.5}$$

$$+V(e^\theta(x - B^{-1/2}D_y) - \frac{1}{2B^2}, e^{-\theta}(B^{-1}D_x + B^{-1/2}y))$$

which is now clearly well defined for $\theta$ in a small complex neighbourhood of 0. We still denote by $P(B,\theta)$ its closed extension from $C_0^\infty$. The spectrum of $P(B,\theta)$ has a nice structure: For $|\theta|$ small, $\Im\theta > 0$, the essential spectrum of $P(B,\theta)$ is equal to the spectrum of $P_0(B,\theta) \equiv B(e^{-2\theta}D_y^2 + e^{2\theta}y^2) + e^\theta x$ shifted to the left by $1/(4B^2)$, which consists of an infinite number of parallel lines: $\{(2n+1)B - 1/(4B^2) + e^\theta\lambda; \lambda \in \mathbf{R}\}$, $n = 0, 1, 2, ....$ Outside these lines, there may exist disrete eigenvalues created by the potential $V$. In this paper we define the discrete eigenvalues of $P(B,\theta)$ as resonances of $P(B)$. For the moment, we are not able to prove the absence of eigenvalues for the operator $P(B)$. So in this paper we make no distinguish between real eigenvalues and physical resonances of $P(B)$. In the following we shall study the resonances of $P(B)$ outside Landau levels $(2n+1)B, n \in \mathbf{N}$. We shall show in geometrically simple cases how the quantum effect of a strong magnetic field combines the classical contribution of total electric potential to create resonances in strong magnetic field.

Let $W(x,y) = x + V(x,y)$ be the total electric potential. For $E \neq 0$, put $\Omega_E = \{(x,y); W(x,y) = E\}$. Assume that $\Omega_E = \{(x_0,y_0)\} \cup \Gamma$, where $\Gamma$ is a connected curve, $(x_0,y_0) \notin \Gamma$ and $W$ has a local nondegenerat extremum (local maximum or minimum with definite Hessian) at $(x_0,y_0)$:

$$W''(x_0,y_0) > 0(\text{ or } < 0). \tag{1.6}$$

and that the classical Hamiltonian $W(x,\xi)$ is non-trapping on $\Gamma$:

$$\{W(x,\xi), x.\cdot\xi\} = x\partial_x(x,\xi) - \xi\partial_\xi W(x,\xi) \neq 0 \text{ on } \Gamma. \tag{1.7}$$

Here $\{.,.\}$ denotes the Poisson bracket. The main result of this paper is the following

**Theorem 1.1** *Under the assumptions (1.4), (1.6) and (1.7), for each $n \in \mathbf{N}$, define*

$$U_n = \{z \in \mathbf{C}; \Re z \in ](2n+1)B + E - \frac{C}{B}, (2n+1)B + E + \frac{C}{B}[, \Im z \in ] - \frac{1}{CB}, \frac{1}{CB}[\},$$

*where $C > 1$ can be arbitrarily large outside a discrete set in $\mathbf{R}$. Then for $B$ large enough, the resonances of $P(B)$ in $U_n$ exist and are all given by complete expansions in powers of $B^{-1}$:*

$$E_{nk}(B) \sim B(2n+1) + E$$

$$+B^{-1}/2\{\pm(2k+1)(\lambda\mu)^{1/2}+(2n+1)(\lambda+\mu)/2\}+\sum_{j\geq 2}B^{-j}e_{\pm,nkj},$$

*where $e_{\pm,nkj}\in\mathbf{R}$, $\lambda$ and $\mu$ are eigenvalues of the Hessian $W''(x_0,y_0)$ and the sign + ( - ) corresponds to a local minimum ( maximum, resp.). In particular, the resonances of $P(B)$ in $U_n$ are all algebraically simple and the width of resonances is of order $O(B^{-\infty})$.*

**REMARKS.** (a). Theorem 1.1 shows that in two-dimensional magnetic resonances, there is no significant difference between a local maximum and a local minimum. It also shows that magnetic resonances have an almost ladder structure: modulo an error of order $O(B^{-1})$, one can obtain the resonances near the n-th Landau level just by translating those near the first Landau level by $2nB$.

(b). Our method in this work applies to general ( non bounded) electric potential W with good geometrical properties at the infinity. We restrict ourselves to the particular structure $W=x+V$ only for the physical interest of this model. See [CLLPT].

(c). In higher even dimensional case, assume that the constant magnetic field is of maximal rank. Then we can still prove the results of Theorem 1.1. But in odd dimensional case, the situation is complete different. In particular, we do not see how to study the magnetic Stark resonances in three dimensional case.

To illustrate the results of Theorem 1.1, let us look at an example:

$$W(x,y)=x+c/(x^2+y^2+R), c\neq 0, R>0.$$

For appropriate values of $c<0$ and R ( e.g., $c=-2$ and $R=1$), we can check that $\nabla W=0$ has two real solutions $(x_1,0),(x_2,0)$ with $x_2<x_1$. W has a local nondegenerate minimum at $(x_1,0)$, while $(x_2,0)$ is a saddle point for $W$. Put $E_j=W(x_j,0)$. Then we can verify that all the assumptions (1.5)-(1.7) are satisfied at the energy $E_1$. So our results show the existence of resonances near the energy $(2n+1)B+E_1$, for each $n$. Remark that for $E\in]E_1,E_2[$, $\Omega_E=\gamma_E\cup\Gamma_E$, where $\gamma_E$ is a compact closed curve and $\Gamma_E$ is unbounded. In the extreme case $E=E_2$, these two curves touch at $(x_2,0)$. We believe that near each given energy $E\in]E_1,E_2]$, there are also resonances of $P(B)$ when $B$ is large enough.

The organization of this work is as follows: In Section 2, we reduce the resonance problem in strong magnetic field to a problem in semiclassical limit with the semiclassical parameter $h = 1/B$. This reduction is already used by Bellissard and Helffer-Sjöstrand ( [B], [HS2]) in self-adjoint case with bounded total electric potential. In Section 3, we study the spectral properties of an h-pseudo-differential operator whose leading symbol is $W(e^{\theta}x, e^{-\theta}\xi)$. In particular, we shall prove the exponential decay of the eigenfunctions for this operator and establish a resolvent estimate outside the set of possible resonances. With these preparations, we shall prove the existence of resonances in Section 4 by studying a suitable Grushin problem and give the complete asymptotic expansion in Section 5 by constructing an asymptotic solution near the extreme point. In the last two Sections, many arguments are similar to semiclassical resonances of usual Schrödinger operators, so we shall sometime not give the full details of the proofs.

