SEMICLASSICAL ANALYSIS OF A NONLINEAR EIGENVALUE PROBLEM AND NON ANALYTIC HYPOELLIPTICITY

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Abstract. We give a semiclassical analysis of a nonlinear eigenvalue problem arising from the study of the failure of analytic hypoellipticity and obtain a general family of hypoelliptic, but not analytic hypoelliptic operators.

1. Introduction

We are interested in a family of operators of the type

$$H(x, D_x, \lambda) = -\Delta + (P(x) - \lambda)^2 \,, \tag{1.1}$$

where $x \mapsto P(x)$ is a polynomial of degree m. In the study of the failure of analytic hypoellipticity, one approach consists in showing that the nonlinear eigenvalue problem

$$H(x, D_x, \lambda)v = 0 \tag{1.2}$$

has at least one non trivial solution $(\lambda, v) \in C \times \mathcal{S}(\mathbb{R}^n)$ $(v \neq 0)$. This has been used by many authors including Baouendi-Goulaouic, Helffer, Christ [3, 4, 5, 6], Hanges-Himonas [8], Chanillo [1] and quite recently by Chanillo-Helffer-Laptev [2], where the reader can find a more extensive list of references. All the results lead to the formulation of a conjecture by Trèves [22] giving a necessary and sufficient condition of analytic hypoellipticity extending [13]. It could be a rather natural conjecture that, when $x \mapsto P(x)$ is an homogeneous elliptic polynomial on $\mathbb{R}^n$ of order $m > 1$, (1.2) has at least one non-trivial solution. This result is proved ([3, 16]) for $n = 1$ and for $m \geq 2n$ when $n = 2, 3$ ([2]). Our aim is to provide a semiclassical approach to this problem. Our analysis concerns actually a more general class of operators of the form

$$\sum_{j=1}^{p} D_{x_j}^2 + \big(P(x_1, \dots, x_p)D_{x_{p+1}} - D_{x_{p+2}}\big)^2 + \big(Q(x_1, \dots, x_p)D_{x_{p+1}}\big)^2 \ . \tag{1.3}$$

When specialized to the case (1.2), we obtain the following

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Theorem 1.1. .
Let n be even. Let P be any real polynomial of degree $m \geq 2$ such that the homogeneous part of degree m is elliptic. Then the nonlinear eigenvalue problem (1.2) has at least one non trivial solution.

Its now standard corollary to analytic hypoellipticity (see for example [2]) is :

Corollary 1.2. .
Let P be elliptic homogeneous of degree $m \geq 2$ on $\mathbb{R}^p$, with an even $p > 0$, then the operator on $\mathbb{R}^{p+2}$:

$$P(x, D_x) := \sum_{j=1}^{p} D_{x_j}^2 + \big(P(x_1, \dots, x_p) D_{x_{p+1}} - D_{x_{p+2}}\big)^2 \ ,$$

is not analytic hypoelliptic at 0.

Remark 1.3. .
As we shall discuss more deeply in Section 5, G. Métivier already showed in [12], *that $P(x, D_x)$ is not hypoanalytic in any open set. This is indeed a sum of squares with an odd number of linearly independent vector fields. But the corollary presented here and its proof give a stronger information at* 0.

Other examples in any dimension related to (1.3) are also given in Section 4. These examples give an additional light to the general conjecture of [22].

Recently, Chanillo-Helffer-Laptev ([2]) used Lidskii's Theorem to prove the existence of nonlinear eigenvalues of (1.2) permitting to recover some known results for $n = 1$ and to give new examples in dimension $n \geq 2$.

The proof of Theorem 1.1 is based on the semiclassical analysis combined with the ideas of Chanillo-Helffer-Laptev ([2]). We follow closely the reduction of Chanillo-Helffer-Laptev which is recalled in Section 2. In this approach via Lidskii's theorem, the existence of a nonlinear eigenvalue of (1.2) is reduced to prove, for some k, the non vanishing of the trace of the k-th power of a linear operator $\mathcal{D}$:

$$\mathrm{Tr}\, \mathcal{D}^k \neq 0.$$

Our approach is to introduce artificially a semiclassical parameter h in this operator (see Section 3). Then, the existence of a nonlinear eigenvalue of (1.2) is reduced to prove that for some k and for some h,

$$\mathrm{Tr}\, \mathcal{D}(h)^k \neq 0,$$

where $\mathcal{D}(h)$ is an h-pseudo-differential operator. In other words, one can say that, while Chanillo-Helffer-Laptev try to prove that, for some k,

$$\sum_j \lambda_j^{-k} \neq 0 \ , \tag{1.4}$$

where λ_j are the nonlinear eigenvalues of (1.2) (which can not be real under our assumptions), we want to find here h and k such that

$$\sum_j (h^{m/(m+1)} \lambda_j + 1)^{-k} \neq 0. \tag{1.5}$$

It is standard (see [17]) that $\operatorname{Tr} \mathcal{D}(h)^k$ has a complete semiclassical expansion when $h \to 0$, which is theoretically computable. The question becomes now to find a non-zero term in these expansions which we call the semiclassical criterion. The computation of the leading term given in Section 4 gives the complete answer for n even. Although we do not have a general result for all n odd, we believe that the semiclassical criterion given below can be used to show that for each given value of n odd, (1.2) has at least one non trivial solution. The introduction of a semiclassical parameter h permits us to overcome the difficulty related to the non commutativity of operators encountered in ([2]).

