On the existence of the N-body Efimov effect

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Abstract

In this paper, we prove the existence of the Efimov effect for N-body quantum systems with $N \geq 4$. Under the conditions that the bottom of the essential spectrum, E_0, of the N-body operator is attained by the spectra of a unique three-cluster Subhamiltonian and its three associated two-cluster Subhamiltonians, and that at least two of these two-cluster Subhamiltonians have a resonance at the threshold E_0, we give a lower bound of the form $C_0|\log(E_0 - \lambda)|$ for the number of eigenvalues on the left of λ, $\lambda < E_0$, where C_0 is a positive constant depending only on the reduced masses in the three-cluster decomposition. We also obtain a lower bound on the number of discrete eigenvalues in coupling constant perturbation.

1 Introduction

The Efimov effect describes an interesting and unexpected phenomenon for three-body Schrödinger operators which can be roughly stated as follows. When the essential spectrum of the three-particle Hamiltonian is the positive real axis, and when at least two of its two-body Subhamiltonians have a resonance at zero, the discrete spectrum of the three-body Schrödinger operator is infinite, even if the interactions are very short-range. This phenomenon is striking if one compares it with the results on the finiteness of eigenvalues of two-body Schrödinger operators or N-body Schrödinger operators whose bottom of essential spectrum is only reached by the spectrum of two-cluster Subhamiltonians. See [11, 13, 14, 33]. Since its discovery by V. Efimov in 1970 ([12]), many works are devoted to this subject. See, for example, [2, 3, 4, 6, 9, 10, 21, 22, 25, 28, 29, 30, 31, 35].

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In particular, D. Yafaev gave a mathematically rigorous proof for the existence of such phenomenon ([35]), and A. V. Sobolev obtained the asymptotics of the number of eigenvalues near zero([28]).

For N-body systems with $N \geq 4$, R. D. Amado and F. C. Greenwood ([5]) discussed the contribution of zero energy eigenvalue and zero energy resonance of $(N-1)$-particle subsystems to the discrete spectrum of the total system and concluded that there is no Efimov effect in the case $N \geq 4$. The subsequent mathematical researches on this subject are concerned with many-body systems whose essential spectrum is the positive real axe and the results obtained are that two-particle zero energy resonances do not create an infinite number of eigenvalues when $N \geq 4$. In this work, we shall prove that threshold energy resonances of two-cluster Subhamilonians, which can be subsystems of any particle numbers, at the bottom of essential spectrum of an N-body Schrödinger operator, $N \geq 4$, do create an infinite number of discrete eigenvalues.

Let us introduce some notation used in this work. Let P denote the N-body Schrödinger operator with the mass-center removed from the total energy operator

$$-\sum_{j=1}^{N} \frac{1}{2m_j}\Delta_{x_j} + \sum_{1\leq i<j\leq N} V_{ij}(x_i - x_j), \quad x_j \in \mathbb{R}^3, 0 < m_j < \infty, \tag{1.1}$$

where x_j and m_j denote the position and mass of the j-th particle, V_{ij} is assumed to be real and relatively compact with respect to $-\Delta$ in $L^2(\mathbb{R}^3)$ and satisfies the decay

$$|V_{ij}(y)| \leq C_{ij} < y >^{-\rho}, \quad \rho > 2, \tag{1.2}$$

for $y \in \mathbb{R}^3$. V_{ij} is assumed to be bounded, but $L^2_{loc}(\mathbb{R}^3)$-singularities can easily be included. P is regarded as a self-adjoint operator in $L^2(\mathbf{X})$, where $\mathbf{X}$ is the $3(N-1)$ dimensional real vectorial space: $\mathbf{X} = \{(x_1, \cdots, x_N) \in \mathbb{R}^{3N}; \sum_{j=1}^{N} m_j x_j = 0\}$. Let $\mathcal{A}$ denote the set of all cluster decompositions of the N-particle system. For $a \in \mathcal{A}$, let $\#a$ denote the number of clusters in a and P^a the Subhamiltonian associated with a. For $i, j \in \{1, \cdots, N\}$, $i \neq j$, we write $(ij) \in a$ if i and j belong to a same cluster in a. If a, b are two cluster decompositions, we write $b \subset a$ if b is a refinement of a. Let a be a k-cluster decomposition $a = (a_1, \cdots, a_k)$. We denote $\mathbf{X}^a = \{x \in \mathbf{X}; \sum_{l\in a_j} m_l x_l = 0, j = 1, \cdots, k\}$ and $\mathbf{X}_a = \{x \in \mathbf{X}; x_i = x_j \text{ if } (ij) \in a_m \text{ for some } m \in \{1, \cdots, k\}\}$. $\mathbf{X}^a$ and $\mathbf{X}_a$ give an orthogonal decomposition for $\mathbf{X}$ relative to the quadratic form $q(x) = \sum_j 2m_j |x_j|^2, x \in \mathbf{X}$. Denote by $\mathbf{X}^*$ (resp., $\mathbf{X}^{a*}, \mathbf{X}_a^*$) the dual space of $\mathbf{X}$ (resp., $\mathbf{X}^a, \mathbf{X}_a$). $\mathbf{X}^*$ is equipped with the dual metric, q^*, of q. In this paper, when coordinates are needed in calculation, we always use in orthonormal basis with respect to q in $\mathbf{X}$ and with respect to q^* in $\mathbf{X}^*$.

Let π^a and π_a denote the orthogonal projection from $\mathbf{X}$ onto $\mathbf{X}^a$ and $\mathbf{X}_a$, respectively. For $x \in \mathbf{X}$, let $x^a = \pi^a x$ and $x_a = \pi_a x$ so that we have the orthogonal decomposition: $x = x^a + x_a$ with $x^a \in \mathbf{X}^a$ and $x_a \in \mathbf{X}_a$. The N-body Schrödinger operator P introduced above can be written in the form $P = P_0 + V(x)$ where P_0 is the

Laplace-Beltrami operator on the Euclidean space $(\mathbf{X}, q)$ and $V(x) = \sum_{a\in\mathcal{A}} V_a(x^a)$ with $V_a(x^a) = V_{ij}(x_i - x_j)$ if a is an $(N-1)$-cluster decomposition and $(ij) \in a$; $V_a(x^a) = 0$, otherwise. Let $-\Delta^a$ ($-\Delta_a$, resp.) denote the restriction of P_0 on $\mathbf{X}^a$ (on $\mathbf{X}_a$, resp.). For $a \in \mathcal{A}$, denote $P^a = -\Delta^a + \sum_{b\subseteq a} V_b(x^b), P_a = P^a - \Delta_a, I_a(x) = \sum_{b\not\subseteq a} V_b(x^b)$. P^a is the Subhamiltonian associated with the cluster decomposition a and I_a is the sum of all inter-cluster interactions. Let $\mathcal{T} = \cup_{a\in\mathcal{A},\#a\geq 2}\sigma_p(P^a)$ be the set of thresholds of P. The HVZ theorem says that the bottom of the essential spectrum, $E_0 \equiv \inf \sigma_{ess}(P)$, of P is given by

$$E_0 = \min_{a\in\mathcal{A},\#a\geq 2} \inf \sigma(P^a) = \min_{a\in\mathcal{A},\#a=2} \inf \sigma(P^a).$$

The purpose of this paper is to study the infiniteness of the discrete spectrum of N-body Schrödinger operators due to threshold energy resonances. Assume that E_0 is attained by a unique three-cluster threshold, b, with $b = (b_1, b_2, b_3)$.

$$\inf \sigma(P^b) = E_0 = \inf \sigma_{ess}(P). \tag{1.3}$$

Let $a_k = (b_i \cup b_j, b_k)$, where i, j and k take different values in $\{1, 2, 3\}$. Then, we have necessarily $\sigma(P^{a_k}) = [E_0, \infty[$. In this paper, we assume that these are the only possible cluster decompositions for which the spectra of the corresponding subsystems attain the bottom of essential spectrum of total Hamiltonian, *i. e.*,

$$\inf \sigma(P^a) = E_0, a \in \{b, a_1, a_2, a_3\}; \quad \inf \sigma(P^a) > E_0, a \notin \{b, a_1, a_2, a_3\}, \#a \geq 2. \tag{1.4}$$

By the HVZ Theorem, the assumption (1.4) implies that E_0 is in the discrete spectrum of P^b for $N \geq 4$, and therefore necessarily, $E_0 < 0$. Assume that the inter-cluster interactions related to b are attractive:

$$V_a(x^a) \leq 0, \text{ for } a \not\subseteq b. \tag{1.5}$$

This assumption, together with the condition that 0 is not an eigenvalue of any two-body Subhamiltionian, was often used in early works on the three-body Efimov effect ([35, 25, 28]). They are removed in [30] by an argument of perturbation for two-body Subhamiltonians. This kind of perturbation is more subtle for many-particle Subhamiltonians. In addition, the condition (1.5) is used in our work to prove that E_0 is not an eigenvalue of two-cluster Subhamiltonians.

Unless otherwise stated, the conditions (1.2), (1.4) and (1.5) are assumed in this work.

For $c, d \in \mathcal{A}$ with $c \subset d$, we define $H^{m,s;c,d}$, $m, s \in \mathbb{R}$, be the weighted Sobolev space on $\mathbf{X}^d$ equipped with the norm

$$||u||_{m,s;c,d} = (\int_{\mathbf{X}^d} |(1 + |x_c^d|^2)^{s/2}(1 - \triangle_{x^d})^{m/2}u|^2 dx^d)^{\frac{1}{2}}. \tag{1.6}$$

If c and d are just the one-cluster decomposition, the indication about the cluster decompositions will be dropped. For example, for $\#d = 1$, we just write $H^{m,s;c,d}$ as $H^{m,s;c}$. Let $< \cdot, \cdot >$ ($(\cdot, \cdot)_a, < \cdot, \cdot >_a$, resp.) denote the scalar product on $L^2(\mathbf{X})$ (on $L^2(\mathbf{X}^a)$,

$L^2(\mathbf{X}_a)$, resp.). $H^{1,-s;c,d}$, $s \geq 0$, to the dual space of $H^{-1,s;c,d}$ with the usual L^2 product as the dual product. Let $\mathcal{L}(m,s;m',s';c,d) = \mathcal{L}(H^{m,s;c,d}, H^{m',s';c,d})$. It will be convenient to regard P as an operator in $\mathcal{L}(1,0;-1,0)$.