# 2 Reduction to a Semiclassical Problem

Let $E \neq 0$. To study the resonances of $P(B)$ near $(2n+1)B + E$, we shall make use of a strong field reduction by n-th eigenfunction of the harmonic oscillator. The idea is the same as in [B] and [HS2]. Since the operator in our case is neither selfadjoint nor semibounded, we need some additional estimates.

Let $\phi_n$ be the n-th real Hermite function:

$$(D_y^2 + y^2)\phi_n = (2n+1)\phi_n, \|\phi_n\| = 1.$$

Put $\psi_n(\theta) = e^{\theta/2}\phi_n(e^{\theta}y)$. Then $\psi_n(\theta)$ is holomorphic in $\theta$. By deformation of the integral contour, one sees that $< \psi_n(\theta), \psi_n(\bar{\theta}) >= 1$ for all $\theta$ near 0. Let $\Pi : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be the projection defined by $\Pi u =< u, \psi_n(\bar{\theta}) >_Y \psi_n(\theta)$, here the scalar product $< ., . >_Y$ denotes the integration in $y$ variable. We denote by the same letter the natural extension of $\Pi$ on $L^2(\mathbf{R}^2)$. Put:

$$\hat{\Pi} = 1 - \Pi, \hat{P} = \hat{\Pi}P(B,\theta)\hat{\Pi} \text{ and } \hat{P}_0 = \hat{\Pi}P_0(B,\theta)\hat{\Pi}.$$

Let $\hat{R}(z)$ and $\hat{R}_0(z)$ denote the reduced resolvent of $\hat{P}$ and $\hat{P}_0$ on the range of $\hat{\Pi}$. We begin with an estimate over the free resolvent.

**Lemma 2.1** *For $\theta \in \mathbf{C}$ with $|\theta|$ small and $\Im\theta > 0$, let $\Omega_1 = \{z \in \mathbf{C}; |\Re z - (2n+1)B| < 2\delta B, |\Im z| < (1-\delta)\Im\theta B/2\}, 0 < \delta < 1$. Then we have:*

$$\|\hat{R}_0(z)\| \le C/B, \text{ uniformly in } z \in \Omega_1. \tag{2.1}$$

*Let $\hat{R}_0(x;z), z \in \Omega_1$, denote the resolvent of $\hat{\Pi}(B(e^{-2\theta}D_y^2 + e^{2\theta}y^2) + e^\theta x)\hat{\Pi}$ on the range of $\hat{\Pi}$ in $L^2(\mathbf{R}_y)$, where $x$ is regarded as a parameter. Then*

$$\|\hat{R}_0(x;z)\|_Y \le C/B, \quad \text{uniformly in } z \in \Omega_1 \text{ and } x \in \mathbf{R}. \tag{2.2}$$

*Here $\|.\|_Y$ denotes the norm of bounded operators on $L^2(\mathbf{R}_y)$.*

**Proof.** Making the change of the variable $x \to Bx$, $P_0(B,\theta)$ becomes $BP_0(1,\theta)$. The corresponding reduced resolvent of $P_0(1,\theta)$ is bounded on $B^{-1}\Omega_1$, which is now B-independent. (2.1) follows easily. To prove (2.2), we utilize the transformation $x \to \epsilon x + x_0$. Then the operator $P_0(B,\theta)$ is unitarily equivalent with $B(e^{-2\theta}D_y^2 + e^{2\theta}y^2) + e^\theta(\epsilon x + x_0)$. Let $\hat{R}_0(\epsilon, x_0; z)$ be the corresponding reduced resolvent defined as above. It follows from (2.1):

$$\|\hat{R}_0(\epsilon, x_0; z)\| \le C/B,$$

uniformly in $x_0$ and $\epsilon > 0$. For each fixed $B, z, x_0$, $\hat{R}_0(\epsilon, x_0; z)$ converges strongly to an x-independent operator on $L^2(\mathbf{R}^2)$. The limit is clearly $\hat{R}_0(x_0; z)$. So we obtain that as operator on $L^2(\mathbf{R}^2)$, $\|\hat{R}_0(x_0, z)\| \le C/B$, uinformly in $x_0$. Since $\hat{R}_0(x_0, z)$ acts only on y-variable, (2.2) follows easily. $\square$

From (2.2), it results that $x \to \hat{R}_0(x;z)$ is a bounded symbol, each derivation on $x$ gaining a facotr $B^{-1}$. In fact by rescaling $x$ variable, one can write:

$$\hat{R}_0(x,z) = B^{-1}G(^{-1}x, B^{-1}z),$$

where $G(\zeta, \lambda), \lambda \in B^{-1}\Omega_1$, is a operator valued holomorphic function of $\zeta$ in a small $\theta$-dependent sector in $\mathbf{C}$. It is also clear that for $x \in \mathbf{R}, |x| < C$, $\hat{R}_0(x,z)$ is well defined for $\theta = 0$ and $z \in \mathbf{R} \cap \Omega_1$, when B is large enough. In this case the reduced free resolvent is a self-adjoint operator in $y$ variable.

**Corollary 2.2** *For $B >> 1$, the reduced resolvent $\hat{R}(z)$ exists for $z \in \Omega_1$ and satisfies:*

$$\|\hat{R}(z)\| \le C/B, \text{ uniformly in } z \in \Omega_1. \tag{2.3}$$

(2.3) follows immediately from (2.1), since the potential V is bounded. By (2.2) we can show that $\hat{R}(z)$ is an h-$\Psi DO$ with bounded symbol. See [$R$] for the notion of $h-\Psi DOs$. The details are omitted. Now we can proceed as in [HS2] to reduce the problem in strong magnetic field to a semiclassical one. In the following, we shall write $h = 1/B$. Consider the Grushin problem for $P(B,\theta) - z$

$$\mathcal{P}(z) = \begin{pmatrix} P(B,\theta) - z & \psi_n(\theta) \\ < ., \psi_n(\bar{\theta}) >_Y & 0 \end{pmatrix} : \mathbf{D} \oplus L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2) \oplus L^2(\mathbf{R}_x). \quad (2.4)$$

Here $\mathbf{D}$ is the domain of $P_0(B,\theta)$ and $< ., . >_Y$ denotes the scalar product in $L^2(\mathbf{R}_y)$.

