INVERSE SCATTERING WITH FIXED ENERGY FOR DILATION-ANALYTIC POTENTIALS

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ABSTRACT. For N-body Schrödinger operators $P = -\Delta + \sum_{1\le i<j\le N} V_{ij}$, $N \ge 3$, we show that the free-channel to free-channel scattering matrix $S_0(\lambda)$ with one energy $\lambda > 0$ uniquely determines all V_{ij} in the class of dilation analytic Schwartz potentials. The same question for two-body Schrödinger operators remains open.

1. INTRODUCTION

The uniqueness of quantum inverse scattering problems for Schrödinger operators has a long history. See [3, 6, 12] and the references quoted therein. For the inverse scattering with fixed energy for two-body Schrödinger operators, one usually requires the potential to be exponentially decaying [9, 11, 13, 14, 17]. In the two dimensional case, one needs in addition the smallness of potentials. If these conditions are not satisfied, the uniqueness of inverse scattering with fixed energy may fail. See [7, 8] for such examples. However, the exponential decay on potential is only a sufficient condition for the uniqueness of inverse scattering with fixed energy. See [10] for the uniqueness of a class of smooth rational potentials in dimension $d \ge 3$. For inverse problem in N-body scattering, see [4, 5, 15, 23, 24] for high energy case, and [18, 19, 20, 21] for fixed energy case with exponentially decreasing potentials. In this note, we shall show that the N-body scattering matrix, $N \ge 3$, at a fixed energy determines uniquely all potentials in the class of dilation-analytic Schwartz functions.

Let us introduce some notation commonly used in N-body problems. Let P denote the N-body Schrödinger operator with the mass-center

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removed from the total energy operator

$$-\sum_{j=1}^{N}\frac{1}{2m_j}\Delta_{x_j}+\sum_{1\le i<j\le N}V_{ij}(x_i-x_j), \tag{1.1}$$

where $x_j\in\mathbb{R}^d$, $d\ge 2$, denotes the position and $m_j>0$ the mass of the j-th particle, and V_{ij} is assumed to be real smooth satisfying

$$|V_{ij}(y)|\le C_{ij}\langle y\rangle^{-\rho},\quad \rho>1, \tag{1.2}$$

for $y\in\mathbb{R}^d$. Here $\langle y\rangle=(1+|y|^2)^{\frac12}$. P is regarded as a self-adjoint operator in $L^2(\mathbb{X})$, where $\mathbb{X}$ is the $(N-1)d$ dimensional space: $\mathbb{X}=\{(x_1,\cdots,x_N)\in\mathbb{R}^{Nd};\sum_{j=1}^N m_jx_j=0\}$. Let $\mathcal{A}$ denote the set of all cluster decompositions of the N-particle system. For $a\in\mathcal{A}$, let $\#a$ denote the number of clusters in a and P^a the subhamiltonian associated with a. For $i,j\in\{1,\cdots,N\}$, $i\ne j$, we write $(ij)\in a$ if i and j belong to a same cluster in a. If a,b are two cluster decompositions, we write $b\subset a$ if b is a refinement of a. Let a be a k-cluster decomposition $a=(a_1,\cdots,a_k)$. We denote $\mathbb{X}^a=\{x\in\mathbb{X};\sum_{l\in a_j}m_lx_l=0, j=1,\cdots,k\}$ and $\mathbb{X}_a=\{x\in\mathbb{X}; x_i=x_j \text{ if } (ij)\in a_m \text{ for some } m\in\{1,\cdots,k\}\}$. $\mathbb{X}^a$ and $\mathbb{X}_a$ give an orthogonal decomposition for $\mathbb{X}$ relative to the quadratic form $q(x)=\sum_j 2m_j|x_j|^2, x\in\mathbb{X}$. Denote by $\mathbb{X}^*$ (resp., $\mathbb{X}^{a*},\mathbb{X}_a^*$) the dual space of $\mathbb{X}$ (resp., $\mathbb{X}^a,\mathbb{X}_a$). $\mathbb{X}^*$ is equipped with the dual metric, q^*, of q. Let $\mathbb{S}_0$ ($\mathbb{S}_a$, $\mathbb{S}^a$, resp.) denote the unit sphere in $\mathbb{X}$ (in $\mathbb{X}_a$, $\mathbb{X}^a$, resp.). Let π^a and π_a denote the orthogonal projection from $\mathbb{X}$ onto $\mathbb{X}^a$ and $\mathbb{X}_a$, respectively. For $x\in\mathbb{X}$, let $x^a=\pi^ax$ and $x_a=\pi_ax$ so that we have the orthogonal decomposition: $x=x^a+x_a$ with $x^a\in\mathbb{X}^a$ and $x_a\in\mathbb{X}_a$. The N-body Schrödinger operator P introduced above can be written in the form $P=P_0+V(x)$ where P_0 is the Laplace-Beltrami operator on the Euclidean space $(\mathbb{X},q)$ and $V(x)=\sum_{a\in\mathcal{A}}V_a(x^a)$ with $V_a(x^a)=V_{ij}(x_i-x_j)$ if a is an $(N-1)$-cluster decomposition and $(ij)\in a$; $V_a(x^a)=0$, otherwise. Let $-\Delta^a$ ($-\Delta_a$, resp.) denote the restriction of P_0 on $\mathbb{X}^a$ (on $\mathbb{X}_a$, resp.). For $a\in\mathcal{A}$, denote $P^a=-\Delta^a+\sum_{b\subseteq a}V_b(x^b), P_a=P^a-\Delta_a, I_a(x)=\sum_{b\not\subseteq a}V_b(x^b)$. P^a is the subhamiltonian associated with the cluster decomposition a and I_a is the sum of all inter-cluster interactions. The free-channel to free-channel scattering operator is defined by