Definition. *We say that E_0 is a* resonance *of P^{a_j} if the equation $P^{a_j}u = E_0 u$ has a solution $u \in H^{1,-s}(\mathbf{X}^{a_j}) \setminus L^2(\mathbf{X}^{a_j})$ for any $s > \frac{1}{2}$.*

Under the assumptions (1.2) and (1.4), the analysis of threshold resonances made in [34] is valid for P^{a_j}. One knows that if E_0 is a resonance of P^{a_j}, then it is simple and the corresponding resonant state u behaves like

$$u(x^{a_j}) = O(e^{-\delta|x^b|}|x_b^{a_j}|^{-1}), \quad |x^{a_j}| \to \infty, \quad \delta > 0. \tag{1.7}$$

One can show under the conditions (1.2), (1.4) and (1.5), E_0 is not an eigenvalue of P^{a_j} for $j = 1,2,3$. The main result of this work is the following.

Theorem 1.1 *Let the conditions (1.2) with $\rho > 2$, (1.4) and (1.5) be satisfied. Assume that at least two of the two-cluster Subhamiltonians P^{a_j}, $j = 1,2,3$, have a resonance at E_0. Let $N(\lambda)$ denote the number of the eigenvalues of P below $\lambda < E_0$. Then, there exists $C_0 > 0$ depending only on the reduced masses of b_1, b_2, b_3 such that*

$$N(\lambda) \geq C_0 |\log|\lambda - E_0||, \lambda \to E_0. \tag{1.8}$$

Clearly, the positivity of C_0 implies that there exists an infinite number of discrete eigenvalues for the N-body Schrödinger operator satisfying the conditions of Theorem 1.1. Comparing with the known criteria for the finiteness or infiniteness of discrete spectrum of N-body Schrödinger operators ([13, 14]), one can say that these eigenvalues are due to the existence of resonances of two-cluster Subhamiltonians at E_0. This phenomenon is similar to the Efimov effect known in three-body problems.

The following result on the coupling constant perturbation gives the variation of the number of discrete eigenvalues when potentials approach those satisfying the conditions of Theorem 1.1.

Theorem 1.2 *Under the conditions of Theorem 1.1, assume in addition that $V_a \in C_0^\infty$ for all $a \not\subseteq b$. Let $P(\mu) = P_b + \mu I_b$, $\mu \geq 1$. Then for $\mu > 1$, the discrete spectrum of $P(\mu)$ is finite. Let $\mathcal{N}(\mu)$ denote the number of discrete eigenvalues of $P(\mu)$. One has*

$$\mathcal{N}(\mu) \geq 2C_0|\log(\mu - 1)| - C_1, \tag{1.9}$$

when $\mu \to 1_+$. Here $C_0 > 0$ is the same as in Theorem 1.1 and C_1 is a constant independent of μ.

For three-body problems, the coupling constant limit is studied in [30] for $\mu < 1$.

Let us now mention some earlier works related to the Efimov effect in N-body problems with $N \geq 4$. For $N = 3$, see [3, 24, 28, 30] and the references therein. For $N \geq 4$, R. D. Amado and F. C. Greenwood argued in [5] that the zero energy eigenvalue and zero energy resonance of $(N-1)$-particle subsystems do not produce an infinite number of eigenvalues in N-body problems when $N \geq 4$. In [32], S. A. Vugal'ter and G. M. Zhislin proved that two-particle resonances at zero energy do not produce an infinite number of discrete eigenvalues when $N \geq 4$. The work of G. Karner [18] also concerned the case where the essential spectrum of the total Hamiltonian is $\mathbb{R}_+$. In [1], F. Ahia gave a new proof of the result of S. A. Vugal'ter and G.M. Zhislin [32]. As remarked by these authors, the question of whether or not the Efimov effect exists for $N \geq 4$ has not been settled.

We emphasize that in spite of the titles, our result is not in discrepancy with [5]. In contrast to [5], the bottom of the essential spectrum, E_0, of the N-body Schrödinger operator in our case is always strictly negative when $N \geq 4$. A mathematical proof of the finiteness of discrete eigenvalues in the situation of [5] is still missing for $N \geq 4$. In N-body problems, there are many spectral configurations for thresholds. An important step in the study of the Efimov effect is the understanding of contribution of threshold energy resonances (see [35, 25, 28, 29, 30] for the role of zero energy resonances of two-body Subhamiltonians in three-body problems). N-body threshold energy resonances have not yet been fully analyzed for $N \geq 3$. The result of [34] for two-cluster threshold energy resonances play an important role in this paper.

The proof of Theorem 1.1 relies heavily on the ideas and the results given in [28] and [34]. The results of [34] are used to obtain the leading term of the resolvent of two-cluster Subhamiltonians near E_0. We follow closely A. V. Sobolev's method to transform the eigenvalue problem to a Toeplitz operator of the Fourier type ([15]). Our main task here is to reduce the additional space dimensions to obtain an integral operator already studied in [28]. Because of the lack of compactness, we only obtain a lower bound on the number of eigenvalues.

The organization of this work is as follows. In Section 2, we reduce the eigenvalue problem by the principle of Birman-Schwinger. In Section 3, the asymptotics of the resolvent of two-cluster Subhamiltonians P^{a_j} are derived from the results proved in [34]. In Section 4, we give a lower bound on the number of eigenvalues. The assumption on the existence of resonances at E_0 is not stable under perturbation. In Section 5, we study coupling constant limit for a family of N-body operators $P(\mu)$ and prove Theorem 1.2.

2 A reduction

Let b and $a_j, j = 1, 2, 3$, be given in the assumption (1.4). Write $I_b = \sum_{a \not\subseteq b} V_a$ as

$$I_b = -W_1 - W_2 - W_3, \tag{2.1}$$

where

$$W_j = -\sum_{a \not\subseteq b, a \subseteq a_j} V_a, j = 1, 2, 3.$$

Then $W_j \geq 0$ and one has

$$P_{a_j} = P_b - W_j. \tag{2.2}$$

Denote $R_j(z) = (P_{a_j} - z)^{-1}$.

For a bounded self-adjoint operator A, we define $n(\lambda, A)$ as

$$n(\mu, A) = \sup\{\dim F; < Au, u > > \mu, u \in F, \|u\| = 1\}.$$

$n(\mu, A)$ is equal to the infinity if μ is in the essential spectrum and if $n(\mu, A)$ is finite, it is equal to the number of the eigenvalues of A bigger than μ.

Proposition 2.1 *Let* $\lambda < E_0$. *One has*

$$N(\lambda) = n(1, A(\lambda)) \tag{2.3}$$

where

$$A(\lambda) = \begin{pmatrix} 0 & A_{12}(\lambda) & A_{13}(\lambda) \\ A_{21}(\lambda) & 0 & A_{23}(\lambda) \\ A_{31}(\lambda) & A_{32}(\lambda) & 0 \end{pmatrix}$$

with

$$A_{ij}(\lambda) = (1 + W_i^{\frac{1}{2}} R_i(\lambda) W_i^{\frac{1}{2}})^{\frac{1}{2}} W_i^{\frac{1}{2}} R_b(\lambda) W_j^{\frac{1}{2}} (1 + W_j^{\frac{1}{2}} R_j(\lambda) W_j^{\frac{1}{2}})^{\frac{1}{2}}$$

for $i \neq j$.

Proof. This Proposition is deduced by the arguments of [28]. Since $P = P_b + I_b$, and $P_b - \lambda$ is positive and invertible for $\lambda < E_0$, one has

$$\lambda \in \sigma_d(P) \Longleftrightarrow 0 \in \sigma_d(1 + R_b(\lambda)^{\frac{1}{2}} I_b R_b(\lambda)^{\frac{1}{2}})$$

and

$$u \in D(P) \text{ and } < (P - \lambda)u, u >< 0 \Longleftrightarrow$$

$$< (R_b(\lambda)^{\frac{1}{2}} |I_b| R_b(\lambda)^{\frac{1}{2}} - 1)v, v > > 0 \text{ and } v = (P_b - \lambda)^{\frac{1}{2}} u.$$

It follows that $N(\lambda) = n(1, R_b(\lambda)^{\frac{1}{2}} |I_b| R_b(\lambda)^{\frac{1}{2}})$. Decompose

$$R_b(\lambda)^{\frac{1}{2}} |I_b| R_b(\lambda)^{\frac{1}{2}} = \sum_{j=1}^{3} R_b(\lambda)^{\frac{1}{2}} W_j R_b(\lambda)^{\frac{1}{2}} = B^* B,$$

with $B : L^2 \to (L^2)^3$ defined by

$$B = (W_1^{\frac{1}{2}} R_b(\lambda)^{\frac{1}{2}}, W_2^{\frac{1}{2}} R_b(\lambda)^{\frac{1}{2}}, W_3^{\frac{1}{2}} R_b(\lambda)^{\frac{1}{2}}) \equiv (B_1, B_2, B_3).$$