**Theorem 2.3** *For $B$ large enough, the Grushin problem (2.4) is invertible for $z \in \Omega_1$ and the inverse is given by*

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{+-}(z) \end{pmatrix} \quad (2.5)$$

*where*

$$\begin{aligned} E(z) &= \hat{R}(z)a(z), E_-(z) = < a(z)., \psi_n(\bar{\theta}) >_Y, \\ E_+(z) &= -\hat{R}(z)a(z)\hat{\Pi}V_\theta\psi_n(\theta) + \psi_n(\theta)(z - (2n+1)B \\ & \quad -\frac{1}{4B^2} - e^\theta x - < V_\theta\psi_n(\theta), \psi_n(\bar{\theta}) >_Y), \\ E_{+-}(z) &= z - \{(2n+1)B - \frac{1}{4B^2} + < V_\theta\psi_n(\theta), \psi_n(\bar{\theta}) >_Y \\ & \quad + e^\theta x - < a(z)\hat{\Pi}V_\theta\psi_n(\theta)., \psi_n(\bar{\theta} >_Y \} \end{aligned}$$

*Here*

$$a(z) = (1 + [V_\theta, \Pi]\hat{R}(z))^{-1} \text{ and}$$

$$V_\theta = V(e^\theta(x - B^{-1/2}D_y) - 1/(2B^2), e^{-\theta}(B^{-1}D_x + B^{-1/2}y)).$$

*In particular, $z$ is an eigenvalue of $P(B,\theta)$ iff 0 is an eigenvalue of $E_{+-}(z)$.*

**Proof.** We use an approximate inverse $\mathcal{E}_0(z)$ for $\mathcal{P}(z)$:

$$\mathcal{E}_0(z) = \begin{pmatrix} \hat{R}(z) & \psi_n(\theta) \\ < ., \psi_n(\bar{\theta}) >_Y & E^0_{+-} \end{pmatrix}$$

where

$$E^0_{+-} = z - \{(2n+1)B - 1/(4B^2) + e^\theta x + < V_\theta \psi_n(\theta), \psi_n(\bar{\theta}) >_Y\}.$$

Since as operators from $L^2(\mathbf{R}_x)$ to $L^2(\mathbf{R}^2)$,

$$V_\theta \psi_n - \Pi(V_\theta \psi_n) = [V_\theta, \Pi] = O(B^{-1/2}),$$

we can check:

$$\mathcal{P}(z)\mathcal{E}_0(z) = I + O(B^{-1/2}), \text{ and } \mathcal{E}_0(z)\mathcal{P}(z) = I + O(B^{-/2}).$$

This proves the inversibility of $\mathcal{P}(z)$. The formula for the inverse follows from a detailed computation. $\square$

It is clear that the map $z \to \mathcal{E}(z)$ is holomorphic in $z \in \Omega_1$. For the later use, we give some more informations about $E_{+-}(z)$. Denote:

$$A(h) = e^\theta x + < V_\theta \psi_n(\theta), \psi_n(\bar{\theta} >_Y, h = \frac{1}{B}.$$

Then we can show as in [HS2] that $A(h)$ is an h-$\Psi DO$:

$$A(h) = a(e^\theta x, he^{-\theta} D_x; h),$$

where $a(., .; h)$ is holomorphic in $x$ and $\xi$ and admits a complete expansion in powers of h:

$$a(x, \xi; h) \sim \sum_{j \geq 0} h^j a_j(x, \xi), \text{ where} \tag{2.6}$$

$$a_0(x, \xi) = x + V(x, \xi), a_1(x, \xi) = (2n+1)\Delta V(x, \xi)/4.$$

Next, we remark that $\Pi$ is a Weyl $\Psi DO$ and $[V_\theta, \Pi] = h^{/2} R_1(h)$, with $R_1(h)$ uniformly bounded in $L^2(\mathbf{R}^2)$. So by Corollary 2.2, we can estimate:

$$< a(z)\hat{\Pi} V_\theta \psi_n(\theta), \psi_n(\bar{\theta}) >_Y = O(h^2)$$

in $\mathcal{L}(L^2)$. From (2.3), we derive that $\hat{R}(z)$ is an h-$\Psi DO$ in x variable, whose symbol is bounded operator in y-variable. This shows that $< a(z)\hat{\Pi}V_\theta\psi_n(\theta), \psi_n(\bar{\theta}) >_Y$ is an h-$\Psi DO$:

$$< a(z)\hat{\Pi}V_\theta\psi_n(\theta), \psi_n(\bar{\theta}) >_Y = h^2 r(e^\theta x, he^{-\theta}D; h, z),$$

where the symbol $r(.,.;h,z)$ has an asymptotic expansion of the form:

$$r(e^\theta x, e^{-\theta}\xi; h, z) \sim \sum_{j\geq 0} h^j r_j(e\theta x, e^{-\theta}\xi; he^\theta x; hz), z \in \Omega_1 \tag{2.7}$$

with $r_j$ decaying like $O(|(x,\xi)|^{-j-1-\epsilon'})$, uniformly in $h$ and $z$. Remark that it is necessary to take $\Im\theta \neq 0$ in order to make $r(.,.;h,z)$ globally defined. But it is clear that (2.3) is also valid for $x \in \mathbf{R}, |x| < C$, and $h > 0$ small enough even when $\theta = 0$. From this, one can derive that $r(x,\xi;h,z)$ is well defined for $|x| < C, \xi \in \mathbf{R}$ and is real-valued if $z \in \mathbf{R} \cap \Omega_1$. The later follows from the self-adjointness of the original operator $P(B)$.

# 3 Spectral Properties of A(h)

We study in this Section the leading part $A(h)$ of $E_{+-}(z)$, which should determine the leading term of the resonances of $P(B)$ in $\Omega$. By (2.6), one sees that the essential spectrum of $A(h)$ is $\{e^\theta\lambda; \lambda \in \mathbf{R}\}$ and the discrete eigenvalues of $A(h)$ are essentially independent of $\theta$ and therefore should be considered as resonances of the $\Psi DO$ $a(x, hD; h)$. In the following, $\theta$ is fixed with $|\theta|$ small and $\Im\theta = \omega > 0$. For the simplicity of notations, we only consider the case $\Re\theta = 0$.