$$S_0=W_{+,0}^*W_{-,0}, \tag{1.3}$$

where

$$W_{\pm,0}=\text{s-}\lim_{t\to\pm}e^{itP}e^{-itP_0},\quad \text{in}\quad L^2(\mathbb{X}).$$

In many-body problems, there exists generically an infinite number of scattering channels and to each couple of scattering channels, one can define a scattering operator which describes the associated scattering process. It is also interesting to study inverse scattering problems related to other channel scattering operators. See [5, 18, 20, 24].

Let $\mathcal{F}_0 : L^2(\mathbb{X}) \to L^2(\mathbb{R}_+; L^2(\mathbb{S}_0))$ be the spectral representation for P_0: $\mathcal{F}(\lambda)P_0\mathcal{F}(\lambda)^* = \lambda$, $\lambda \in \mathbb{R}_+$. Since S_0 commutes with P_0, S_0 is also diagonalized by $\mathcal{F}_0$. The free-channel to free-channel scattering matrix is defined by

$$S_0(\lambda) = \mathcal{F}(\lambda)S_0\mathcal{F}(\lambda)^*, \quad a.e. \quad \lambda \in \mathbb{R}_+. \tag{1.4}$$

It is still unknown whether $S_0(\lambda)$ can be defined pointwise in λ as a bounded operator on $L^2(\mathbb{S}_0)$. However, using micolocalized resolvent estimates or the propagation of singularities for N-body operators, one can show that the distributional kernel (*i.e.* the scattering amplitude), $S_0(\lambda; \omega, \omega')$, of $S_0(\lambda)$ is well-defined for each λ for angular momenta in $\mathbb{S}_0 \setminus (\cup_{\#a \geq 2} \mathbb{X}_a)$. The N-body inverse scattering is to determine the potentials V_{ij}, $1 \leq i < j \leq N$, from the scattering amplitude $S_0(\lambda; \omega, \omega')$ with a fixed energy λ.

We are interested in the inverse scattering for dilation-analytic Schwartz potentials.

Definition 1.1. *Let $V(y)$ be $-\Delta$-bounded as multiplication on $L^2(\mathbb{R}^d)$. V is said to be dilation-analytic, if there is $\epsilon_0 > 0$ such that the family of operators $\mathbb{R} \ni \theta \to V(e^\theta y)(-\Delta_y + 1)^{-1}$ extends to a compact operator-valued holomorphic function for $\theta \in \mathbb{C}$ with $|\Im\theta| < \epsilon_0$.*

Our result is the following

Theorem 1.2. *Let $N \geq 3$ and $d \geq 2$. Assume that $V_{ij} \in \mathcal{S}(\mathbb{R}^d)$, $1 \leq i < j \leq N$. Suppose that*

$$V_{ij}(y) = U_{ij}(y) + W_{ij}(y) \tag{1.5}$$

where U_{ij} is dilation-analytic and W_{ij} decays exponentially: $W_{ij}(y) = O(e^{-a|y|})$ for some $a > 0$. Then the scattering matrix $S_0(\lambda)$ for one fixed energy $\lambda > 0$ determines all potentials V_{ij}, $1 \leq i < j \leq N$.