2. Chanillo-Helffer-Laptev's approach

We rewrite $H(x, D_x, \lambda)$ in the form

$$H(x, D_x, \lambda) = L - 2\lambda M + \lambda^2 \,, \tag{2.1}$$

with

$$L = -\Delta + P(x)^2 \,,\ M = P(x) \,. \tag{2.2}$$

The operator L is invertible and its inverse is a pseudo-differential operator (see [17]). It is also easy to give sufficient conditions for determining whether the operator

$$A := L^{-1} \tag{2.3}$$

belongs to a given Schatten class (see the appendix in [2]). Then the initial problem is reduced to the spectral analysis of

$$(I - 2\lambda B + \lambda^2 A)u = 0 \,. \tag{2.4}$$

with

$$B = A^{1/2} P A^{1/2} \tag{2.5}$$

Chanillo-Helffer-Laptev ([2]) used a rather standard approach to transform this nonlinear spectral problem to a linear ones and then apply Lidskii's Theorem to prove the existence of a non trivial solution.

We first recall the reduction to the linear spectral problem. It is easy to see that it is enough to show that the operator $\mathcal{D}$ defined by

$$\mathcal{D} := \begin{pmatrix} 2B & A^{1/2} \\ -A^{1/2} & 0 \end{pmatrix} \tag{2.6}$$

has a non zero eigenvalue μ. The first component of the corresponding eigenvector is an eigenvector of the problem (1.2) with $\mu = \frac{1}{\lambda}$.

If B and A are compact, $\mathcal{D}$ is compact but the main difficulty is that $\mathcal{D}$ is not selfadjoint. The Lidskii Theorem says that

Theorem 2.1. .
Let $\mathcal{C}$ be a trace class operator. Let $\lambda_j(\mathcal{C})$ denote the sequence of non zero eigenvalues of $\mathcal{C}$. Then

$$\sum_j \lambda_j(\mathcal{C}) = \operatorname{Tr} \mathcal{C} \,.$$

Here the eigenvalues are repeated according to their algebraic multipicity.

By Lidskii's Theorem, if $\operatorname{Tr} \mathcal{C} \neq 0$, $\mathcal{C}$ has at least one non zero eigenvalue.

Corollary 2.2. (Chanillo-Helffer-Laptev [2]).
Let $\mathcal{D}$ be a compact operator. Assume that there exists $k \in \mathbf{N}^*$ such that $\mathcal{D}^k$ is of trace class and $\operatorname{Tr} \mathcal{D}^k \neq 0$. Then (1.2) has at least one non trivial solution.

Chanillo-Helffer-Laptev used this criterion in the case $k = 2, 3, 4$ and obtained a family of interesting results in non analytic hypoellipticity. The non commutativity between A and B makes it hard to apply their method with higher rang criteria (although some results can be obtained in the same vein for $k = 6, 8$).

3. The semiclassical criterion

As explained in the introduction, our goal is to introduce a semiclassical parameter h in the operators A and B and to apply the theory of h-pseudo-differential operators to give a complete semiclassical asymptotic expansion of the trace $\mathcal{D}(h)^k$. We are then led to find conditions under which the leading term is non zero.

Initially, the family of operators to study is of the form

$$H(x, D_x, \lambda) = -\Delta + (P(x) - \lambda)^2 \ , \tag{3.1}$$

where $x \mapsto P(x)$ is a real polynomial on $\mathbb{R}^n$ of order $m \geq 2$. Write P in the form

$$P = P_m + P_{m-1} + \cdots + P_0 \ ,$$

where P_j is a homogeneous polynomial of degree j. We assume that P_m is elliptic on $\mathbb{R}^n$:

$$P_m(x) \neq 0, \quad \forall x \neq 0. \tag{3.2}$$

To be definite, we assume $P_m(x) > 0$, $\forall x \neq 0$. The case $P_m(x) < 0$, $\forall x \neq 0$ is similar. This condition on P_m requires m to be even and thus excludes polynomials of odd degree when $n = 1$.