Let $E \neq 0$ so that (1.6) and (1.7) are satisfied. We propose to study the eigenvalus of $A(h)$ near $E$. Without loss, we can assume that $E$ is a local non-degenerate minimum of $W(x,y)$. Otherwise we need only consider $-A(h)$ near $-E$. By a translation, we can also assume that $(x_0, y_0) = (0,0)$. Let $B(\delta) = \{(x,\xi) \in \mathbf{R}^2; |x| + |\xi| < \delta\}$ and $w = (e^\theta x, e^{-\theta}\xi)$. Then

$$a_0(w) - E = a_0(\Re w) - E + i\Im w \nabla a_0(\Re w) + r(w),$$

where the rest $r(w)$ is bounded by $|r(w)| < C\omega^2 < (x,\xi) >^{-\epsilon'}$. Outside $B(\delta)$, $a_0(\Re w) - E$ can vanish only when $\Re w \in \Gamma$ ( see (1.6) ) . By the assumption on $V$, $\Gamma$ has an asymptote $\{x = E\}$. Near $\Gamma$, by (1.7), $\Im w.\nabla a_0(\Re w) \neq 0$. Note that for $|\xi|$ large enough, one has $|\{a_0, x.\xi\}| \geq |E|/2 > 0$ near $\Gamma$. Consequently, we obtain:

$$|a_0(w) - E| \geq c_0\{|a_0(\Re w) - E| + \omega\} \tag{3.1}$$

for $(x, \xi) \notin B(\delta)$ , $\omega > 0$ small enough and for some $c_0 > 0$. By a perturbation argument in $\xi$ variable, we can show that (3.1) remains true for $|\Im \xi| \leq \phi(x)$ and $(x, \Re\xi) \notin B(\delta)$, where $\phi$ is a bounded positive real function such that: (1). the symbol of $A(h)$ is holomorphic in $|\Im\xi| < \phi(x) + C$, for some $C > 0$; (2). $\phi(x) > C(\theta) > 0$ for all $x \in \mathbf{R}$, with $C(\theta)$ independent of $\theta$ for $x$ outside a small neighbourhood of $\Gamma_1 = \{x; (x, \xi) \in \Gamma,$ for some $\xi\}$ and $C(\theta) = c'\omega$ for $x$ near $\Gamma_1$. Making symplectic changes of coordinates, we can assume that the Hessian of $a_0$ at (0, 0) is diagonal. So in $B(\delta)$, one has:

$$a_0(w) = E + (\lambda e^{2\theta} x^2 + \mu e^{-2\theta} \xi^2)/2 + O((x, \xi)^3), \tag{3.2}$$

where $\lambda$ and $\mu$ are the two eigenvalues of the Hessian of $a_0$ at (0, 0). By (3.1) and (3.2), we can construct a real smooth function $f$ such that $f(x) > 0$ for $x \neq 0$, $f(x) = cx^2$ for x near 0 and

$$|f'(x)| \leq \phi(x) \text{ for all } x. \tag{3.3}$$

If we choose $c > 0$ small, then we can show that:

$$\begin{aligned} |a_0(e^\theta x, e^{-\theta}(\xi + if'(x))) \quad &- \quad E| \geq C\omega, \text{ for } (x, \xi) \notin B(\delta) \qquad (3.4)\\ \Re\{a_0(e^\theta x, e^{-\theta}(\xi + if'(x))) \quad &- \quad E\} \geq c'(x^2 + \xi^2), \text{ for } (x, \xi) \in B(\delta). \end{aligned}$$

In the above analysis, $x$ and $\xi$ play the same role. In particular, we can also derive from (3.1) that there exists a positive function $g(\xi)$ with $g(\xi) = c\xi^2$, for $\xi$ near 0 so that (3.4) holds for $\xi \in \mathbf{R}$ and $\Im x = g'(\xi)$.

**Theorem 3.1** *Let $\lambda(h) \in \{z \in \mathbf{C}; |z - E| < Ch\}, (C > 0)$, be an eigenvalue of $A(h)$. Let $u(h)$ be an associated eigenfunction: $A(h)u(h) = \lambda(h)u(h)$, $\|u(h)\| = 1$. Then there exists $C > 0$,*

$$\|e^{f/h}u(h)\| \leq C, \text{ uniformly in } h > 0 \tag{3.5}$$

*Here $f$ is the function constructed above.*

**Proof.** Since $f$ is bounded, $u_f \equiv e^{f/h}u(h)$ is in $L^2$. We want to estimate the h-dependence of its norm. Put: $A_f = e^{f/h}A(h)e^{-f/h}$. Then

$$(A_f - \lambda(h))u(h) = 0. \tag{3.6}$$

Since the symbol $a(.,.;\theta,h)$ of $A(h)$ is holomorphic in a neighbourhood of $\{|\Im\xi| \le \phi(x)\}$, by a change of integral contour in $\xi$, we obtain

$$\begin{aligned} A_f g(x) &= (2\pi h)^{-1}\int\int_{\mathbf{R}^2} e^{(i(x-y).\xi+f(x)-f(y))/h}a((x+y)/2,\xi;\theta,h)g(y)dyd\xi \\ &= (2\pi h)^{-1}\int\int_{\mathbf{R}^2} e^{i(x-y).(\xi-if'(x,y))/h}a((x+y)/2,\xi;\theta,h)g(y)dyd\xi \\ &= (2\pi h)^{-1}\int\int_{\mathbf{R}^2} e^{i(x-y).\xi/h}a((x+y)/2,\xi+if'(x,y);\theta,h)g(y)dyd\xi \end{aligned}$$

for $g \in C_0^\infty$. Here $f'(x,y)$ is defined by $f(x)-f(y) = (x-y)f'(x,y)$. we have used the analyticity and decay assumptuons on $V$ to show that the integral in $\xi$ over $\mathbf{R}$ and over $\Im\xi = f'(x,y)$ are equal. Now it is clear that $A_f$ is an h-$\Psi DO$ , whose h- principal symbol is given by $a_0(e^\theta x, e^{-\theta}(\xi + if'(x)))$.