Remark 1.3. *(a). If $U_{ij} = 0$, the N-body inverse scattering with fixed energy was studied in* [21, 18, 19]. *Because of the restrictions of* [13], *the known result in two-body inverse scattering with fixed energy in $d = 2$ requires the potentials to be small. Here we do not need this smallness assumption.*

(b). As an example of the class of potentials in which Theorem 1.2 holds, we can take

$$V_{ij}(y) = a_{ij} e^{-b_{ij}\langle y \rangle^{\rho_{ij}}} \tag{1.6}$$

where $a_{ij} \in \mathbb{R}$, $b_{ij} > 0$, $0 < \rho_{ij} \leq 1$. The result in the case $0 < \rho_{ij} < 1$ is new. To our knowledge, the result like Theorem 1.2 is unknown for two-body Schrödinger operators.

Theorem 1.2 is a consequence of the structure of the free-channel to free-channel scattering matrix and the analyticity in energy variable of two-body scattering matrix for dilation-analytic potentials.

2. Recovering the scattering matrix of subhamiltonians

For $a \in \mathcal{A}$, $\#a \geq 2$, the free-channel to free-channel scattering operator for the subhamiltonian P^a is defined by

$$S_0^a = (W_{+,0}^a)^* W_{-,0}^a, \tag{2.1}$$

where

$$W_{\pm,0}^a = \text{s-}\lim_{t\to\pm} e^{itP^a} e^{-itP_0^a}, \quad \text{in} \quad L^2(\mathbb{X}^a),$$

where P_0^a is P^a with all potentials equal to zero. Let $S_0^a(\lambda)$ denote the scattering matrix. One has (Theorem 3 of [20])

Theorem 2.1. *Let $V_{ij} \in \mathcal{S}(\mathbb{R}^d)$ for all $1 \leq i < j \leq N$. Then the free-channel to free-channel scattering matrix $S_0(\lambda)$ at one energy λ determines the free-channel to free-channel scattering matrix for all two-body subhamiltonians $S_0^a(\mu)$ ($a \in \mathcal{A}$, $\#a = N-1$) for all energies $0 < \mu < \lambda$.*

The proof of this theorem in [20] relies on the structure of $S_0(\lambda)$. (Note that $S_0(\lambda)$ there is normalized slightly differently, namely geometrically, see the last sentence of this paragraph.) To get a rough idea for this, we first recall that as a consequence of the propagation of singularities (meaning lack of rapid decay at infinity) for tempered distributional eigenfunctions of P, proved in [22], the wave front set of $S_0(\lambda)g$, for distributions g supported away from $\mathbb{X}_a$, $a \neq 0$, lies inside the image of $\mathrm{WF}(g)$ under the generalized broken bicharacteristic relation. In other words the wave front relation of $S_0(\lambda)$ is a subset of the generalized broken bicharacteristic relation. Thus, location of singularities of $S_0(\lambda)g$ can be described rather precisely. Intuitively, one may think of broken bicharacteristics as classical trajectories 'scaled out to infinity', with bound states in subsystems adding a quantum mechanical aspect to their definition; see [20, 22]. In the absence of bound states in *any* proper subsystem, their projection to the sphere $\mathbb{S}_0$ would be piecewise arcs of great circles, of total length π, where the 'breaks' can only happen at the $\mathbb{X}_a$, and are such that the tangent vectors of the incoming and outgoing segments at a break point in $\mathbb{S}_0 \cap \mathbb{X}_a$ differ by a vector normal to $\mathbb{X}_a$. We also note that the distance π along a great circle corresponds to the antipodal map: free particles entering from one direction exit in the opposite direction! Our (i.e. the wave operator) normalization of the scattering matrix composes the relation linking the beginning and end points of these curves with the antipodal map, by factoring out the free motion.

Now, generalized bicharacteristics can be quite complicated, but it is easy to show that certain points p in the wave front relation of $S_0(\lambda)$, which can thus be connected by a generalized broken bicharacteristic, can in fact only be connected by very special ones: namely ones that lie in the complement of $\mathbb{X}_b$ for all b with $\#b \leq N-1$, with the exception of a single point which in turn lies in $\mathbb{X}_a$ for an $(N-1)$-cluster a, with $(ij) \in a$. In other words, these broken bicharacteristics only hit a single $(N-1)$-cluster a, i.e. describe interactions of particles in which (modulo smoothing, hence negligible effects) only particles i and j collided.

In a neighborhood of these special points p, on the other hand, $S_0(\lambda)$ is microlocally a Fourier integral operator, i.e. its singularities can be described very precisely. In fact, as was already exploited in [21, Proposition 6.4], this is the Fourier integral operator given by the free-channel to free-channel scattering matrix of $\Delta_0 + V_{ij}$, at energies μ, $0 < \mu < \lambda$, where μ depends on p. It is then easy to see (and this is the crux of the proof in [20]) that $S_0^a(\mu)$ is determined by $S_0(\lambda)$ for $0 < \mu < \lambda$, since the set of these points p is actually large enough.