To introduce the semiclassical parameter h, we make the dilation $x \to \tau x$ and set $\lambda = (\lambda' - 1)\tau^m$ (One takes $\lambda = (\lambda' + 1)\tau^m$ if P_m is negative). Let

$$H(x, hD; \lambda', h) = -h^2\Delta + (P(x, h) - \lambda')^2 \ , \tag{3.3}$$

where $h = \frac{1}{\tau^{m+1}}$ and

$$P(x, h) = (P_m(x) + 1) + h^{1/(m+1)} P_{m-1}(x) + h^{2/(m+1)} P_{m-2}(x) + \cdots + h^{m/(m+1)} P_0 \ . \tag{3.4}$$

Then, the initial problem

$$H(x, D; \lambda)v = 0 \tag{3.5}$$

has a non trivial solution (λ, v) if and only if

$$H(x, hD; \lambda', h)u = 0 \tag{3.6}$$

has a non trivial solution (λ', u) for some $h > 0$. Let us remark that

$$-h^2\Delta + (P_m + 1)^2 \geq 1.$$

One can then prove that for $h > 0$ small enough, $-h^2\Delta + P(x,h)^2 > 1/2$ and therefore is invertible.

More generally, let us consider the nonlinear eigenvalue problem

$$(-h^2\Delta + (Q(x,h))^2 + (P(x,h) - z)^2)u = 0, \tag{3.7}$$

where $P(x,h)$ is of the above form and

$$Q(x,h) = Q_m(x) + h^{1/(m+1)}Q_{m-1}(x) + h^{2/(m+1)}Q_{m-2}(x) + \cdots + h^{m/(m+1)}Q_0 \ , \tag{3.8}$$

with Q_j a homogenous polynomial of degree j. This kind of operators appears in the study of non analytic hypoellipticity for operators of the form (1.3). We assume that P_m and Q_m are real and there exists $C > 0$ such that

$$C^{-1} < x >^{2m} \leq (P_m(x)+1)^2 + Q_m(x)^2 \leq C < x >^{2m}, \quad x \in \mathbb{R}^n. \tag{3.9}$$

Here $< x >= (1+|x|^2)^{1/2}$. In the case $Q_m = 0$, P_m has to be an elliptic polynomial of degree m so that (3.9) is satisfied. We can show by the argument used above that $-h^2\Delta + P(x,h)^2 + Q(x,h)^2$ is invertible for $h > 0$ small. We define the operators $A = A(h)$ and $B = B(h)$ in this new setting by :

$$A(h) = (-h^2\Delta + P(x,h)^2 + Q(x,h)^2)^{-1}, B(h) = A(h)^{1/2}P(x,h)A(h)^{1/2}. \tag{3.10}$$

For a temperate symbol, possibly h-dependent, $K(x,\xi,h)$, we denote by $K(x,hD,h)$ the h-pseudo-differential operators defined by Weyl quantization

$$K(x,hD,h)u(x) = \frac{1}{(2\pi h)^n}\int\int e^{i(x-y)\cdot\xi/h}K(\frac{x+y}{2},\xi,h)u(y)\,dyd\xi \ , \tag{3.11}$$

$\forall\ u \in \mathcal{S}(\mathbb{R}^n)$. By the h-pseudo-differential calculus [17], we deduce easily that $A(h)$ and $B(h)$ are h-pseudo-differential operators with symbols, satisfying for any $N \in I\!N^*$,

$$a(x,\xi;h) \ = \ \sum_{j=0}^{(m+1)N} h^{\frac{j}{(m+1)}}a_j(x,\xi) + h^{N+\frac{1}{(m+1)}}R_N(a,h) \tag{3.12}$$

$$b(x,\xi;h) \ = \ \sum_{j=0}^{(m+1)N} h^{\frac{j}{(m+1)}}b_j(x,\xi) + h^{N+\frac{1}{(m+1)}}R_N(b,h) \ . \tag{3.13}$$

Here the symbols

$$\begin{aligned} a_0 &= (\xi^2 + (P_m+1)^2 + Q_m(x)^2)^{-1} \ , \\ b_0 &= (\xi^2 + (P_m+1)^2 + Q_m(x)^2)^{-1}(P_m+1) \ , \end{aligned} \tag{3.14}$$

are, by definition, the h-principal symbols of $A(h)$ and $B(h)$, respectively. If we denote by $S^{\rho}_{\phi,\varphi}$ the class of symbols defined as in Robert [17], and if we introduce :

$$\begin{aligned} \rho_j &= (1+\xi^2+x^{2m})^{-1}(1+x^2)^{-\frac{j}{2(m+1)}}(1+\xi^2)^{-\frac{j}{2(m+1)}}, \\ \phi &= (1+x^2)^{1/2}, \\ \varphi &= (1+\xi^2)^{1/2} \ , \end{aligned}$$

then

$$a_j \in S_{\phi,\varphi}^{\rho_j}, \quad b_j \in S_{\phi,\varphi}^{\rho_j <x>^m}.$$

Moreover the remainders $R_N(a,h)$ and $R_N(b,h)$ are respectively a bounded family of symbols in $S_{\phi,\varphi}^{\rho_{N(m+1)+1}}$ and in $S_{\phi,\varphi}^{\rho_{Nm+1}<x>^m}$.