Let $0 \le \chi_j \le 1, j = 1,2$ be smooth functions such that supp $\chi_1 \subset B(\delta), \chi_1 = 1$ near $(0,\ 0)$ and $\chi_1^2 + \chi_2^2 \equiv 1$. By the analysis made above, the symbol of $A_f - \lambda(h)$ is elliptic on supp $\chi_2$. By constructing an approximate inverse for $A_f - \lambda(h)$, we derive from (3.6) that

$$\{\chi_2(x,hD) + O(h)\}u(h) = 0. \tag{3.7}$$

On supp $\chi_1$, we use the second estimate in (3.4):

$$\Re < (A_f - \lambda(h))u^1, u^1 > \ge c < (h^2D^2 + x^2 - Ch)u^1, u^1 >, \tag{3.8}$$

for some $C, c > 0$. Here $u^1 = \chi_1(x,hD)u_f$. We decompose the last scalar product in (3.8) into twe parts according to: $|x| > Rh^{1/2}$ and $|x| \le Rh^{1/2}$, $R >> 1$. Then one has:

$$\Re < (A_f - \lambda(h))u^1, u^1 > \ge (R-C)h < u^1, u^1 > -(R+C)h\int_{\{|x|<Rh^{1/2}\}}|u^1(x)|^2dx.$$

Since $u(h)$ is normalized and $f(x)/h$ is bounded when $|x| < Rh^{1/2}$, we can show that the last integral in the above inequality is bounded by $C_1$ uniformly in h. Since $(A_f - \lambda(h))u^1 = O(h)u_f$, we derive from (3.6) and (3.8) that

$$(R-C)\|u^1\|^2 \le C_R + c\|u_f\|\|u^1\|. \tag{3.9}$$

From here we derive: $\|u^1\|^2 \leq C'_R + c'/R\|u_f\|^2$. Since $\chi_1(x,hD)^2 + \chi_2(x,hD)^2 \geq 1/2$ in the sense of selfadjoint operators for $h > 0$ small enough, it results from (3.7) and (3.9):

$$\|u_f\|^2 \leq C''_R + c''/R\|u_f\|^2.$$

(3.5) follows when we choose R large enough. $\square$

**REMARK.** Since the Fourier transform exchanges the role of $x$ and $\xi$, making use of the remark preceeding Theorem 3.1, we can show that (3.5) also holds in $\xi$ variable:

$$\|e^{g(hD)/h}u(h)\| \leq C$$

uniformly in h, where g is given before Theorem 3.1. This shows that the energy of eigenfunctions are microlocally exponentially concentrated near (0,0).

Theorem 3. 1 is important, because it allows us to determine the eigenvalues of $A(h)$ just by computing the microlocalized version of $A(h)$ near (0,0). From now on, we denote: $D = \{z \in \mathbf{C}; |z - E| \leq C_0 h\}$, where $C_0$ is a large fixed constant. Near (0,0), $a_0(x,\xi) = E + (\lambda x^2 + \mu\xi^2)/2 + O((x,\xi)^3)$. Put

$$A_0(h) = (\lambda e^{2\theta}x^2 + \mu h^2 e^{-2\theta}D^2)/2 + h(2n+1)\Delta V(0,0)/4.$$

The eigenvalues of $A_0(h)$ are independent of $\theta$ and are given by $he_k, k = 0, 1, 2...$, where

$$e_k = (2k+1)(\lambda\mu)^{1/2}/2 + (2n+1)(\lambda+\mu)/4. \tag{3.10}$$

Define $D_0 = \{z \in D; |z - he_k| > \epsilon h, \text{ for any } k\}$. The following result shows that there is no spectrum of $A(h)$ in $D_0$.

**Proposition 3.2** *For $h > 0$ small enough, $D_0$ is contained in the resolvent set of $A(h)$ and $\|(A(h) - z)^{-1}\| \leq ch^{-1}, z \in D_0$.*

**Proof.** Let $\chi_j, j = 1, 2$, be a partition of unity on $\mathbf{R}^2$ with $\chi_1 = 1$ near 0 and 0 outside an $\epsilon$- neighbourhood of 0. Let $B_1$ the h-$\Psi DO$ with symbol $\chi_2(a_0(\theta) - z)^{-1}$, where $a_0(\theta)$ is the leading symbol of $A(h)$ and $z \in D$. Then

$$B_1(A(h) - z) = \chi_2(x, hD) + O(h), \text{ for any } z \in D. \tag{3.11}$$

Let $R_0 = (A_0(h) + E - z)^{-1}$. Then a scaling argument shows

$$\|x^j (hD)^k R_0\| \le C h^{(k+j)/2-1} \tag{3.12}$$

for $j + k \le 2$. On supp $\chi_1$, we compare $A(h)$ with $A_0(h)$:

$$\chi_1(x, hD) R_0 (A(h) - z) = \chi_1(x, hD)\{1 + R_0(A(h) - A_0(h) - E)\} \tag{3.13}$$

Let $\chi_3$ be equal to 1 near support of $\chi_1$ and 0 outside a $2\epsilon$-neighbourhood of (0, 0). Making use of (3.12) and an inductive argument, we can show that $\chi_1(x, hD) R_0 (1 - \chi_3(x, hD)) = O(h^\infty)$ On supp $\chi_3$, we expand the symbol of $A(h)$ near (0, 0). Again using (3.12) and the fact that $|x| + |\xi| < 2\epsilon$ on supp $\chi_3$, we have : $\|R_0(A(h) - A_0(h) - E)\chi_3(x, hD)\| \le C\epsilon + C_\epsilon h$ where $C > 0$ is independent of $\epsilon$ and h. From (3.11) and (3.13) it results

$$Ch^{-1}\|A(h) - z)u\| \ge (1 - C\epsilon - C_\epsilon h)\|u\| \ge \|u\|/2,$$

for $z \in D_0$. Taking $\epsilon > 0$ and $h > 0$ small enough, this proves that $A(h) - z$ is inversible , because the spectrum of $A(h)$ in $D$ is discrete. The resolvent estimate follows easily. □