This argument is rather sketchy; we refer the reader to [20] for the detailed proof.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following result on the analyticity in the energy variable of two-body scattering matrix.

Theorem 3.1. *Let $P = -\Delta + V(x)$ be a two-body Schrödinger operator on $\mathbb{R}^d$ with potential $V(x) = U(x) + W(x)$ where W is smooth and exponentially decaying $W(x) = O(e^{-a|x|})$ for some $a > 0$, and U is smooth dilation-analytic short-range: $\exists \rho > 1$ such that*

$$|U(e^\theta x)| \leq C\langle x\rangle^{-\rho}, \quad x \in \mathbb{R}^d$$

for θ in a small complex neighborhood of $0 \in \mathbb{C}$. Then, one has

(a). The resolvent $R(z) = (P-z)^{-1}$, $\Im z > 0$, extends meromophically into the domain $D = \{z \in \mathbb{C}; |\Im z| < \epsilon_0 \Re z, \Im z > -b\}$ for some $\epsilon_0 > 0$ and $b > 0$.

(b) The scattering matrix $S(\lambda)$, $\lambda > 0$, for the pair $(-\Delta, -\Delta + V(x))$ extends meromorphically into D with the same poles as $R(z)$.

This result is known under different conditions. See in [1] for short-range dilation analytic potential ($W = 0$) for any dimension and [2] for $d \geq 3$ under more general conditions than those of Theorem 3.1. Being unable to find an appropriate reference for the case $d = 2$, we indicate here how to modify the arguments of [2] to treat the case $d = 2$.

The space dimension plays a role in the study of local singularities, which arise due to the *global* use of polar coordinates, namely their use even near the origin. Unlike in the case $d \geq 3$, the operators

$$\hat{D}_j = \frac{\partial}{\partial x_j} + \frac{d-1}{2r^2}x_j - ik\frac{x_j}{r}, \quad r = |x|, \quad j = 1, \cdots, d,$$

introduced in [2] are not defined from H^2 to L^2 when $d = 2$. Also, the multiplication by e^{-ikr}, $k \in \mathbb{C}^-$, does not map H^2 to H^2. In fact, for $d = 2$, $\frac{1}{r}$ is not $-\Delta$-bounded, but only $-\Delta$-form bounded with relative bound 0. So when $d = 2$, $\hat{D}_j$ is not continuous from H^2 to L^2, but is still continuous from H^1 to H^{-1}, and the multiplication by e^{-ikr} is continuous from H^1 to H^1. If we regard P as self-adjoint operator defined with form-domain H^1, the limiting absorption principle then holds in $\mathcal{L}(H^{-1,s}; H^{1,-s})$ for any $s > 1/2$. So to treat the case $d = 2$, we can replace the weighted H^2 (resp., L^2) spaces used in [2] by H^1 (resp., H^{-1}) spaces with the same weight. With this modification, the compositions of operators used in [2] are well defined.

The condition $d \geq 3$ also appears in the proof of [2, Lemma 1.6] which asserts that as operator-valued function in $\mathcal{L}(L^2(\mathbb{S}^{d-1}), H^{2,-s}(\mathbb{R}^d))$ for $s > 1/2$, $T_0^{+*}(\overline{k})$ defined by

$$(T_0^{+*}(\overline{k})f)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{S}^{d-1}} e^{-ikr+ikx\cdot\omega} f(\omega)\, d\omega, \tag{3.1}$$

where $r = |x|$ and $f \in L^2(\mathbb{S}^{d-1})$, extends continuously to $\tilde{\mathbb{C}}^- = \{z \in \mathbb{C}; \Im z \leq 0, z \neq 0\}$ and is analytic in $\mathbb{C}^- = \{z \in \mathbb{C}; \Im z < 0\}$. The authors of [2] used the expansion of the kernel of $T_0^{+*}(\overline{k})$ in terms of spherical harmonics and Bessel functions, where the condition $d \geq 3$ appears. Their proof is actually still valid in the case $d = 2$. Here we present an easy alternative proof for the analyticity of $T_0^{+*}(\overline{k})$ in $\mathbb{C}^-$ for $d \geq 2$. Let us estimate the complex derivative

$$(\partial_k T_0^{+*}(\overline{k})f)(x) = -i(2\pi)^{-\frac{d}{2}} \int_{\mathbb{S}^{d-1}} e^{-ikr+ikx\cdot\omega} r(1-\omega\cdot\theta) f(\omega)\, d\omega,$$