The Chanillo-Helffer-Laptev's k-criterion becomes now the following

Lemma 3.1. .
Let $\mathcal{D}(h)$ be defined as in (2.6) with A, B replaced by $A(h)$, $B(h)$. Let us assume there exists k such that $\mathcal{D}(h)^k$ is an operator of trace class and that

$$\mathrm{Tr}\ \mathcal{D}(h)^k \neq 0 \tag{3.15}$$

for some $h > 0$. Then the nonlinear spectral problem (3.7) has at least one non trivial solution.

To apply this lemma, we prove the following

Theorem 3.2. .
Let us assume the condition (3.9) for P and Q with $m \geq 1$. Let $k > n(m+1)/m$, $n \geq 1$. Then $\mathcal{D}(h)^k$ is an operator of trace class for all $h > 0$ sufficiently small. We have, for any N, the following asymptotic expansion

$$\mathrm{Tr}\ \mathcal{D}(h)^k = (2\pi h)^{-n}\{\sum_{j=0}^{N(m+1)} h^{\frac{j}{(m+1)}} H_{j;n,k} + \mathcal{O}(h^{N+\frac{1}{(m+1)}})\}, \tag{3.16}$$

when $h \to 0$. Here $H_{j;n,k}$ is independent of h and N, and can be computed from the symbol of $\mathcal{D}(h)^k$. In particular,

$$H_{0;n,k} = \int_{\mathbb{R}^{2n}} \mathrm{tr}\ (\sigma_k(x,\xi))\ dxd\xi, \tag{3.17}$$

σ_k being the h-principal symbol of $\mathcal{D}(h)^k$.

Proof.
We note that $A(h)^{1/2}$ and $B(h)$ are h-pseudo-differential operators with symbol in $S_{\phi,\varphi}^{\rho_0^{1/2}}$. Therefore $\mathcal{D}(h)^k$ is an h-pseudo-differential operator with matrix-valued symbol in the class $S_{\phi,\varphi}^{\rho_0^{\frac{k}{2}}}$. Since $m \geq 1$, we have :

$$\rho_0^{\frac{k}{2}} \in L^1(\mathbb{R}^{2n}) \text{ if } k > n(m+1)/m\ .$$

Consequently, $\mathcal{D}(h)^k$ is a trace class operator when $k > n(m+1)/m$. If we denote by $\sigma_k(x,\xi;h)$ the total symbol of $\mathcal{D}(h)^k$, it has a complete semiclassical expansion similar to $a(h)$ beginning with σ_k, the h-principal symbol of $\mathcal{D}(h)^k$. The semiclassical expansion of the trace follows from the formula

$$\mathrm{Tr}\ \mathcal{D}(h)^k = (2\pi h)^{-n} \int\int \mathrm{tr}\ (\sigma_k(x,\xi;h))\ dxd\xi.$$

Here tr denotes the trace of 2×2 matrices. ∎

A consequence of Lemma 3.1 and Theorem 3.2 is the following

Corollary 3.3. (The semiclassical criterion).
Let (3.2) be satisfied. If there exist $k \in \mathbb{N}$ with $k > (m+1)n/m$ and $j \in \mathbb{N}$ such that $H_{j;n,k} \neq 0$, then the nonlinear eigenvalue problem (3.7) has at least one non trivial solution (z,u) for each $h > 0$ sufficiently small.

4. An application of the semiclassical criterion: the classical criterion

In this section, we apply the semiclassical criterion at the classical level, that is for $j = 0$.

Proposition 4.1. .
Assume that $Q_m = 0$ and $P_m \geq 0$ is elliptic. Let $k > (m+1)n/m$. Then one has

$$H_{0;n,k} = 0, \quad \text{if } n \text{ is odd,} \tag{4.1}$$

$$H_{0;n,k} = 2(-1)^\ell C_n \frac{(n-1)!}{(k-1)(k-2)\cdots(k-n)} C(P_m), \quad \text{if } n = 2\ell. \tag{4.2}$$

Here C_n is the volume of S^{n-1} and

$$C(P_m) = \int_{\mathbb{R}^n_x} (P_m(x)+1)^{-k}\, dx > 0\ .$$

In particular, we observe that :

$$H_{0;n,k} \neq 0 \text{ when } k > (m+1)n/m \tag{4.3}$$

for all n even. As a consequence, we get Theorem 1.1.