**Theorem 3.3** *Let $D$ be defined as above with $C_0 \ne e_k, \forall k \in \mathbf{N}$. Then for $h > 0$ small enough, there is a one-to-one correspondence between the eigenvalues of $A(h)$ in $D$ and the $e_k$'s such that $e_k < C_0$. After an appropriate arrangement, the j-th eigenvalue $E_j(h)$ of $A(h)$ is given by:*

$$E_j(h) = E + he_j + O(h^{3/2}). \tag{3.14}$$

**Proof.** With the notations of Proposition 3.2, one has:

$$(A(h) - z)^{-1} = B_1(z) + \chi_1(x, hD) R_0$$

$$+\{O(h) - \chi_1(x, hD) R_0 (A(h) - A_0 - E)\chi_3(x, hD)\}(A(h) - z)^{-1}$$

for $z \in D_0$. Now fix an $e_j$ with $e_j < C_0$. Denote by $\gamma$ the contour $\{|z - E - he_j| = \epsilon h\}$ with $\epsilon > 0$ small enough. Define $\Pi(h)$ by

$$\Pi(h) = \frac{1}{2\pi i} \oint_\gamma (z - A(h))^{-1} dz.$$

Since $B_1(z)$ is holomorphic inside $\gamma$, it results from the above resolvent expression and Proposition 3.2 that $\Pi(h) = \chi_1(x, hD)\Pi_0(h) + o(1)$, where $\Pi_0(h)$ is the spectral projector of $A_0$ associated to $e_j$ and $o(1)$ is a bounded operator whose norm can be made arbitrarily small by the argument used in the proof of Proposition 3.2. Since $\Pi(h)$ is a projector, we deduce that $\Pi(h)$ is of the same rank as $\chi_1(x, hD)\Pi_0(h)$: Rank $\Pi(h) = 1$. This shows that $A(h)$ has just a simple eigenvalue inside $\gamma$. Using the exponential decay of the eigenfunctions, we can show that this eigenvalue is given by (3.14). The details are omitted. $\square$

By constructing an asymptotic solution for each eigenvalue of $A(h)$ in $D$, we can show that the eugenvalues of $A(h)$ in $D$ have a complete asymptotic expansion in powers of h. The details are now standard and therefore omitted here. See also the construction of asymptotic solutions given in Section 5.

# 4 Existence of Resonances

With the preparations of Section 3, we can easily prove the existence of resonances of $P(B)$ near the eigenvalues of $A(h)$. To show this, we work in a neighbourhood of an eigenvalue $E_j(h)$ of $A(h)$ given by (3.14). Let $\Omega$ be defined by: $\Omega = \{z; |z - E_j(h)| < \epsilon h\}$ where $\epsilon > 0$ is small enough. Let $u$ be an eigenfunction of $A(h)$: $A(h)u(h) = E_j(h)u$, $\|u\| = 1$. Let $A(h)^*$ be the adjoint of $A(h)$. Take $u^*$ an eigenfunction of $A(h)^*$ such that

$$A(h)^* u^* = \bar{E}_j(h) u^*, < u, u^* >= 1.$$

Let $\Pi'$ be the spectral projector of $A(h)$ associated to the eigenvalue $E_j(h)$ defined by: $\Pi' f =< f, u^* > u$. Put $\hat{\Pi}' = 1 - \Pi'$ and $A' = \hat{\Pi}' A(h) \hat{\Pi}'$. Then by Theorem 3.3 one can show that the reduced resolvent $R'(z) \equiv (A' - z)^{-1}\hat{\Pi}'$ exists and is holomorphic for $z \in \Omega$.

**Lemma 4.1** *For $h > 0$ small enough, we have: $\|R'(z)\| \leq Ch^{-1}$ , uniformly in $z \in \Omega$.*

Lemma 4.1 follows from the similar estimate on the reduced resolvent in $\Omega$ of $A_0(h)$ introduced in Section 3. The details are the same as the proof of

Proposition 3.2 and are omitted here. Let $E_{+-}(z)$ be constructed in Theorem 2.3. To find the values of $z \in \Omega$, for which 0 is an eigenvalue of $E_{+-}(z)$, let us consider the Grushin problem:

$$\mathcal{P}^1(z) = \begin{pmatrix} E_{+-}(z) + (2n+1)B & u \\ < ., u^* > & 0 \end{pmatrix} : D \oplus \mathbf{C} \to L^2(\mathbf{R}) \oplus \mathbf{C}, \tag{4.1}$$

where $D$ is the domain of $E_{+-}(z)$ which is equal to the domain of the multiplication by x. Since $E_{+-}(z) = z - ((2n+1)B + A(h) + O(h^2))$ uniformly in $z \in \Omega$, we can easily construct an inverse for (4.1). Consider the operator:

$$\mathcal{E}_0(z) = \begin{pmatrix} -R'(z) & u \\ < ., u^* > & E_j(h) - z \end{pmatrix} : L^2(\mathbf{R}) \oplus \mathbf{C} \to D \oplus \mathbf{C}.$$

By Lemma 3.1 it is easy to check that

$$\mathcal{E}_0(z)\mathcal{P}^1(z) = 1 + O(h) \text{ and } \mathcal{P}^1(z)\mathcal{E}_0(z) = 1 + O(h), z \in \Omega.$$

So the inverse of $\mathcal{P}^1(z)$ exists and $\mathcal{P}^1(z)^{-1} = \mathcal{E}_0(z) + O(h)$.

**Theorem 4.2** *For $h > 0$ small enough, there exists only one resonance $z_j(h)$ of $P(B)$ in $\{z \in \mathbf{C}; |z - (2n+1)B - E_j(h)| < \epsilon h\}$. In addition $z_j(h)$ is algebraically simple and is given by*

$$z_j(h) = (2n+1)B + E + he_j + O(h^2). \tag{4.2}$$

**Proof.** To give the existence and the location of resonances inside $\Omega$, we need only compute the zeros of the element $F(z)$ in the right lower corner of $\mathcal{P}^1(z)^{-1}$, which is now a holomorphic function of $z \in \Omega$. See [HS1]. A direct computation gives:

$$\mathcal{P}^1(z)^{-1} = \mathcal{E}_0(z)(1 + K(z))^{-1},$$

where $K(z)$ is given by

$$K(z) = \begin{pmatrix} r(z,h)R'(z) & r(z,h)u \\ 0 & 0 \end{pmatrix}$$

with $r(z,h) = -\{E_{+-}(z) + (2n+1)B - z + A(h)\}$. Since $r(z,h)u = O(h^2)$ as operator from $\mathbf{C}$ to $L^2$, we obtain $F(z) = E_j(h) - z + < r(z,h)u, u^* > +O(h^2) = E_j(h) - z + O(h^2)$. Consequently, we derive that $F(z)$ has just a simple zero near $E_j(h)$ which is given by $E_j(h) + O(h^2)$ ( see also [HS1] ). The desired result follows from Theorem 3.3. $\square$

From the proof of Theorem 4.1, it is clear that the resonances of $P(B)$ in $(2n+1)B + D$, where $D$ is defined in Section 3, are all given by the formula (4.2) with $e_j < C_0$.