Then B^*B and BB^* have the same non-zero eigenvalues with the same multiplicity, and the same essential spectrum. It follows that for any $\mu > 0$,

$$n(\mu, B^*B) = n(\mu, BB^*). \tag{2.4}$$

Consequently,

$$N(\lambda) = n(1, BB^*). \tag{2.5}$$

BB^* can be decomposed as $BB^* = D + K$, where

$$D = \begin{pmatrix} B_1B_1^* & 0 & 0 \\ 0 & B_2B_2^* & 0 \\ 0 & 0 & B_3B_3^* \end{pmatrix}, \quad K = \begin{pmatrix} 0 & B_1B_2^* & B_1B_3^* \\ B_2B_1^* & 0 & B_2B_3^* \\ B_3B_1^* & B_3B_2^* & 0 \end{pmatrix}.$$

Since $\sigma(P^{a_j}) = [E_0, \infty[$, $1 - W_j^{\frac{1}{2}} R_b(\lambda) W_j^{\frac{1}{2}}$ is invertible and

$$(1 - W_j^{\frac{1}{2}} R_b(\lambda) W_j^{\frac{1}{2}})^{-1} = 1 + W_j^{\frac{1}{2}} R_{a_j}(\lambda) W_j^{\frac{1}{2}} > 0. \tag{2.6}$$

So $1 - D$ is invertible and positive definite. A direct calculation shows that $n(1, BB^*) = n(1, (1-D)^{-\frac{1}{2}} K (1-D)^{-\frac{1}{2}})$ and that $(1-D)^{-\frac{1}{2}} K (1-D)^{-\frac{1}{2}} = A(\lambda)$. ■

Note that in three-body case, the operator $A(\lambda)$ is compact. This is no longer true if $N \geq 4$, since for $N \geq 4$ and $i \neq j$, $W_i^{\frac{1}{2}} W_j^{\frac{1}{2}}$ is independent of the x^b variables, $W_i^{\frac{1}{2}} R_b(\lambda) W_j^{\frac{1}{2}}$ is not compact on $L^2(\mathbf{X})$.

3 Resolvent asymptotics for Subhamiltonians

Definition. ([23, 17]) *We say that E_0 is a* resonance *of P if the equation $Pu = E_0 u$ has a solution $u \in H^{1,-s}(\mathbf{X}) \setminus L^2(\mathbf{X})$ for any $s > \frac{1}{2}$. We call E_0 a regular point of P if E_0 is neither eigenvalue, nor resonance of P; an exceptional point of the first kind (respectively, the second kind; the third kind) of P if it is a resonance but not an eigenvalue (respectively, an eigenvalue but not a resonance; both an eigenvalue and a resonance) of P.*

Note the $b \subset a_j$ and P^b can be regarded as a two-cluster Subhamiltonian of P^{a_j} on $\mathbf{X}^{a_j}$. The assumption (1.3) implies that P^b is the unique Subhamiltonian (relatively to the many-particle systems described by P^{a_j}) whose spectrum attains the bottom of the essential spectrum of P^{a_j}. Therefore, the condition (1.3) of [34] is satisfied for P^{a_j}. The following results of [34] are important in this work.

Theorem 3.1 *Assume (1.2) with $\rho > 2$ and (1.3). Let $\varphi_0 \in D(P^b)$ be the eigenfunction of P^b associated to E_0*

$$P^b \varphi_0 = E_0 \varphi_0, \quad \|\varphi_0\| = 1. \tag{3.1}$$

Let $j = 1,2,3$. If $u \in H^{1,-s}(\mathbf{X}^{a_j}), \forall s > \frac{1}{2}$ and $P^{a_j} u = E_0 u$, then, u is a resonant function of P^{a_j} if and only if

$$\int_{\mathbf{X}^{a_j}} W_j(x^{a_j})(x) u(x^{a_j}) \varphi_0(x^b) \, dx^{a_j} \neq 0. \tag{3.2}$$

The resonance at E_0, if it does exist, is simple and the resonant state u has the following behavior at the infinity

$$u(x^{a_j}) = O(e^{-c_0|x^b|} |x_b^{a_j}|^{-1}), \quad c_0 > 0. \tag{3.3}$$

The property (3.3) follows from the proof of Theorem 1.1 of [34]. See Proposition 3.3 and Corollary 4.6 of [34]. In fact, there is an explicit relation between the resonant state u and that of a three-dimensional Schödinger operator, v,

$$u = \epsilon_+(E_0) v \qquad \text{or} \qquad u = \mathcal{E}_+(E_0) v$$

according to whether or not E_0 is eigenvalue of an auxiliary operator. (3.3) follows from the known behavior of three dimensional zero energy resonant state, and the properties of $\epsilon_+(E_0)$ and $\mathcal{E}_+(E_0)$. See (3.13) below for $\epsilon_+(E_0)$.

Let $\delta > 0$ be small. Denote $U_\delta(E_0) = \{z \in \mathbb{C}; \Im z > 0, |z - E_0| < \delta\}$ and $U_\delta = U_\delta(0)$. Let $R^m(z) = (P^{a_m} - z)^{-1}$ and $R_m(z) = (P_{a_m} - z)^{-1} = (P^{a_m} - \Delta_{x_{a_m}} - z)^{-1}$. The asymptotics of the resolvent $R^m(z) = (P^{a_m} - z)^{-1}$, $m = 1,2,3$, can be stated as follows.

Theorem 3.2 *Assume (1.2) with $\rho > 2$ and (1.3). Let $\zeta = z - E_0$ and $(\zeta)^{\frac{1}{2}}$ be the branch such that $\Im(\zeta)^{\frac{1}{2}} > 0$ for $\Im\zeta > 0$. Set $l = 0$ (resp., $l = 1,2,3$) if E_0 is a regular point (resp., an exceptional point of the first, the second, the third kind) of P^{a_m}. Then for $k_l \in \mathbb{Z}$, $k_l \geq -2$, depending on ρ and l and $s > 0$ depending k_l and ρ, one has the following asymptotic expansion in $\mathcal{L}(-1,s;1,-s;b,a_m)$,*

$$R^m(z) = \sum_{j=-2}^{k_l} \zeta^{j/2} B_{j,m}^{(l)} + O(|\zeta|^{k_l/2+\epsilon}), \quad ,\zeta \to 0. \tag{3.4}$$

for some $\epsilon > 0$ depending on s and ρ. More precisely, let $\rho_0 = [(\rho+1)/2] \in \mathbb{N}$ be the biggest integer $\leq (\rho+1)/2$. Then, when $l = 0$, one has $k_0 = \rho_0 - 1$ and

$$B_{-2,m}^{(0)} = B_{-1,m}^{(0)} = 0. \tag{3.5}$$

When $l = 1$, one has $k_1 = \rho_0 - 2$ and

$$B_{-2,m}^{(1)} = 0, \quad B_{-1,m}^{(1)} = i < \cdot, u_m > u_m, \tag{3.6}$$

and u_m is an E_0 resonant function of P^{a_m} normalized by

$$\frac{1}{2\sqrt{\pi}}\int_{\mathbf{X}^{a_m}} -W_m(x^{a_m})u_m(x^{a_m})\varphi_0(x^b)\, dx^{a_m} = 1. \tag{3.7}$$

For $l = 2, 3$, let $\rho > 3$. Then, $k_l = \rho_0 - 4$ and

$$B^{(2)}_{-2,m} = B^{(3)}_{-2,m} = -\Pi_{E_0,m}, \tag{3.8}$$

$\Pi_{E_0,m}$ being the orthogonal projection onto the eigenspace of P^{a_m} associated with the eigenvalue E_0. If $\rho > 3$ and $l \in \{0, 1, 2, 3\}$, s and ϵ satisfy: $(2\rho_0 - 1)/2 < s < \rho - (2\rho_0 - 1)/2$ and $0 < \epsilon \le \min\{\frac{1}{2}, (2s' - 2\rho_0 + 1)/4\}$, for any $s' < s$.

For $N \ge 4$, $W_m^{\frac{1}{2}}$ does not give the weight $< x_b^{a_m} >^{-s}$ *uniformly* in x^b, which necessary to obtain from Theorem 3.2 the asymptotics of $W_m^{\frac{1}{2}}R_m(\lambda)W_m^{\frac{1}{2}}$ near E_0. But a slight modification of the detailed analysis given in [34] enables us to overcome this difficulty. The following result gives the asymptotic behavior of $R_m(\lambda)$ for λ near E_0.

Theorem 3.3 *Assume (1.2) with $\rho > 2$, (1.4) and (1.5). Let $m \in \{1, 2, 3\}$.*

(a). E_0 is not an eigenvalue of P^{a_m}.

(b). Assume that E_0 is a resonance for P^{a_m}. Let u_m be the unique resonant state normalized by (3.7). Then, $\phi_m \in H^{1,s}(\mathbf{X}^{a_m})$ for any $s < \frac{\rho-1}{2}$. Put

$$\phi_m = W_m^{\frac{1}{2}}u_m, \quad \nu = E_0 - \lambda. \tag{3.9}$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be supported in small neighborhood of 0, $0 \le \chi \le 1$ and $\chi(s) = 1$ for s near 0. Then, one has

$$W_m^{\frac{1}{2}}R_m(\lambda)W_m^{\frac{1}{2}} = \chi(-\Delta_{x_{a_m}})^2(-\Delta_{x_{a_m}} + \nu)^{-\frac{1}{2}}((\cdot, \phi_m)_{a_m} \otimes \phi_m) + r_m(\lambda), \tag{3.10}$$

in $\mathcal{L}(L^2(\mathbf{X})$. The remainder $r_m(\lambda)$ can be decomposed as

$$r_m(\lambda) = r_m^{(1)}(\lambda) + r_m^{(2)}(\lambda),$$

where $r_m^{(1)}(\lambda)$ is a family of operators uniformly bounded for λ in small neighborhood of E_0, and $r_m^{(2)}(\lambda)$ is a family of bounded operators continuous in $\lambda < E_0$ satisfying that for any $\epsilon \in]0, \frac{1}{2}]$ with $\epsilon < (\rho - 2)/2$, $< x_{a_j} >^s (-\Delta_{x_{a_m}} + \nu)^{\frac{1}{2}-\epsilon} r_m^{(2)}(\lambda) < x_{a_k} >^s$ is a continuous $\mathcal{L}(L^2)$-valued function for $\lambda \le E_0$, $j, k \ne m$, and $0 \le s < \frac{\rho-1}{2}$.