Proof of Proposition 4.1.
The condition (3.9) is satisfied. We compute

$$\int\int \operatorname{tr}\ (\sigma_k(x,\xi))\ dxd\xi. \tag{4.4}$$

where σ_k is the h-principal symbol of $\mathcal{D}(h)^k$. Although the symbolic calculus for matrix-valued h-pseudo-differential operators is complicated, the h-principal symbol of $\mathcal{D}(h)^k$ can be easily computed. Since the h-principal symbol of $\mathcal{D}(h)$ is

$$\begin{pmatrix} 2b_0 & a_0^{1/2} \\ -a_0^{1/2} & 0 \end{pmatrix}$$

the h-principal symbol of $\mathcal{D}(h)^k$ is

$$\sigma_k = \begin{pmatrix} 2b_0 & a_0^{1/2} \\ -a_0^{1/2} & 0 \end{pmatrix}^k .$$

Therefore,

$$\operatorname{tr}\ \sigma_k = (b_0 + \sqrt{b_0^2 - a_0}\ )^k + (b_0 - \sqrt{b_0^2 - a_0}\ )^k\ . \tag{4.5}$$

We recall that

$$a_0 = (\xi^2 + (P_m + 1)^2)^{-1}, \quad b_0 = (\xi^2 + (P_m + 1)^2)^{-1}(P_m + 1).$$

By the change of variables $\xi \to (P_m + 1)\eta$, we obtain that

$$\int\int \operatorname{tr}(\sigma_k)(x, \xi)\, dx d\xi = C(P_m) \int_{\mathbb{R}^n_\eta} (1 + \eta^2)^{-k}((1 + i|\eta|)^k + (1 - i|\eta|)^k)\, d\eta. \tag{4.6}$$

This shows that

$$H_{0;n,k} = 2C_n C(P_m) \Re \int_0^\infty (1 + r^2)^{-k}(1 + ir)^k r^{n-1}\, dr, \quad k > (m + 1)n/m, \tag{4.7}$$

C_n being the volume of S^{n-1}.

Let

$$L(n, k) = \int_0^\infty (1 + r^2)^{-k}(1 + ir)^k r^{n-1}\, dr = \int_0^\infty (1 - ir)^{-k} r^{n-1}\, dr.$$

An integration by parts gives :

$$\begin{aligned} L(n, k) &= i\frac{n-1}{k-1} L(n - 1, k - 1) \\ &= i^{n-2} \frac{(n - 1)(n - 2)\cdots 2}{(k - 1)(k - 2)\cdots(k - n + 2)} L(2, k - n + 2)\ . \end{aligned} \tag{4.8}$$

Since

$$L(2, j) = -\frac{1}{(j - 1)(j - 2)}\ ,$$

we have

$$H_{0;n,k} = 2C_n \Re\{i^n \frac{(n - 1)!}{(k - 1)(k - 2)\cdots(k - n + 1)(k - n)}\} C(P_m). \tag{4.9}$$

This proves (4.1) and (4.2). ∎

Using the classical criterion, we can construct a family of examples when n is odd.

Corollary 4.2. .
Let $n = n_1 + n_2$ with n_1 even and $n_2 \geq 1$. Let P, R be real elliptic homogeneous polynomials[1] *of degree m on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. For $x \in \mathbb{R}^n$, set $x = (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then, the operator :*

$$-\Delta_x + (R(x'')D_{y_1})^2 + (P(x')D_{y_1} - D_{y_2})^2\ ,$$

is not analytic hypoelliptic at 0 in $\mathbb{R}^{n+2}$.

Proof.
It suffices to show that the nonlinear eigenvalue problem

$$\left(\sum_{j=1}^n D_{x_j}^2 + (R(x''))^2 + (P(x') - z)^2 \right) u = 0$$

[1]The important property is only that : $-\Delta_{x''} + R(x'')^2$ on $\mathbb{R}^{n_2}$ has an eigenvalue λ_0.

has a non trivial solution (z,u). We can look for $v(x')$ satisfying, for some z,

$$\left(-\Delta_{x'} + \lambda_0 + (P(x') - z')^2\right) v = 0$$

where $\lambda_0 > 0$ is an eigenvalue of $-\Delta_{x''} + R(x'')^2$. The corresponding semiclassical operator is $-h^2\Delta_{x'} + \lambda_0 h^{2m/(m+1)} + \left(P(x') - z\right)^2$ on $\mathbb{R}^{n_1}$. We can apply Proposition 4.1 for $Q = Q_0 = \lambda_0$ to conclude. ∎

Remark 4.3. .
One can play with λ_0. We can indeed consider a sequence λ_n of eigenvalues of $-\Delta_{x''} + R(x'')^2$ tending to $+\infty$ and to choose a corresponding sequence h_n of h's tending to 0, such that $\lambda_n h_n^{\frac{2m}{m+1}} = 1$. This could permit to relax the condition of ellipticity of P, particularly if we add a Q like in (3.7).