# 5 Construction of Asymptotic Solutions

By the results of Section 2, we can write

$$E_{+-}(z) = z - (2n+1)B - A(\theta,h) - h^2 r(e^\theta x, he^{-\theta}D; h, z),$$

where $r(z,\theta,h) \equiv r(e^\theta x, he^{-\theta}D; h, z)$ is an h-$\Psi DO$. Different from (2.7), the insignificant term $-1/(4B)^2$ is now included in $r(z,\theta,h)$. Let $\Omega$ be the neighbourhood of $E_j(h)$ defined in Section 4. We want to construct an asymptotic solution for the non-linear eigenvalue problem:

$$A(\theta,h) + h^2 r(z,\theta,h))u = zu, z \in \Omega. \tag{5.1}$$

The same problem was studied in [M], where the leading part of $A(h)$ is a semiclassical Schrödinger operator. So we shall only give details on the construction of the first few terms for an asymptotic solution of (5.1). The arguments of [M] can be carried over for the remaining parts.

**Lemma 5.1** *For any $R > 0$, there exist $h_0 > 0$ and an h-$\Psi DO$ $r'(z,\theta,h)$ depending holomorphically on $\theta$ near $0 \in \mathbf{C}$ such that for any $\chi \in C_0^\infty$ with supp $\chi \subset ]-R,R[$, one has:*

$$\|\chi(x)(r(z,\theta,h) - r'(z,\theta,h))\| = O(h^\infty), \tag{5.2}$$

*for $\Im\theta > 0$. In addition, $r'(z,\theta,h)$ is selfadjoint if $z$ and $\theta$are real.*

**Proof.** Remark that $r(z, \theta, h)$ is only defined when $\Im\theta > 0$. But by the remark following (2.7), its symbol $r(x, \xi; z, \theta, h)$ is well defined for $|x| < R, z \in \Omega$ and $\theta$ in a small neighbourhood of 0, if $h > 0$ is small enough. So we only need to take $r'(z, \theta, h)$ an h-$\Psi DO$ with symbol $\chi'(x) r(x, \xi; z, \theta, h)$, where $\chi'$ is a cut-off function for $[-R, R]$. $\square$

By Lemma 5.1, we are led to consider the problem:

$$(A(0, h) + h^2 r'(z, 0, h))u = zu, z \in \Omega, \tag{5.3}$$

where $A(0, h)$ and $r'(z, 0, h)$ are now selfadjoint, if $z$ is real. We want to construct an asymptotic solution of (5.3) in the form

$$z \sim \sum h^j \lambda_j, u \sim e^{-d(x)/h} \sum h^j u_j,$$

with $\lambda_j \in \mathbf{R}$. To begin with, we determine the phase function $d(.)$. Write the total symbol of $A(0, h) + h^2 r'(z, 0, h)$ as:

$$a_0(x, \xi) + h a_1(x, \xi) + \sum_{j \geq 2} h^j a_j(x, \xi; z),$$

where $a_0, a_1$ are given by (2.6). Near (0,0), $a_0$ can be written in the form:

$$a_0(x, \xi) = E + \frac{1}{2}(\lambda x^2 + \mu \xi^2) + O((x, \xi)^3).$$

Here $\lambda, \mu > 0$, since we have assumed that $a_0$ has a local non-degenerate minimum at (0,0). On the set $a_0(x, i\xi) = E$ for $(x, \xi)$ near (0, 0), one has:

$$\xi = \pm(\lambda/\mu)^{1/2} x + o(x). \tag{5.4}$$

Note that on the above set, $\xi \neq 0$ if $x \neq 0$, because $(x, 0)$ is not in the energy surface $\Omega_E$ for $x$ near 0. On the set $a_0(x, i\xi) = E$, one has $\partial_\xi a_0(x, i\xi) \neq 0$ for $x \neq 0$ and $x$ near 0. This means that in a small neighbourhood of 0, there exists a unique function $\xi(x)$ defined for $x \neq 0$ such that $x \Re\xi(x) > 0$ for $x$ real and

$$a(x, i\xi(x)) = E \text{ for } x \neq 0 \text{ small.} \tag{5.5}$$

From (5.4), it results that $\xi(x) = (\lambda/\mu)^{1/2} x + o(x)$. So $\xi(x)$ can be defined by continuity at $x = 0 : \xi(0) = 0$. Making use of the analyticity of $V$, we can show that $\xi(x)$ thus defined extends to a holomorphic function of $x$ in a small complex neighbourhood of 0. Now choose $d(.)$ so that

$$d'(x) = \xi(x), d(0) = 0$$

Then $d(.)$ is holomorphic in $x$ near 0 and $\Re d(x) > 0$ for $x$ real with $x \neq 0$. We extend $d(.)$ smoothly so that it equals a small positive constant outside a neighbourhood of 0. Now as shown in Section 3, $e^{d/h}A(0,h)e^{-d/h}$ is an h-$\Psi DO$ and we can formally expand it in the form:

$$e^{d/h}A(0,h)e^{-d/h} \sim \sum_j h^j b_j(x,D),$$

where $b_j(x,D)$ is a differential operator of order j. In particular, we have:

$$b_0(x,D) = a_0(x, id'(x)) \tag{5.6}$$

$$b_1(x,D) = -i(\partial_\xi a_0(x,id'(x))\partial_x + 1/2d''(x)\partial_\xi^2(x,id'(x)) + a_1(x,id'(x))$$

Note that $a_0(x,id'(x)) = E$ for $x$ near 0. Now to find a formal solution for $A(0,h)u = zu$ in the form

$$z \sim \sum h^j \lambda_j, u \sim e^{-d/h} \sum h^j u_j, \text{ with } \lambda_0 = E,$$

we need only to solve the transport equations

$$(b_1(x,D) - \lambda_1)u_{l-1} = -\sum_{k+j=l,k>1} (b_k(x,D) - \lambda_k)u_j, l = 1,2,3,... \tag{5.7}$$