Proof. (a). Assume that $\eta \in L^2(\mathbf{X}^{a_m})$ is a non trivial solution of $(P^{a_m} - E_0)\eta = 0$, or

$$(P^b - \Delta_{x_b^{a_m}} - E_0)\eta = W_m\eta.$$

Under the assumption (1.4), η is the ground state of P^{a_m}, therefore $\eta > 0$. Let $\psi(x_b^{a_m}) = (\varphi_0, \eta)_b$ and $f(x_b^{a_m}) = (W_m\varphi_0, \eta)_b$. Here $(\cdot,\cdot)_b$ is the scalar product on $L^2(\mathbf{X}^b)$. Since φ_0 is the ground state of P^b, we have $\psi \in L^2(\mathbf{X}_b^{a_m})$ with $\dim \mathbf{X}_b^{a_m} = 3$, and $\psi(x_b^{a_m}) > 0$. By (1.5), $f(x_b^{a_m}) \geq 0$. We obtain

$$-\Delta_{x_b^{a_m}}\psi = f \geq 0.$$

Theorem A.3.2 of [27] (with $V_+ = 0$) shows that $\psi \notin L^2$. This contradiction proves that E_0 can not be an eigenvalue of P^{a_m}.

(b). It is easier to work in the dual variables ξ_{a_m} of x_{a_m}. Let $z = \lambda - |\xi_{a_m}|^2$. Since $P^{a_m} \geq E_0$,

$$(P^{a_m} - z)^{-1}(1 - \chi(|\xi_{a_m}|^2)^2)$$

is uniformly bounded for λ in a small neighborhood of E_0.

Let Π_0 denote the orthogonal projection in $L^2(\mathbf{X}^{a_m})$ induced by the eigenfunction φ_0 of P^b. Let

$$P'_m = \Pi'_0 P^{a_m} \Pi'_0, \qquad \text{with } \Pi'_0 = (1 - \Pi_0).$$

(3.9) can be easily derived from (3.4) or (4.8) of [34] (with P replaced by P^{a_m} and P^a by P^b) according to the cases whether or not E_0 is eigenvalue of P'_m. In the case where E_0 is not an eigenvalue of P'_m, (3.4) of [34] says that for z near E_0,

$$(P^{a_m} - z)^{-1} = \epsilon(z) - \epsilon_+(z)\epsilon_{+-}(z)^{-1}\epsilon_-(z), \tag{3.11}$$

for z near E_0, where

$$\begin{aligned}
\epsilon(z) &= R'_m(z) \equiv (P'_m - z)^{-1}\Pi'_0, && (3.12)\\
\epsilon_+(z) &= R'_m(z)W_m(\varphi_0 \otimes \cdot) + (\varphi_0 \otimes \cdot), && (3.13)\\
\epsilon_-(z) &= (\cdot, \varphi_0)_b + (W_m R'_m(z)\cdot, \varphi_0)_b, && (3.14)\\
\epsilon_{+-}(z) &= (z - E_0) - (-\triangle_{x_b^{a_m}} - (W_m\varphi_0, \varphi_0)_b && (3.15)\\
&\quad -(W_m R'_m(z) W_m(\varphi_0 \otimes \cdot), \varphi_0)_b).
\end{aligned}$$

$z \to R'_m(z) \in \mathcal{L}(L^2(\mathbf{X}^{a_m}))$ is holomorphic near $z = E_0$. Set

$$r_m^{(1)}(\lambda) = R'_m(z) + (P^{a_m} - z)^{-1}(1 - \chi(|\xi_{a_m}|^2)^2). \tag{3.16}$$

$r_m^{(1)}(\lambda)$ is continuous for λ near E_0 and, since $P^{a_m} \geq E_0$,

$$r_m^{(1)}(\lambda) \geq 0, \qquad \lambda \leq E_0.$$

Therefore,

$$W_m^{\frac{1}{2}}(P^{a_m} - z)^{-1}W_m^{\frac{1}{2}} = -W_m^{\frac{1}{2}}\epsilon_+(z)\epsilon_{+-}(z)^{-1}\epsilon_-(z)W_m^{\frac{1}{2}} + O(1), \tag{3.17}$$

in $\mathcal{L}(L^2)$ and for $|z-E_0|$ small enough,

$$\begin{aligned} & -W_m^{\frac{1}{2}}\epsilon_+(z)\epsilon_{+-}(z)^{-1}\epsilon_-(z)W_m^{\frac{1}{2}} \\ = \;& W_m^{\frac{1}{2}}(z-E_0)^{-\frac{1}{2}}(i(W_m^{\frac{1}{2}}\cdot,u_m)_{a_m}\otimes u_m)+O(|z-E_0|^{-\frac{1}{2}+\epsilon}), \end{aligned} \tag{3.18}$$

in $\mathcal{L}(L^2)$ for any $0<\epsilon<\frac{\rho-2}{2}$. See Theorem 3.2. By Theorem 3.1,

$$W_m^{\frac{1}{2}}u_m \in H^{1,s}(\mathbf{X}^{a_m}), \qquad \forall\, 0\le s<\frac{\rho-1}{2}.$$

Let $r_m^{(2)}(\lambda)$ denote the remainder in (3.18). To obtain the improved remainder estimate $r_m^{(2)}(\lambda)$, we remark that since φ_0 is exponentially decreasing in $|x^b|$,

$$W_m^{\frac{1}{2}}\epsilon_+(z): \quad L^2(\mathbf{X}_b^{a_m};< x^{a_m} >^{s-\frac{\rho}{2}}\, dx_b^{a_m}) \to L^2(\mathbf{X}^{a_m};\, < x^{a_m} >^{s}\, dx^{a_m})$$

is continuous. This means that we gain a decrease of $< x^{a_m} >^{-\frac{\rho}{2}}$. Similar property holds for $\epsilon_-(z)W_m^{\frac{1}{2}}$. Recall that $\epsilon_{+-}(z)^{-1}$ is studied in [34] through another Grushin problem and that the its most singular part is calculated on the one-dimensional space spanned by the resonant state u_m. Since by Theorem 3.1

$$u_m \in H^{1,-s}(\mathbf{X}^{a_m}), \quad \forall s>\frac{1}{2},$$

we can prove that

$$r_m^{(2)}(\lambda): H^{k,-s}(\mathbf{X}^{a_m}) \to H^{k,s}(\mathbf{X}^{a_m})$$

is continuous for $k=-1,0,1$ and for any $0\le s<\frac{\rho}{2}-\frac{1}{2}$. Since a_m is a two-cluster decomposition associated with b and due to the exponential decay of φ_0, we obtain the desired decay in x_{a_j} variables for $j\neq m$.

In the case $E_0\in\sigma_d(P'_m)$, we can use (4.8) of [34] to prove the same result. Remark that in this case, the Grushin problems studied are matrices of higher order related to the eigenfunctions of P'_m with eigenvalue E_0. These eigenfunctions are exponentially decreasing on $\mathbf{X}$. ■

Remark that if E_0 is not a resonance of P^{a_m}, part (a) of Theorem 3.3 implies that it is a regular point and the proof of (b) of Theorem 3.3 shows that $W_m^{\frac{1}{2}}R_m(\lambda)W_m^{\frac{1}{2}}$ is bounded and continuous in $\lambda\le E_0$. From (3.10), we obtain the following

Corollary 3.4 *Assume that E_0 is a resonance of P^{a_m}. Let $\phi_m=W_m^{\frac{1}{2}}u_m$. Then,*

$$\begin{aligned} & (1+W_m^{\frac{1}{2}}R_m(\lambda)W_m^{\frac{1}{2}})^{\frac{1}{2}} \\ = \;& \frac{1}{\|\phi_m\|}\chi(-\Delta_{x_{a_m}})(-\Delta_{x_{a_m}}+\nu)^{-\frac{1}{4}}[(\cdot,\phi_m)_{a_m}\otimes\phi_m]+Q_m(\lambda), \end{aligned} \tag{3.19}$$

in $\mathcal{L}(L^2(\mathbf{X}))$ for λ near E_0, $\nu=E_0-\lambda$. Here

$$Q_m(\lambda)=Q_m^{(1)}(\lambda)+Q_m^{(2)}(\lambda)$$

with $Q_m^{(1)}(\lambda)$ uniformly bounded for $\lambda \leq E_0$, and $Q_m^{(2)}(\lambda)$ bounded for all $\lambda < E_0$ and satisfying that

$$\nu \to < x_{a_j} >^s (-\Delta_{x_{a_m}} + \nu)^{\frac{1}{2}-\epsilon} Q_m^{(2)}(\lambda) < x_{a_k} >^s \in \mathcal{L}(L^2)$$

is continuous in $\nu = E_0 - \lambda \leq 0$, for $j, k \neq m$, $0 \leq s < \frac{\rho-1}{2}$, $0 < \epsilon < \frac{\rho-2}{2}$.