Let us now consider the more general case $Q_m \neq 0$. For (P_m, Q_m) satisfying (3.9), we can check that the classical criterion still works in some cases. Let us remark that

$$\text{tr } \sigma_k = (b_0 + \sqrt{b_0^2 - a_0}\,)^k + (b_0 - \sqrt{b_0^2 - a_0}\,)^k \tag{4.10}$$

with now

$$a_0 = (\xi^2 + (P_m+1)^2 + Q_m^2)^{-1}, \quad b_0 = (\xi^2 + (P_m+1)^2 + Q_m^2)^{-1}(P_m+1).$$

Let

$$T = ((P_m+1)^2 + Q_m^2)^{1/2}, \quad \tau_1 = (P_m+1)/T, \quad \tau_2 = |Q_m|/T. \tag{4.11}$$

By the change of variables $\xi \to T\eta$, we obtain that

$$\int \text{tr } (\sigma_k)(x,\xi)\, d\xi = 2T^{n-k}\Re \int_{\mathbb{R}^n_\eta} (1+\eta^2)^{-k}(\tau_1 + i(\tau_2^2 + \eta^2)^{1/2})^k\, d\eta. \tag{4.12}$$

The case $n = 1$. .
This case can be also thoroughly analyzed by the method of [2]. When $n = 1$, we obtain here for $k = 3$,

$$H_{0;1,3} = -\frac{3\pi}{2}\int_{\mathbb{R}_x} \tau_1\tau_2^2 T^{-2} dx. \tag{4.13}$$

It is clear that if $P_m \geq 0$ (this requires m to be even) and $Q_m \neq 0$, then, $\tau_1 \geq 0$ and $H_{0;1,3} < 0$. If P_m changes sign, $H_{0;1,3}$ may be vanishing.

The case $n \geq 2$. .
When $n \geq 2$, $k > (m+1)n/m$,

$$H_{0;n,k} = 2C_n \Re \int_{\mathbb{R}^n_x} T(x)^{n-k}\int_0^\infty (1+r^2)^{-k}(\tau_1 + i(\tau_2^2 + r^2)^{1/2})^k r^{n-1}\, dr. \tag{4.14}$$

Set

$$C_{n,k} = \int_0^\infty (1+r^2)^{-k}(\tau_1 + i(\tau_2^2 + r^2)^{1/2})^k r^{n-1}\, dr. \tag{4.15}$$

By a change of variable, we obtain

$$C_{n,k} = \int_{\tau_2}^{\infty} (\tau_1 - it)^{-k} (t^2 - \tau_2^2)^{(n-2)/2} t \, dt.$$

The subcase $n = 2$. .
In the case $n = 2$, it is easy to see that

$$C_{2,k} = \frac{(\tau_1 + i\tau_2)^{k-2}}{k-2} - \frac{\tau_1(\tau_1 + i\tau_2)^{k-1}}{k-1}. \tag{4.16}$$

Since $\tau_1^2 + \tau_2^2 = 1$, we can see that

$$\Re C_{2,3} = \tau_1(1 + 2\tau_2^2)/2 \geq 0, \text{ if } \tau_1 \geq 0 \tag{4.17}$$

$$\Re C_{2,4} = -\frac{1}{2} + 2\tau_1^2 - \frac{4}{3}\tau_1^4. \tag{4.18}$$

This shows $H_{0;2,3} > 0$ for $n = 2$ when $P_m \geq 0$. The classical criterion can be used in this case for $m > 3$ when $n = 2$.

To study the case $n = 2$ and $m = 2$, we need information on the sign of $H_{0;2,4}$ which depends on the relation between P_2 and Q_2. An elementary computation gives :

Lemma 4.4. .

(1) *If* $\tau_1(x_1, x_2)^2 \leq (3 - \sqrt{3})/4$ *for all* (x_1, x_2), *then* $H_{0;2,4} > 0$.
(2) *If* $\tau_1(x_1, x_2)^2 \geq (3 - \sqrt{3})/4$ *for all* (x_1, x_2), *then* $H_{0;2,4} < 0$.

The subcase $n = 3$. .
When $n = 3$, $k = 5$, we have

$$H_{0;3,5} = \frac{5\pi}{8} \int_{\mathbb{R}_x^3} \tau_2^4 \tau_1 T^{-2} dx > 0, \tag{4.19}$$

so long as Q_m is not identically zero and $P_m \geq 0$.

Some applications. .
We obtain the following :

Proposition 4.5. .
(a). Let $n = 1$. *The operator*

$$P_m(x, D) := D_x^2 + (D_s - cx^m D_t)^2 + (x^m D_t)^2$$

is not analytic hypoelliptic at $0 \in \mathbb{R}^3$ *for all* $m \geq 2$ *even and* $c \in \mathbb{R}$.