For $x$ near 0, these transport equations can be easily solved by an induction in $l$. Note that $b_1(x,D)$ can be written as:

$$b_1(x,D) = xc_0(x)\partial_x + c_1(x),$$

where $c_j(0) \neq 0$, for $j = 0,1$. If we look for solution to (5.7) with $l = 1$ in the form $u_0 = x^m\phi_0(x)$ with $\phi_0(0) \neq 0$, then we must take

$$\lambda_1 = mc_0(0) + c_1(0)$$

which is nothing else but $e_m$ given in (3.10). In this case $\phi_0(x)$ is uniquely determined by

$$(xc_0(x)\partial_x + mc_0(x) + c_1(x) - \lambda_1)\phi_0 = 0, \phi_0(0) = 1.$$

Clearly $\phi_0$ exists and is holomorphic in $x$ near 0. In the case $m = 0$, one can directly apply the method of [H] to determine $u_l$ for $l \geq 1$. In general case, we look for solutions to (5.7) with $l \geq 2$ in the form $u_{l-1} = p_{l-1}(x) + x^m \phi_{l-1}(x)$, where $p_{l-1}$ is a polynomial of degree $m-1$. Assume that we have determined $\lambda_j$ for all $j < l$ and $u_j$ for all $j < l-1$. To determine $\phi_{l-1}$ and the coefficients of $p_{l-1}$, we expand the functions in (5.7) into the form

$$a_0 + a_1 x + \ldots + a_{m-1} x^{m-1} + x^m f_m(x),$$

where $a_j$ are constants. Then (5.7) becomes

$$(x c_0(x)\partial_x + c_1(x) - \lambda_1)(p_{l-1} + x^m \phi_{l-1}) = q_l(x) + x^m (g_l(x) + \lambda_l \phi_0(x)) \quad (5.8)$$

where $q_l$ is a polynomial of degree $m-1$. Since $kc_0(0) + c_1(0) - \lambda_1 \neq 0$ for $k \leq m-1$, (5.8) uniquely determines $a_j, j = 0, 1, \ldots, m-1$. In addition, comparing the coefficients before $x^m$ of the both sides of (5.8)( after having expanded each term in powers of $x$), we see that $\lambda_l$ is given by:

$$\lambda_l = -g_l(0) + \sum_{j=1}^{m-1} (m-j) c_0^{(j)}(0) a_{m-j}/j! + \partial_x^m (c_1 p_{l-1})(0)/m!$$

and $\phi_{l-1}$ should verify

$$(x c_0(x)\partial_x + m c_0(x) + c_1(x) - \lambda_1)\phi_{l-1} = r_{ml} + g_l + \lambda_l \phi_0 \qquad (5.9)$$

with $r_{lm}(0) = -\lambda_l - g_l(0)$. We can determine $\phi_{l-1}$ from (5.9) by adding the condition $\phi_{l-1}(0) = 0$. This shows that for each m, we can construct an asymptotic solution near 0 to $A(0,h)u = zu$ in the form

$$z^m \sim \sum h^j \lambda_j^m, u^m \sim e^{-d/h} \sum h^j u_j^m$$

with $\lambda_0^m = E$ and $\lambda_1^m = e_m$. All $\lambda_j^m$ are real, since $A(0,h)$ is self-adjoint. Now put $z_0 = z^m, v_0 = u^m$. We can now construct by an induction in k an asymptotic solution of the linear eigenvalue problem in $z_k \in \Omega$:

$$(A(0,h) - z_k + h^2 r'(z_{k-1}, 0, h))v_k \sim O(e^{-d/h} h^\infty), \qquad (5.10)$$

for $k = 1, 2 \ldots$, in the form: $z_k \sim \sum h^j \lambda_{jk}, v_k \sim e^{-d/h} \sum h^j v_{jk}$ with $\lambda_{k0} = E$ and $\lambda_{k1} = e_m$. By the argument of [M], one can show that we do not need to modify the $k$ (resp. $k+1$) first terms in the semiclassical expansions of $v_{k-1}$ (resp. $z_k$) in order to obtain an asymptotic solution $(v_k, z_k)$ to (5.10). In addition, all $\lambda_{kj}$ are real and all $v_{kj}$ are holomorphic near 0. Define $z'(h)$ and $u'(h)$ such that:

$$z'(h) \sim \sum_{j=0}^{\infty} h^j \lambda_{jj}, u'(h) \sim h^{-m/4} e^{-d(x)/h} \sum_{j=0}^{\infty} h^j v_{jj}(x).$$

This gives us an asymptotic solution to (5.3). To obtain an asymptotic solution to (5.1), we put:

$$u(x,h,\theta) = \chi(x) u'(e^{\theta} x, h), \Im\theta > 0, \tag{5.11}$$

where $\chi$ is a cut-off function which equals 1 near 0. Applying Lemma 5.1, we have proven the following

**Theorem 5.2** *Let $\Omega$ be a small neighbourhood of $E_j(h)$ defined in Section 4. There exist $z'(h) \in \Omega$ in the form $z'(h) \sim \sum h^j \mu_j, \mu_j \in \mathbf{R}$ , and $u(h,\theta)$ with $\|u(h,\theta)\| \geq c > 0$, such that:*

$$E_{+-}((2n+1)B + z'(h)) u(h,\theta) = O(h^{\infty}).$$

Now to finish the proof of Theorem 1.1, we just need to study a Grushin problem for $E_{+-}((2n+1)B+z), z \in \Omega$, similar to (4.1), but with $u(h,\theta)$ given by (5.11) and $u(h,\theta)^* = u(h,\bar{\theta}) / < u(h,\theta), u(h,\bar{\theta} >$. The details are omitted.

Finally we remark that several questions related to the present work remain open. In particular, it seems that the absence of eigenvalues for the class of Hamiltonians considered in this paper is not yet established. It is also interesting to give precise estimates on the width of resonances which we believe is exponentially small. Finally we indicate that the imaginary part of resonances might be positive or negative, depending on whether $E > 0$ or $E < 0$. This situation should be compared with the resonances of Dirac operator, where the choice is often made so that the product of the energy and the width of resonances is positive.

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