Proof. Remark that by Theorem 3.1, $\phi_m \in L^{2,s}$, $s < \frac{\rho-1}{2}$ and is non zero. Since

$$\begin{aligned} A &= 1 + W_m^{\frac{1}{2}} R_m(\lambda) W_m^{\frac{1}{2}} \\ B &= \chi(-\Delta_{x_{a_m}})^2 (-\Delta_{x_{a_m}} + \nu)^{-\frac{1}{2}} [(\cdot, \phi_m)_{a_m} \otimes \phi_m] \end{aligned}$$

are positive self-adjoint operators for $\lambda < E_0$, their square roots are well defined. One can calculate that

$$((\cdot, \phi_m)_{a_m} \otimes \phi_m)^{\frac{1}{2}} = \frac{1}{\|\phi_m\|} (\cdot, \phi_m)_{a_m} \otimes \phi_m.$$

Set

$$Q_m^{(1)}(\lambda) = (B + 1 + r_m^{(1)}(\lambda))^{\frac{1}{2}} - B^{\frac{1}{2}}, \quad Q_m^{(2)}(\lambda) = A^{\frac{1}{2}} - (B + 1 + r_m^{(1)}(\lambda))^{\frac{1}{2}}.$$

Since $r_m^{(1)}(\lambda) \geq 0$, one has $B + 1 + r_m^{(1)}(\lambda) \geq 1$. Its square root is well defined. It is well-known ([8]) that for two positive operators C, D in a Hilbert space, one has the inequality

$$\|C^{\frac{1}{2}} - D^{\frac{1}{2}}\| \leq \|C - D\|^{\frac{1}{2}}$$

The estimate on $Q_m^{(1)}(\lambda)$ is evident. To estimate $Q_m^{(2)}(\lambda)$, we work with $Q_m^{(2)}(\xi_{a_m}, \lambda)$ the Fourier transform in x_{a_m}-variables of $Q_m^{(2)}(\lambda)$ and in weighted L^2-spaces on $\mathbf{X}^{a_m}$. Applying (b) of Theorem 3.3, one sees that in this setting,

$$\|Q_m^{(2)}(\xi_{a_m}, \lambda)\| \leq \|r_m^{(2)}(\lambda)\|^{\frac{1}{2}} \leq C(\nu + |\xi_{a_m}|^2)^{-\frac{1}{4}+\frac{\epsilon}{2}}.$$

Corollary 3.4 follows, because $Q_m(\lambda) = Q_m^{(1)}(\lambda) + Q_m^{(2)}(\lambda)$. ■

4 Lower bound on the number of eigenvalues

For $\lambda < E_0$, let $\nu = E_0 - \lambda$. Set

$$F(\lambda) = \begin{pmatrix} 0 & F_{12}(\lambda) & F_{13}(\lambda) \\ F_{21}(\lambda) & 0 & F_{23}(\lambda) \\ F_{31}(\lambda) & F_{32}(\lambda) & 0 \end{pmatrix} : (L^2(\mathbf{X}))^3 \to (L^2(\mathbf{X}))^3,$$

with

$$F_{ij}(\lambda) = (1 + W_i^{\frac{1}{2}} R_i(\lambda) W_i^{\frac{1}{2}})^{\frac{1}{2}} W_i^{\frac{1}{2}} \Pi_0 (-\Delta_b + \nu)^{-1} W_j^{\frac{1}{2}} (1 + W_j^{\frac{1}{2}} R_j(\lambda) W_j^{\frac{1}{2}})^{\frac{1}{2}},$$

for $i \neq j$. Recall that Π_0 is the orthogonal projection in $L^2(\mathbf{X})$ induced by φ_0: $\Pi_0 = (\cdot, \varphi_0)_b \otimes \varphi_0$. $F(\lambda)$ is compact.

If E_0 is a resonance of P^{a_j}, we set

$$G_j(\nu) = \frac{1}{\|\phi_j\|} \chi(-\Delta_{x_{a_j}})(-\Delta_{x_{a_j}} + \nu)^{-\frac{1}{4}}((\cdot, \phi_j)_{a_j} \otimes \phi_j). \tag{4.1}$$

If E_0 is not a resonance of P^{a_j}, we just set $G_j(\nu) = 0$ and $Q_j(\lambda) = (1 + W_j^{\frac{1}{2}} R_j(\lambda) W_j^{\frac{1}{2}})^{\frac{1}{2}}$. Then, we have in the both cases

$$(1 + W_j^{\frac{1}{2}} R_j(\lambda) W_j^{\frac{1}{2}})^{\frac{1}{2}} = G_j(\nu) + Q_j(E_0 - \nu),$$

where Q_j verifies the remainder estimate of Corollary 3.4. See the remark after the proof of Theorem 3.3. In the following, we shall only work with indices i and j for which both P^{a_i} and P^{a_j} have a resonance at E_0. If E_0 is not a resonance of one of these two Sub-hamiltonians, the leading term we want to study vanishes.

Let $G(\nu)$ and $Q(\nu)$ denote the diagonal matrices with the entries $G_j(\nu)$ and $Q_j(E_0 - \nu)$, $j = 1, 2, 3$, respectively. Let $H(\nu)$ be the matrix with the entries $H_{ij}(\nu)$, $i, j = 1, 2, 3$, where

$$H_{ij}(\nu) = 0, \text{ if } i = j; \quad H_{ij}(\nu) = W_i^{\frac{1}{2}} \Pi_0 (-\Delta_b + \nu)^{-1} W_j^{\frac{1}{2}}, \text{ if } i \neq j. \tag{4.2}$$

Write

$$F(\lambda) = G(\nu)H(\nu)G(\nu) + \{G(\nu)H(\nu)Q(\nu) + Q(\nu)H(\nu)G(\nu) + Q(\nu)H(\nu)Q(\nu)\}. \tag{4.3}$$

We shall first establish the asymptotics of $n(1, G(\nu)H(\nu)G(\nu))$ as $\nu \to 0_+$. Then, Theorem 1.1 will be deduced by a perturbation argument based on the following lemma.

Lemma 4.1 *Let $T(\nu) = T_0(\nu) + T_1(\nu)$, where $T_0(\cdot)$ ($T_1(\cdot)$) is a self-adjoint compact operator-valued function in $\nu > 0$ ($\nu \geq 0$). Assume that there is some function $f(\nu)$ with $f(\nu) \to 0$ when $\nu \to 0_+$ such that $\lim_{\nu \to 0_+} f(\nu) n(\mu, T_0(\nu)) = l(\mu)$, with $l(\mu)$ continuous in μ. Then, the same limit exists for $T(\nu)$: $\lim_{\nu \to 0_+} f(\nu) n(\mu, T(\nu)) = l(\mu)$.*

For the proof of Lemma 4.1, see Lemma 4.9 of [28]). The main technical point to apply Lemma 4.1 is the following

Proposition 4.2 *Operator $G(\nu)H(\nu)G(\nu)$ (resp., $G(\nu)H(\nu)Q(\nu) + Q(\nu)H(\nu)G(\nu) + Q(\nu)H(\nu)Q(\nu)$) is compact and continuous in $\nu > 0$ (resp., in $\nu \geq 0$).*

Proof. Since $W_i^{\frac{1}{2}}\Pi_0(-\Delta_b+\nu)^{-1}W_j^{\frac{1}{2}}$ is a compact operator on $L^2(\mathbf{X})$, the result for $\nu>0$ is clear. It suffices to show that as bounded operator-valued function,

$$G(\nu)H(\nu)Q(\nu)+Q(\nu)H(\nu)G(\nu)+Q(\nu)H(\nu)Q(\nu)$$

can be continuously extended to $\nu=0$. The limit operator is then necessarily compact. Let us only consider the operator $G(\nu)H(\nu)Q(\nu)$. The other two terms can be treated in the same way. The (i,j)-entry of this operator is $G_i(\nu)H_{ij}(\nu)Q_j(E_0-\nu)$ which can be written as

$$\begin{aligned}&\Pi_i W_i^{\frac{1}{2}}\chi(-\Delta_{x_{a_i}})\Xi_{a_i}(\nu)^{\frac{1}{4}}\Pi_0\Xi_b(\nu)W_j^{\frac{1}{2}}Q_j^{(1)}(E_0-\nu)\\ +\;&\Pi_i[W_i^{\frac{1}{2}}\chi(-\Delta_{x_{a_i}})\Xi_{a_i}(\nu)^{\frac{1}{4}}\Pi_0\Xi_b(\nu)W_j^{\frac{1}{2}}\Xi_{a_j}(\nu)^{\frac{1}{4}-\frac{\epsilon}{2}}]\cdot[\Xi_{a_j}(\nu)^{-\frac{1}{4}+\frac{\epsilon}{2}}Q_j^{(2)}(E_0-\nu)],\end{aligned}$$

where

$$\Pi_i=\frac{1}{\|\phi_i\|}(\cdot,\phi_i)_{a_i}\otimes\phi_i,\quad \Xi_a(\nu)=(-\Delta_{x_a}+\nu)^{-1}.$$

Here we assume that E_0 is a resonance of P^{a_i}. Otherwise, $G_i(\nu)=0$. By Corollary 3.4, $Q_j^{(1)}(E_0-\nu)$ and $<x_{a_k}>^s\Xi_{a_j}(\nu)^{-\frac{1}{4}+\frac{\epsilon}{2}}Q_j^{(2)}(E_0-\nu)$, $k\neq j$, are continuous up to $\nu=0$, for some $\epsilon>0$ and for any $1/2<s<\frac{\rho-1}{2}$.