(b). Let $n = 2$. *Let* P, Q *be real homogenous polynomials of degree* $m \geq 2$ *satisfying (3.9) and* $P \geq 0$. *When* $m = 2$, *we assume in addition that one of the above conditions on* τ_1 *is satisfied. Then the operator*

$$D_x^2 + D_y^2 + (D_s - P(x, y)D_t)^2 + (Q(x, y)D_t)^2, \quad (x, y) \in \mathbb{R}^3, \tag{4.20}$$

is not analytic hypoelliptic at 0 *in* $\mathbb{R}^4$.

(c). Let $n = 3$. Let P, Q be real homogenous polynomials on $\mathbb{R}^3$ of degree $m \geq 2$ satisfying (3.9). Assume that $Q \neq 0, P \geq 0$. Then the operator

$$-\Delta_x + (D_s - P(x)D_t)^2 + (Q(x)D_t)^2 \tag{4.21}$$

is not analytic hypoelliptic at 0 in $\mathbb{R}^5$.

We believe that the condition on τ_1 in (b) is technical. As examples of (P, Q) satisfying the conditions of (b) of Proposition 4.5, we can take $P(x, y) = (x^2 + y^2)^\ell$, $Q(x, y) = (xy)^\ell$, $\forall\ \ell \geq 1$, because in the case $\ell = 1$, one can check that $\tau_1(x, y)^2 \geq 4/5 > (3 - \sqrt{3})/4$. We shall come back at the analysis of this example corresponding to $\ell = 1$ in the next section.

Remark 4.6. .
Let us briefly compare the results obtained here and those of [2]*. Apparently, in the case $Q = 0$, the "classical" criterion does not produce any result for $n \geq 1$ odd. But the semiclassical criteria may still work by considering a coefficient of higher order of the expansion of the trace. As indicated by J. Sjöstrand, a similar approach is used by L. Nedelec in* [15] *for getting a lower bound for the number of resonances of an h-pseudodifferential system. The same condition on the dimension appears. The "quantum" criterion given in* [2] *works for $n = 1, 2, 3$ but with a stronger condition on m when $n > 1$. Moreover it has been observed in Remark 4.4 in* [2] *that the condition of ellipticity of P can be replaced by a weaker condition. The last point is that the homogeneity of P plays an important role in the dilation argument of* [2]*, while in the semiclassical approach, the lower order parts can be included. This appears to be useful in the dimension reduction.*

5. Comparison with Métivier's results

In this section, we would like to analyze the links between our results and the previous results obtained by G. Métivier [11, 13, 14].

5.1. **A first family of examples.**

Let us first consider the operator on $\mathbb{R}^{n+2}$:

$$H(X, D_X) = -\Delta + (P(x)D_{x_{n+1}} - D_{x_{n+2}})^2 \,,$$

where $X = (x, x_{n+1}, x_{n+2})$, P is an homogeneous positive elliptic polynomial of degree $m \geq 2$ on $\mathbb{R}^n$.

If we take the "microlocal spirit" we observe that H is an operator with double characteristics, whose principal symbol is the function :

$$(T^*\mathbb{R}^{n+2} \setminus 0) \ni (X, \Xi) \mapsto |\xi|^2 + (P(x)\xi_{n+1} - \xi_{n+2})^2 \,.$$

The symbol vanishes exactly at order 2 on the submanifold

$$\Sigma = \{(X, \Xi) \mid \xi = 0 \,,\ P(x)\xi_{n+1} - \xi_{n+2} = 0 \,,\ \xi_{n+1} \neq 0\} \,.$$

This submanifold is of codimension $n + 1$. Let us now analyze the "symplecticity" of Σ. We recall that Σ is said to be symplectic if the restriction to Σ of the canonical 2-form is non degenerate. An easy way for verifying the symplecticity is to consider the $(n + 1) \times (n + 1)$ matrix u_i, u_j where $u_i(X, \Xi) = \xi_i$ for $i = 1, \cdots, n$ and

$u_{n+1}(X, \Xi) = P(x)\xi_{n+1} - \xi_{n+2}$ and to show that it is not degenerate. An immediate computation shows that its rank at a given point is 2 if $\nabla P \neq 0$ and 0 if $\nabla P = 0$.

When P is elliptic and homogeneous, we get that the rank is constant outside 0 and equal to 2. There are two cases :

(1) When $n = 1$, we get that Σ is symplectic except at the points of Σ such that $x = 0$. The result of Trèves-Tartakoff-Métivier-Sjöstrand [20, 21, 12, 19] shows that the operator is microlocally analytic hypoelliptic outside of Σ (ellipticity) and in the neighborhood of the points of Σ such that $x \neq 0$. In this case, the operator is not analytic hypoelliptic at any point $(0, x_{n+1}, x_{n+2})$.