where $x = r\theta$. Let $k = a - ib \in \mathbb{C}^-$, $b > 0$. Splitting the above integral into two pieces according to $|1-\omega\cdot\theta| > r^{-1+\epsilon}$ and $|1-\omega\cdot\theta| \leq r^{-1+\epsilon}$

for $r \geq 1$ and $0 < \epsilon < 1$, one obtains

$$|\partial_k(T_0^{+*}(\overline{k})f)(x)|^2 \leq C\|f\|^2 e^{-2br^\epsilon} r^2$$
$$+C \int_{|1-\omega\cdot\theta|\leq r^{-1+\epsilon}} |f(\omega)|^2 d\omega \int_{\mathbb{S}^{d-1}} e^{-2br(1-\omega\cdot\theta)} |r(1-\omega\cdot\theta)|^2 \, d\omega$$

Since $\omega \mapsto (1-\omega\cdot\theta)$ has a unique non degenerate minimum on the sphere at $\omega = \theta$, one has

$$\int_{\mathbb{S}^{d-1}} e^{-2br(1-\omega\cdot\theta)} |r(1-\omega\cdot\theta)|^2 \, d\omega \leq C r^{-\frac{(d-1)}{2}}$$

and

$$\int_{\mathbb{S}_\theta^{d-1}} \int_{|1-\omega\cdot\theta|\leq r^{-1+\epsilon}} |f(\omega)|^2 d\omega d\theta \leq C r^{-(1-\epsilon)\frac{(d-1)}{2}} \|f\|^2.$$

For $s > 1/2$, taking $0 < \epsilon < \frac{2(2s-1)}{d-1}$, one has

$$\|\langle x\rangle^{-s} \partial_k T_0^{+*}(\overline{k})f)\|^2 \leq C(b,s)\|f\|^2 \tag{3.2}$$

It follows that $\mathbb{C}^- \ni k \to T_0^{+*}(\overline{k}) \in \mathcal{L}(L^2(\mathbb{S}^{d-1}), L^{2,-s}(\mathbb{R}^d))$ is analytic for $s > 1/2$. The continuity of $T_0^{+*}(\overline{k})$ in $\tilde{\mathbb{C}}^-$ near $\mathbb{R}\setminus\{0\}$ can be proved by standard technique of the limiting absorption principle. An easy argument shows that the same is true if we regard $T_0^{+*}(\overline{k})$ as $\mathcal{L}(L^2(\mathbb{S}^{d-1}), H^{1,-s}(\mathbb{R}^d))$-valued function. This shows the validity of the second part of [2, Lemma 1.6] for $d \geq 2$ with $\mathcal{L}(L^2(\mathbb{S}^{d-1}), H^{2,-s}(\mathbb{R}^d))$ replaced by $\mathcal{L}(L^2(\mathbb{S}^{d-1}), H^{1,-s}(\mathbb{R}^d))$. The first part of [2, Lemma 1.6] follows by taking the adjoint.

The other arguments of [2] can be adapted with minor modifications to prove Theorem 3.1 for $d = 2$.

Proof of Theorem 1.2. Theorem 1.2 follows easily from Theorems 2.1 and 3.1. In fact, let $a \in \mathcal{A}$ be an $N-1$ cluster decomposition where the particles i and j are in the same cluster. Then $S_0^a(\mu)$ is just the two-body scattering matrix of $(-\Delta_y, -\Delta_y + V_{ij}(y))$ and by Theorem 2.1, $S_0^a(\mu)$, $0 < \mu < \lambda$ is uniquely determined by $S_0(\lambda)$. On the other hand, under the conditions of Theorem 1.2, the positive eigenvalues of $-\Delta_y + V_{ij}(y)$ are absent. Theorem 3.1 implies that the scattering matrix $S_0^a(\lambda)$ is analytic on $]0,+\infty[$. Therefore, $S_0^a(\mu)$ for $\mu \in]0,\lambda[$ determines uniquely $S_0^a(\mu)$ for all $\mu \in \mathbb{R}_+$. It is well-known that in the case $d \geq 2$, V_{ij} can be recovered from the asymptotics of $S_0^a(\mu)$, $\mu \to +\infty$ (see

[4, 6, 16]). Therefore, for $d \geq 2$, all V_{ij} are uniquely determined by one scattering matrix $S_0(\lambda)$. □

To end this note, let us indicate that in [8], the authors constructed rapidly decaying transparent potentials for two-body Schrödinger operators in two-dimensional case. R. Novikov told the second author that it may be possible to study the analyticity this class of transparent potentials. Results of inverse scattering are dimension-dependent. An interesting open question is to see if the result of Theorem 1.2 holds for two-body Schrödinger operators in dimension $d \geq 3$.

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