Remark that by Theorem 3.1, $\Pi_i<x_{a_j}>^s$ is bounded on L^2 for $1/2<s<\frac{\rho-1}{2}$. It suffices to show that for $\epsilon>0$ small,

$$\Pi_i W_i^{\frac{1}{2}}\Pi_0\Xi_b(\nu)\Xi_{a_i}(\nu)^{\frac{1}{4}}W_j^{\frac{1}{2}}\qquad\text{and}$$

$$\Pi_i W_i^{\frac{1}{2}}\Xi_{a_j}(\nu)^{\frac{1}{4}-\frac{\epsilon}{2}}\Pi_0\Xi_b(\nu)\Xi_{a_i}(\nu)^{\frac{1}{4}}W_j^{\frac{1}{2}}<x_{a_i}>^{-\frac{\rho-1}{2}+\epsilon}$$

are bounded and continuous in ν up to $\nu=0$ for $i\neq j$. Remark that the $W_{a_i}^{\frac{1}{2}}\Pi_0=O(<x_{a_j}>^{-\frac{\rho}{2}})$ for $i\neq j$ and that since $\dim\mathbf{X}_{a_j}=3$,

$$\mathbb{R}_+\ni\nu\to<x_{a_j}>^{-s}(-\Delta_{x_{a_j}}+\nu)^{-s'/2}\in\mathcal{L}(L^2(\mathbf{X}_{a_j}))$$

is continuous in $\nu\geq 0$ for $s=s'=0$ and for any $s>3/2,\ s'<3/2$. In fact, in the later case, the operator

$$<x_{a_j}>^{-s}(-\Delta_{x_{a_j}}+\nu)^{-s'/2}\chi(-\Delta_{x_{a_j}})$$

is a Hilbert-Schmidt operator for all $\nu\geq 0$ and

$$<x_{a_j}>^{-s}(-\Delta_{x_{a_j}}+\nu)^{-s'/2}(1-\chi(-\Delta_{x_{a_j}}))$$

is uniformly bounded in $\nu\geq 0$. Here $\chi(\cdot)$ is the cut-off around 0 used in Theorem 3.3. A complex interpolation gives that for $0\leq s'<3/2,\ s'<s$,

$$\mathbb{R}_+\ni\nu\to<x_{a_j}>^{-s}(-\Delta_{x_{a_j}}+\nu)^{-s'/2}\in\mathcal{L}(L^2(\mathbf{X}_{a_j}))\tag{4.4}$$

is uniformly bounded in $\nu \geq 0$. Since $\rho > 2$, we can apply (4.4) to prove that for $0 < \epsilon < \frac{\rho-1}{2}$ small enough,

$$\begin{aligned} &\Pi_i W_i^{\frac{1}{2}} \Pi_0 \Xi_b(\nu) \Xi_{a_i}(\nu)^{\frac{1}{4}} W_j^{\frac{1}{2}} \qquad \text{and} \\ &\Pi_i W_i^{\frac{1}{2}} \Xi_{a_j}(\nu)^{\frac{1}{4}-\frac{\epsilon}{2}} \Pi_0 \Xi_b(\nu) \Xi_{a_i}(\nu)^{\frac{1}{4}} W_j^{\frac{1}{2}} < x_{a_i} >^{-\frac{\rho-1}{2}+\epsilon} \end{aligned} \tag{4.5}$$

are uniformly bounded in $\mathcal{L}(L^2)$ for $\nu \geq 0$. Here we used the estimate

$$\Xi_b(\nu) \leq C \Xi_{a_i}(\nu)^{\tau} \Xi_{a_j}(\nu)^{1-\tau}, \qquad C > 0,$$

for all ξ_b and $\tau \in [0,1]$. It follows that $G_i(\nu) H_{ij}(\nu) Q_j(E_0 - \nu)$ is uniformly bounded for $\nu \geq 0$.

It is now easy to use an argument of approximation to prove that $G_i(\nu) H_{ij}(\nu) Q_j(E_0 - \nu)$ is continuous up to $\nu = 0$ in operator norm. The details are omitted. ■

Let us now recall some results from [28] which are important in our work. Let $\hat{S}(\sigma) : (L^2(\mathbb{S}^2))^3 \to (L^2(\mathbb{S}^2))^3$ be the integral operator with the kernel

$$\hat{S}(\theta, \omega; \sigma) = \begin{pmatrix} 0 & \hat{S}_{12}(t;\sigma) & \hat{S}_{13}(t;\sigma) \\ \hat{S}_{21}(t;\sigma) & 0 & \hat{S}_{23}(t;\sigma) \\ \hat{S}_{31}(t;\sigma) & \hat{S}_{32}(t;\sigma) & 0 \end{pmatrix}, \quad t = \theta \cdot \omega, \quad \theta, \omega \in \mathbb{S}^2, \tag{4.6}$$

with

$$\hat{S}_{ij}(t;\sigma) = (2\pi)^{-1} \delta_i \delta_j u_{ij} e^{i\tau_{ij}\sigma} \frac{\sinh(\sigma \arccos\ s_{ij} t)}{(1 - s_{ij}^2 t^2)^{\frac{1}{2}} \sinh(\pi\sigma)}, \quad i \neq j.$$

Here $\delta_i = 1$ if E_0 is a resonance of P^{a_i}; 0, otherwise. The constants $u_{ij}, s_{ij}, \tau_{ij}$ depend only on the ratio of reduced masses in the clusters b_1, b_2, b_3. In particular, $u_{ij} > 0$ and $|s_{ij}| < 1$. See (4.15) and (4.22). For $\mu > 0$, define

$$\mathcal{U}(\mu) = \frac{1}{4\pi} \int_{\mathbb{R}} n(\mu, \hat{S}(\sigma))\, d\sigma. \tag{4.7}$$

This function was studied in detail in [28] and is very important for the proof of the existence of the Efimov effect. In particular, it is proved in [28] that $\mathcal{U}(\cdot)$ is continuous and that if at least two of the three δ_j's are non zero,

$$\mathcal{U}(1) > 0 \tag{4.8}$$

Theorem 4.3 *For $\mu > 0$, one has*

$$\lim_{\nu \to 0_+} |\log \nu|^{-1} n(\mu, G(\nu) H(\nu) G(\nu)) = \mathcal{U}(\mu). \tag{4.9}$$

Proof. We denote

$$\Pi_j = \frac{1}{\|\phi_j\|} (\cdot, \phi_j)_{a_j} \otimes \phi_j, \quad \Gamma_j(\nu) = \chi(-\Delta_{x_{a_j}})(-\Delta_{x_{a_j}} + \nu)^{-\frac{1}{4}}.$$

Then, $\Pi_j^2 = (\cdot, \phi_j)_{a_j} \otimes \phi_j$ and $G_j(\nu) = \Gamma_j(\nu)\Pi_j(\nu)$. Define

$$\mathcal{E}_{-,j} = (\cdot, \phi_j)_{a_j} : L^2(\mathbf{X}) \to L^2(\mathbf{X}_{a_j}), \text{ and}$$

$$\mathcal{E}_{+,j} = \phi_j \otimes \cdot : L^2(\mathbf{X}_{a_j}) \to L^2(\mathbf{X}),$$

so that $\Pi_j^2 = \mathcal{E}_{+,j}\mathcal{E}_{-,j}$. Let $\mathcal{F}_j : L^2(\mathbf{X}_{a_j}) \to L^2(\mathbf{X}^*_{a_j})$ be the three-dimensional unitary Fourier transform defined by

$$\mathcal{F}_j u(\xi_{a_j}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{X}_{a_j}} e^{-ix_{a_j}\cdot\xi_{a_j}} u(x_{a_j})\, dx_{a_j}.$$

Let $\gamma_j(\nu)$ denote the multiplication by the function $\chi(|\xi_{a_j}|^2)(|\xi_{a_j}|^2+\nu)^{-\frac{1}{4}}$. Let Π (resp., $\Gamma(\nu)$, $\gamma(\nu)$, $\mathcal{E}_\pm$, $\mathcal{F}$) denote the diagonal matrices with the entries Π_j (resp., $\Gamma_j(\nu)$, $\gamma_j(\nu)$, $\mathcal{E}_{\pm,j}$, $\mathcal{F}_j$, $j = 1,2,3$). By an elementary argument, one can see that the non-zero eigenvalues of $G(\nu)H(\nu)G(\nu) = \Pi\Gamma(\nu)H(\nu)\Gamma(\nu)\Pi$ are the same as those of $\Gamma(\nu)H(\nu)\Gamma(\nu)\Pi^2$ with the same multiplicities. The later has the same non-zero eigenvalues as those of the operator

$$\mathcal{E}_-\Gamma(\nu)H(\nu)\Gamma(\nu)\mathcal{E}_+ \in \mathcal{L}(L^2(\mathbf{X}_{a_1}) \times L^2(\mathbf{X}_{a_2}) \times L^2(\mathbf{X}_{a_3})).$$

So, we obtain for $\mu > 0$,

$$n(\mu, G(\nu)H(\nu)G(\nu)) = n(\mu, \mathcal{E}_-\Gamma(\nu)H(\nu)\Gamma(\nu)\mathcal{E}_+). \tag{4.10}$$

Applying Fourier transform $\mathcal{F}$, it follows that

$$n(\mu, G(\nu)H(\nu)G(\nu)) = n(\mu, \gamma(\nu)\mathcal{F}\mathcal{E}_-H(\nu)\mathcal{E}_+\mathcal{F}^*\gamma(\nu)). \tag{4.11}$$

Let

$$\psi_j(x^{a_j}) = W_j(x^{a_j})u_j(x^{a_j})\varphi_0(x^b), \quad j = 1,2,3. \tag{4.12}$$

By Theorem 3.1 and the assumption (1.2), one has

$$|x^{a_j}|^\delta \psi_j \in L^1(\mathbf{X}^{a_j}), \text{ for any } 0 < \delta < \rho - 2.$$

The normalization of resonant state gives

$$(\psi_j, 1)_{a_j} = \int_{\mathbf{X}^{a_j}} \psi_j(x^{a_j})dx^{a_j} = -2\sqrt{\pi}. \tag{4.13}$$