(2) When $n > 1$, Métivier's result[2] [11, 13] gives that the operator H is not analytic hypoelliptic in any open set in $\mathbb{R}^{n+2}$. What we show here is the more precise result that P is not analytic hypoelliptic at any point $(0, x_{n+1}, x_{n+2})$ which is a finer property. See the introduction of [13] for the discussion between the definitions of analytic hypoellipticity and germ-hypoanalyticity (analytic hypoellipticity in a neighborhood of a point and analytic hypoellipticity at a point).

5.2. **A new class of non analytic hypoelliptic operators.** We now show that maybe more interesting examples can be treated when considering the more general class :

$$H(X, D_X) = -\Delta + (P(x)D_{x_{n+1}} - D_{x_{n+2}})^2 + Q(x)^2 D^2_{x_{n+1}} ,$$

where P and Q are homogeneous polynomials of degree $m > 1$ with $P \geq 0$ and $P^2 + Q^2$ elliptic. When restricting H to x_{n+1} independent distributions, we get an analytic hypoelliptic operator on $\mathbb{R}^{n+1}$:

$$-\Delta_x + (P(x)^2 + Q(x)^2)D^2_{x_{n+2}} ,$$

by applying a theorem of Grushin ([7]).

We have seen in Section 4, that the "classical" criterion can give a result under some additional condition. We will concentrate our analysis to the specific case when :

$$n = 2\ ,\ P(x) = x_1^2 + x_2^2\ ,\ Q(x) = \alpha x_1 x_2\ ,$$

with $\alpha > 0$.

Let us do the same microlocal analysis as in the previous subsection. The characteristic set Σ is now defined as a union of two regular submanifolds of dimension 4 in $\mathbb{R}^8 \setminus 0$:

$$\Sigma = \Sigma_1 \cup \Sigma_2\ ,$$

with

$$\Sigma_j = \{\xi_1 = 0\ ,\ \xi_2 = 0\ ,\ \xi_4 = (x_1^2 + x_2^2)\xi_3\ ,\ x_j = 0, \xi_3 \neq 0\}\ .$$

Moreover Σ_j is symplectic outside $\Sigma_1 \cap \Sigma_2$ and not symplectic at $\Sigma_1 \cap \Sigma_2$.

[2]Note that the operator is hypoelliptic with loss of one derivative in $\mathbb{R}^{n+2} \setminus \{x = 0\}$.

Outside $\Sigma_1 \cap \Sigma_2$, observing that the symbol of H vanishes exactly at order 2 on Σ there, we get again from [20, 21, 12, 19] that H is microlocally analytic hypoelliptic. Métivier's criterion of non-analytic-hypoellipticity can not be applied at the points $(0, 0, x_3, x_4)$ (the operator is indeed not hypoelliptic with loss of one derivative) and it is interesting to see what is obtained through our approach.

Proposition 5.1. .
For $\alpha > 0$ small enough, the operator $H(X, D_X)$ is not analytic hypoelliptic at any point $(0, x_3, x_4)$.

Proof.
We just apply the criterion of the previous section (Proposition 4.5, second case) and the discussion following the statement.

When α is small enough (at least $0 < \alpha \leq 1$), we observe that the quantity $(P_m+1)^2/\left((P_m+1)^2 + Q_m^2\right)$ is sufficiently near 1. In particular, the second condition on τ_1 in Lemma 4.4 is satisfied.

Remark 5.2. .
As explained to us by M. Christ, it is possible to prove, for rather large classes of models depending analytically on an additional parameter α, that some associated Fredholm determinant is analytic in α. One can conclude that, if the operator is not analytic hypoelliptic for some value of α, then it is not analytic hypoelliptic for generic values of α. We refer to [5], Proposition 5.2 for the argument. In the particular case of the above Proposition 5.1, we can present the argument in the following way. Let $H_{0,2,4}(\alpha)$ denote $H_{0,2,4}$ defined as in Theorem 3.2 with $P = x_1^2 + x_2^2$ and $Q = \alpha x_1 x_2$. Using (4.12) and (4.16), one can check that $H_{0,2,4}(\alpha)$ is real analytic in $\alpha > 0$. The proof of Proposition 5.1 shows that $H_{0,2,4}(\alpha) \neq 0$ for $\alpha > 0$ small. Thus, $H_{0;2,4}(\alpha) \neq 0$ for all $\alpha > 0$, except a discrete set in $\mathbb{R}_+$ and therefore Proposition 5.1 remains true in this case. This argument can be used for more general cases. It is indeed easy to check in many situations the analyticity of $H_{0,n,k}$ with respect to the parameter.

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