For $i \neq j$, the (i,j) entry of $\mathcal{F}\mathcal{E}_-H(\nu)\mathcal{E}_+\mathcal{F}^*$ is

$$\mathcal{F}_i((-\Delta_b+\nu)^{-1}(\mathcal{F}_j^*\cdot, \psi_j)_b, \psi_i)_{a_i} : L^2(\mathbf{X}^*_{a_j}) \to L^2(\mathbf{X}^*_{a_i}).$$

We have the decompositions $x^{a_j} = x^b + x_b^{a_j}$, $\xi_b = \xi_{a_j} + \xi_b^{a_j}$ for $x^{a_j} \in \mathbf{X}^{a_j}$, $\xi_b \in \mathbf{X}_b^*$. Let $\mathcal{F}^i$ denote the partial Fourier transform in $x_b^{a_i}$ variables:

$$\mathcal{F}^i u(\xi_b^{a_i}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{X}_b^{a_i}} e^{-ix_b^{a_i}\cdot\xi_b^{a_i}} u(x_b^{a_i})dx_b^{a_i}.$$

By the Parseval equality in $x_b^{a_j}$ variables,

$$\begin{aligned}\mathcal{F}_i((-\Delta_b+\nu)^{-1}(\mathcal{F}_j^*\cdot,\psi_j)_b,\psi_i)_{a_i} &= ((|\xi_b|^2+\nu)^{-1}\mathcal{F}_i[\mathcal{F}^i(\mathcal{F}_j^*\cdot,\psi_j)_b],\mathcal{F}^i\psi_i)_{L^2(\mathbf{X}^b\times\mathbf{X}_b^{a_i*})}\\ &= ((|\xi_b|^2+\nu)^{-1}\hat{\psi}_j,\hat{\psi}_i)_{L^2(\mathbf{X}_b^{a_i*})}.\end{aligned}$$

Here

$$\hat{\psi}_j(\xi_b^{a_j}) = (2\pi)^{-3/2}\int_{\mathbf{X}^{a_j}} e^{-i\xi_b^{a_j}\cdot x_b^{a_j}}\,\psi_j(x^{a_j})\,dx^{a_j}.$$

For $i\neq j$, the (i,j)-entry of the operator $\gamma(\nu)\mathcal{F}\mathcal{E}_-H(\nu)\mathcal{E}_+\mathcal{F}^*\gamma(\nu)$ is given by

$$\int_{\mathbf{X}_b^{a_i*}}\gamma_i(\xi_i,\nu)(|\xi_b|^2+\nu)^{-1}\hat{\psi}_j(\xi_b^{a_j})\overline{\hat{\psi}_i(\xi_b^{a_i})}\gamma_j(\xi_j,\nu)u(\xi_j)\,d\xi_b^{a_i}, \tag{4.14}$$

for $u\in L^2(\mathbf{X}_{a_j}^*)$. To determine the kernel of this operator, we use (ξ_{a_i},ξ_{a_j}), $i\neq j$, as coordinates of $\mathbf{X}_b^*$. This is possible because $\mathbf{X}_b^*=\mathbf{X}_{a_i}^*\oplus\mathbf{X}_{a_j}^*$, $i\neq j$. One can calculate:

$$\begin{aligned}\xi_b^{a_j} &= c_{ij}^j\xi_{a_i}+b_{ij}^j\xi_{a_j}\\ \xi_b^{a_i} &= c_{ij}^i\xi_{a_i}+b_{ij}^i\xi_{a_j}\\ |\xi_b|^2 &= d_i|\xi_{a_i}|^2+d_{ij}\xi_{a_i}\cdot\xi_{a_j}+d_j|\xi_{a_j}|^2\end{aligned} \tag{4.15}$$

Here $c_{ij}^i, b_{ij}^i, c_{ij}^j, b_{ij}^j, d_i>0, d_j>0$ and d_{ij} are non-zero constants depending only on the reduced masses of the clusters b_1,b_2,b_3. These constants can be explicitly calculated by making use of, for example, the clustered Jacobi coordinates related to the cluster decomposition b ([26]). Since $|\xi_b|^2$ is positive definite on $\mathbf{X}_b^*$, we have necessarily

$$|d_{ij}|^2<4d_id_j. \tag{4.16}$$

Making the change of variables $\xi_b^{a_i}\to\xi_{a_j}$ in the integral in $\xi_b^{a_i}$, we see that the kernel of the (i,j)-entry of the operator $\gamma(\nu)\mathcal{F}\mathcal{E}_-H(\nu)\mathcal{E}_+\mathcal{F}^*\gamma(\nu)$ is

$$k_{ij}=m_{ij}\gamma_i(\xi_{a_i},\nu)(|\xi_b|^2+\nu)^{-1}\hat{\psi}_j(\xi_b^{a_j})\overline{\hat{\psi}_i(\xi_b^{a_i})}\gamma_j(\xi_{a_j},\nu). \tag{4.17}$$

Here m_{ij} is a positive constant depending only on the masses, and $\xi_b,\xi_b^{a_i},\xi_b^{a_j}$ are expressed in terms of (ξ_{a_i},ξ_{a_j}) by (4.15). Due to the cut-off $\chi(|\xi_{a_l}|^2)$ in $\gamma_l(\xi_{a_l},\nu)$ for $l=i,j$, the integral operator defined by k_{ij} can be naturally identified with an operator from $L^2(B(1,a_j))$ to $L^2(B(1,a_j))$. Here

$$B(r,a_j)=\{\xi_{a_j}\in\mathbf{X}_{a_j}^*;|\xi_{a_j}|<r\}, r>0.$$

Note that $\hat{\psi}_j(0)=-2^{-\frac{1}{2}}\pi^{-1}$ and

$$\hat{\psi}_j(\xi_b^{a_j})=\hat{\psi}_j(0)+\vartheta(\xi_b^{a_j}), \tag{4.18}$$

where $\vartheta(\xi_b^{a_j})$ is a bounded function behaving like $O(|\xi_b^{a_j}|^\delta)$ as $\xi_b^{a_j} \to 0$. We now remove the cut-off χ and replace $\hat{\psi}_j(\xi_b^{a_j})$ by $\hat{\psi}_j(0)$. Let ς be the characteristic function of $]1, \infty[$. Let

$$K^0(\nu) = \begin{pmatrix} 0 & K_{12}^0(\nu) & K_{13}^0(\nu) \\ K_{21}^0(\nu) & 0 & K_{23}^0(\nu) \\ K_{31}^0(\nu) & K_{32}^0(\nu) & 0 \end{pmatrix}$$

be the operator with the entries

$$K_{ij}^0(\nu) : L^2(B(1, a_j)) \to L^2(B(1, a_i))$$

defined by the integral kernel $k_{ij}^0(\nu)$

$$k_{ij}^0(\nu) = \frac{1}{2\pi^2}\delta_i\delta_j m_{ij}\varsigma(\nu^{-\frac{1}{2}}|\xi_{a_i}|)(|\xi_{a_i}|^2+\nu)^{-\frac{1}{4}}(|\xi_b|^2+\nu)^{-1}(|\xi_{a_j}|^2+\nu)^{-\frac{1}{4}}\varsigma(\nu^{-\frac{1}{2}}|\xi_{a_j}|), \tag{4.19}$$

where $\delta_l = 1$, if E_0 is a resonance of P^{a_l}; 0, otherwise. In (4.19), ξ_b is expressed in terms of the variables ξ_{a_i} and ξ_{a_j}, $i \neq j$. As in Proposition 4.2, it can be shown that $k_{ij} - k_{ij}^0(\nu)$ defines a compact operator continuous in $\nu \geq 0$. So it does not contribute to the leading term of the asymptotics.

The rest of the proof is the same as Theorem 4.7 of [28]. By the dilation $y \to \nu^{\frac{1}{2}}y$, one sees that the operator

$$K^0(\nu) \in \mathcal{L}(L^2(B(1,a_1)) \times L^2(B(1,a_2)) \times L^2(B(1,a_3)))$$

is unitarily equivalent with the operator

$$K^0(1) \in \mathcal{L}(L^2(B(\nu^{-\frac{1}{2}},a_1)) \times L^2(B(\nu^{-\frac{1}{2}},a_2)) \times L^2(B(\nu^{-\frac{1}{2}},a_3))).$$

The integral kernel $k_{ij}^0(1)$ of the (i,j)-th entry of $K^0(1)$ is given by (4.19) with $\nu = 1$:

$$k_{ij}^0(1) = \frac{1}{2\pi^2}\delta_i\delta_j m_{ij}\varsigma(|\xi_{a_i}|)(|\xi_{a_i}|^2+1)^{-\frac{1}{4}}(|\xi_b|^2+1)^{-1}(|\xi_{a_j}|^2+1)^{-\frac{1}{4}}\varsigma(|\xi_{a_j}|).$$

Let $T_0 \in \mathcal{L}(L^2(B(\nu^{-\frac{1}{2}},a_1)) \times L^2(B(\nu^{-\frac{1}{2}},a_2)) \times L^2(B(\nu^{-\frac{1}{2}},a_3)))$ be the integral operator whose integral kernel of the (i,j)-th entry is defined by $t_{ij} = 0$ if $i = j$ and

$$t_{ij} = \frac{1}{2\pi^2}\delta_i\delta_j m_{ij}\varsigma(|\xi_{a_i}|)|\xi_{a_i}|^{-\frac{1}{2}}|\xi_b|^{-2}|\xi_{a_j}|^{-\frac{1}{2}}\varsigma(|\xi_{a_j}|),$$

if $i \neq j$. Replacing $K^0(1)$ by T_0 again gives rise to a remainder which corresponds a compact operator continuous up to $\nu \geq 0$. Let $\mathbb{S}_j$ denote the unit sphere in $\mathbf{X}_{a_j}^*$. Making use of the unitary transformation induced by the change of coordinates $\xi_{a_j} = e^{r_j}\omega_j$, $r_j \in \mathbb{R}, \omega_j \in \mathbb{S}_j$, which maps the set $1 < |\xi_{a_j}| < \nu^{-\frac{1}{2}}$ onto the set $]0, |\log\nu|/2[\times\mathbb{S}_j$, one can check that $T_0(\nu)$ is unitarily equivalent with the operator

$$S_R \in \mathcal{L}(L^2(0,R;L^2(\mathbb{S}_1)) \times L^2(0,R;L^2(\mathbb{S}_2)) \times L^2(0,R;L^2(\mathbb{S}_3))) \tag{4.20}$$

where $R = |\log\nu|/2$ and the entries of S_R are given as follows: $S_{kk} = 0$ and for $i \neq j$, S_{ij} is the operator of integral kernel

$$S_{ij}(r_i - r_j; \omega_i \cdot \omega_j) = \frac{\delta_i \delta_j m_{ij}}{2\pi^2 (d_i e^{(r_i - r_j)} + d_j e^{(r_j - r_i)} + d_{ij} \omega_i \cdot \omega_j)}. \tag{4.21}$$

See the equation (4.15). Set

$$u_{ij} = \frac{m_{ij}}{(d_i d_j)^{\frac{1}{2}}}, \quad s_{ij} = \frac{d_{ij}}{2(d_i d_j)^{\frac{1}{2}}}.$$

Then, $|s_{ij}| < 1$ by (4.16) and

$$S_{ij}(r_i - r_j; \omega_i \cdot \omega_j) = \frac{\delta_i \delta_j u_{ij}}{4\pi^2 (\cosh((r_i - r_j) + \tau_{ij}) + s_{ij} \omega_i \cdot \omega_j)}, \tag{4.22}$$

where

$$\tau_{ij} = \frac{1}{2} \log(d_i / d_j).$$

S_R is a Toeplitz operator of the form $\Pi_R S \Pi_R$ where

$$S \in \mathcal{L}(L^2(\mathbb{R}_+; L^2(\mathbb{S}_1)) \times L^2(\mathbb{R}_+; L^2(\mathbb{S}_2)) \times L^2(\mathbb{R}_+; L^2(\mathbb{S}_3)))$$

is a convolution operator in radial variables with the kernel given by (4.22) and Π_R is the multiplication by the characteristic function of $]0, R[$. This operator is already studied in [28]. Applying Theorem 4.5 of [28], we conclude that

$$\lim_{R \to \infty} R^{-1} n(\mu, S_R) = 2\mathcal{U}(\mu), \tag{4.23}$$

where $\mathcal{U}(\mu)$ is defined by (4.7). Note that $\hat{S}_{ij}(t, \sigma)$ is related to $S_{ij}(r, t)$ by the Fourier transform

$$\hat{S}_{ij}(t, \sigma) = \int_{\mathbb{R}} e^{-i\sigma r} S_{ij}(r, t)\, dr. \tag{4.24}$$

Since $R = \frac{1}{2}|\log\nu|$, we obtain

$$\lim_{\nu \to 0_+} |\log\nu|^{-1} n(\mu, G(\nu) H(\nu) G(\nu)) = \mathcal{U}(\mu).$$

Theorem 4.3 is proved. ■

Proof of Theorem 1.1. It follows from Lemma 4.1, Proposition 4.2 and Theorem 4.3 that

$$\lim_{\nu \to 0_+} |\log\nu|^{-1} n(1, F(E_0 - \nu)) = \mathcal{U}(1) \neq 0.$$

Since $\Pi_0' R_b(\lambda) \Pi_0' \geq 0$ for $\lambda < E_0$, where $\Pi_0' = 1 - \Pi_0$, one has $A(\lambda) \geq F(\lambda)$. Therefore,

$$n(1, A(\lambda)) \geq n(1, F(\lambda)). \tag{4.25}$$

Theorem 1.1 follows Proposition 2.1. ■

Remark. *(a). We do not know if the lower bound on $N(\lambda)$ obtained from $A(\lambda)$ is optimal. To prove that $N(\lambda) \sim n(1, A(\lambda))$, it is sufficient to prove that the number of positive eigenvalues of the operator $A(\lambda) - F(\lambda)$ is bounded by $o(|\log(E_0 - \lambda)|)$ as $\lambda \to E_0$. $A(\lambda) - F(\lambda)$ is not a compact operator. The study of its eigenvalue distribution requires other methods than those used in this work.*

(b). For three-body problems, it is known that if only one of two-body Subhamiltonians has a zero energy resonance, the discrete spectrum of the three-body operator is finite. We can not conclude from this work if the same property is true for N-body problems with $N \geq 4$.

5 Coupling constant limit

The Efimov effect is a critical phenomenon: threshold energy resonances are unstable under perturbation. Therefore, it is natural to study how eigenvalues vary when the potentials are small perturbation of those for which the Efimov effect appear. In this Section, we give a result in coupling constant limit.

Consider a family of N-body Schrödinger operators

$$P(\mu) = P_b + \mu I_b$$

where $P = P(1)$ satisfies the assumptions of Theorem 1.1. We want to give an estimate on the number of discrete eigenvalues of $P(\mu)$ when $\mu \to 1_+$. Let $P^a(\mu)$ denote the Subhamiltonian of $P(\mu)$ associated with $a \in \mathcal{A}$. Set

$$\Sigma_k(\mu) = \min_{a \in \mathcal{A}, \# a = k} \inf \sigma(P^a(\mu)), \quad k = 2, 3.$$

Lemma 5.1 *Assume (1.2), (1.4), (1.5) for P. Assume in addition that $V_a \in C_0^\infty(\mathbf{X}^a)$ and $V_a \neq 0$ for all $a \not\subseteq b$. For $\mu > 1$ sufficiently close to 1, one has*

$$\Sigma_3(\mu) = E_0 \tag{5.1}$$

and

$$\Sigma_2(\mu) = E_0 + C(\mu - 1)^2(1 + o(1)), \quad C < 0. \tag{5.2}$$

In particular, the discrete spectrum of $P(\mu)$ is finite for $\mu > 1$ sufficiently close to 1.

Proof. By the assumption (1.4),

$$\inf \sigma(P^a) > E_0, \text{ if } \# a = 3, a \neq b, \quad \inf \sigma(P^b) = E_0.$$

By an argument of perturbation, we can show that if $\# a = 3, a \neq b$,

$$\inf \sigma(P^a(\mu)) = \inf \sigma(P^a) + O(\mu - 1) > E_0$$

when μ is sufficiently close to 1. If $a = b$, $P^b(\mu) = P^b$ is independent of μ. So, $\inf \sigma(P^b(\mu)) = E_0$. (5.1) follows.

To prove (5.2), we remark that by the above argument, for $\#a = 2, a \neq a_1, a_2, a_3$, one has $\inf \sigma(P^a(\mu)) = \inf \sigma(P^a) + O(\mu - 1) > E_0$. When $a = a_j$, $j = 1, 2, 3$, we want to apply to $P^{a_j}(\mu)$ the result of [19] on coupling constant thresholds which is proved under the assumption that $V_a \leq 0, V_a \in C_0^\infty$, $\forall a \in \mathcal{A}$. From their proof, it is clear that only the inter-cluster interactions are relevent. Remark that

$$P^{a_j}(\mu) = P^b - \Delta_{x_b^{a_j}} + \mu I_b^{a_j}$$

with $I_b^{a_j} = \sum_{a \subseteq a_j, a \not\subseteq b} V_a \leq 0$ and that for the many-body operator P^{a_j}, P^b is the unique Subhamiltion (two-cluster relative to a_j) whose spectrum attains the bottom of the essential spectrum of P^{a_j}. By Theorem 3.1 of [19], for $\mu > 1$ with $\mu - 1$ small enough, $P^{a_j}(\mu)$ admits at least one eigenvalue with an asymptotic expansion in powers of $\mu - 1$ or of $(\mu - 1)^{\frac{1}{2}}$. In particular, for the lowest eigenvalue $E_j(\mu)$ of $P^{a_j}(\mu)$, one has

$$E_j(\mu) = E_0 + c_j(\mu - 1)^2 + O((\mu - 1)^3), c_j < 0, \text{ when } \mu \to 1_+.$$

It follows that

$$\Sigma_2(\mu) = \min_{j=1,2,3} E_j(\mu).$$

(5.2) is proved with $C = \min\{c_1, c_2, c_3\}$. Since $\sigma_{ess}(P(\mu)) = \Sigma_2(\mu) < \Sigma_3(\mu)$, the discrete spectrum of $P(\mu)$ is finite ([11]). ■

Theorem 5.2 *Under the conditions of Theorem 1.1 for $P = P(1)$, assume in addition that $V_a \in C_0^\infty$ and $V_a \neq 0$ for all $a \not\subseteq b$. Let $\mathcal{N}(\mu)$ denote the number of discrete eigenvalues of $P(\mu)$ in $]-\infty, \Sigma_2(\mu)[$. Then, one has*

$$\mathcal{N}(\mu) \geq 2C_0 |\log(\mu - 1)| - C_1, \tag{5.3}$$

when $\mu \to 1_+$. Here $C_0 > 0$ is the same as in Theorem 1.1 and C_1 is a constant independent of μ.

Proof. By (1.5), $P(\mu) = P + (\mu - 1)I_b \leq P$ for $\mu > 1$. It follows from the minimax principle that $\mathcal{N}(\mu) \geq N(\Sigma_2(\mu))$. Applying Theorem 1.1 and (5.2), we obtain

$$\mathcal{N}(\mu) \geq N(\Sigma_2(\mu)) \geq C_0 |\log(|C|(\mu - 1)^2(1 + o(1)))|.$$

This proves Theorem 5.2. ■

Note that the coupling constant limit $\mu \to 1$ with $\mu < 1$ for the three-body Efimov effect was considered in [30].

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