On the Uniqueness of Inverse Scattering for N-Body Systems

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Abstract

In this paper, we study the high energy asymptotics of the free channel-free channel scattering matrix for generalized N-body Schrödinger operator $P = -\Delta + \sum_{a\in\mathcal{A}} V_a(x^a)$ with short range interactions and prove that all the interactions V_a are uniquely determined by the free channel- free channel scattering matrix at high energies.

1 Introduction

There are now many works on inverse scattering problems, either at high energies or at fixed energy, for two-body Schrödinger operators. See, for example, [1, 3, 5] and the references quoted therein. For N-body Schrödinger operators with short range interactions, the direct scattering – the existence and asymptotic completeness of wave operators – is well known since the work of Sigal-Soffer [7]. The purpose of this work is to study an inverse scattering problem for many-particle quantum systems: Given scattering operators for an N-body system, can one determine (and reconstruct) all potentials from the scattering operators at high energies? We shall give an affirmative answer to this question and in fact, we shall prove that all potentials are uniquely determined by just one scattering operator –the free channel-free channel scattering operator– at high energies.

To state more precisely the main result of this work, let us first introduce some notations related to generalized N-body Schrödinger operators. (One will see that the inverse scattering at high energies for physical N-body systems only involving pair interactions is much easier). Let Δ be the Laplacian on the Euclidean space $\mathbf{X} = \mathbf{R}^d$ equiped with a positive definite quadratic form $q(\cdot)$. Let $\mathcal{A}$ be the set of all possible cluster decompositions of an N-body system labelled by $\{1, 2, \cdots, N\}$. For $a, b \in \mathcal{A}$, we shall write $a \subset b$ if a is a refinement of the cluster decomposition b. The N-body Schrödinger operator to be studied in this work is of the form:

$$P = -\Delta + \sum_{a\in\mathcal{A}} V_a(x^a).$$

Here $x^a = \pi^a x$ with π^a the orthogonal projection (relative to q) from $\mathbf{X}$ onto some subspace $\mathbf{X}^a$. Assume the following axioms on the collection $\{\mathbf{X}^a, a \in \mathcal{A}\}$: (i). $\mathbf{X}^a \subseteq \mathbf{X}^b$ iff $a \subseteq b$; (ii). There are elements $a_{\max}$ and $a_{\min}$ in $\mathcal{A}$ such that $\mathbf{X}^{a_{\max}} = \mathbf{X}$ and $\mathbf{X}^{a_{\min}} = \{0\}$; (iii). For any $a, b \in \mathcal{A}$, the union $a \cup b$ is defined in $\mathcal{A}$ with the property that $\mathbf{X}^{a\cup b} = \mathbf{X}^a + \mathbf{X}^b$. Since $x^{a_{\min}} \equiv 0$, $V_{a_{\min}}$ is a constant. To fix the idea, we take $V_{a_{\min}} = 0$.

For each $a \in \mathcal{A}$, we denote by $\mathbf{X}_a$ the orthogonal complement (with respect to the Euclidean structure on $\mathbf{X}$) of $\mathbf{X}^a$ in $\mathbf{X}$: $\mathbf{X} = \mathbf{X}^a \oplus \mathbf{X}_a$. Accordingly, a generic point $x \in \mathbf{X}$ can be decomposed as: $x = x^a + x_a$. We denote Δ^a (Δ_a, resp.) the Laplacian in x^a-variables (x_a-variables, resp.) and $D^a = -i\partial/\partial x^a$, $D_a = -i\partial/\partial x_a$. Put

$$P^a = -\Delta^a + \sum_{b \subseteq a} V_b(x^b), \quad P_a = P^a - \Delta_a \text{ and } I_a(x) = \sum_{b \not\subseteq a} V_b(x^b).$$

Then for any $a \in \mathcal{A}$, one has: $P = P_a + I_a(x)$. Let $\mathcal{T}$ denote the set of all thresholds (including the eigenvalues) of P:

$$\mathcal{T} = \cup_a \sigma_{pp}(P^a).$$

Let a be a non-trivial cluster decomposition (i.e., $a \in \mathcal{A}$ with $\#a \geq 2$). A scattering channel α stands for a collection of data: $\alpha = (a, E_\alpha, \phi_\alpha)$, where $E_\alpha \in \sigma_{pp}(P^a)$ and ϕ_α is an associated normalized eigenfunction:

$$P^a \phi_\alpha = E_\alpha \phi_\alpha, \ \|\phi_\alpha\| = 1.$$

When $a = a_{\min}$, one uses the convention that $P^a = 0$, $P_a = -\Delta$ and in this case, the only scattering channel is the free one: $\alpha = (a_{\min}, 0, 1)$. Let $\mathcal{J}_\alpha : L^2(\mathbf{X}_a) \to L^2(\mathbf{X})$ the channel identification:

$$(\mathcal{J}_\alpha f)(x) = \phi_\alpha(x^a) f(x_a).$$

We assume that $\forall a \in \mathcal{A}$, $V_a(x^a)$ is real and $V_a(x^a), (x^a \cdot \nabla) V_a(x^a)$ and $(x^a \cdot \nabla)^2 V_a(x^a)$ are relatively compact with respect to $-\Delta^a$ in $L^2(\mathbf{X}^a)$ and there exists $\rho > 1$ and $R > 0$,

$$|V_a(y)| \leq C\langle y\rangle^{-\rho}, \quad \forall |y| > R. \tag{1.1}$$

Here $\langle x\rangle = (1+|x|^2)^{1/2}$. Under the assumption (1.1), it is well known that the wave opeartors

$$W_\pm^\alpha = s - \lim_{t\to\pm\infty} U(t)^* U_a(t) \mathcal{J}_\alpha$$

exist for any scattering channel α and are asymptotically complete ([7]). Here $U(t)$ and $U_a(t)$ are unitary groups generated by P and P_a, respectively. Let $\alpha = (a, E_\alpha, \phi_\alpha)$ and $\beta = (b, E_\beta, \phi_\beta)$ be two given scattering channels. Let

$$S_{\beta\alpha} = W_+^{\beta *} W_-^\alpha$$

be the scattering operator from an initial channel α to a final channel β.

The inverse scattering problem for N-body systems consists in recovering informations on the potentials V_a from the knowledge on scattering operators $S_{\beta\alpha}$. Remark that there are usually an infinite number of scattering operators $S_{\beta\alpha}$ and that all but one depend on the spectral properties of sub-Hamiltonians P^a. In particular, if $V_a \geq 0$, $\forall a \in \mathcal{A}$, we just have one scattering operator $S_{\alpha\alpha}$ where α is the free scattering channel. So if we want to solve an inverse scattering problem for a class of potentials including repulsive systems, we only have the free channel-free channel scattering operator at our disposal. In this paper, we shall prove that this scattering operator already contains sufficient informations which allow us to recover all V_a from its high energy asymptotics. In order to avoid difficulties arising from local singularities, we strengthen the condition (1.1) by the following

$$\begin{array}{l}\forall a \in \mathcal{A},\ V_a(x^a) \text{ is real-valued and } (x^a\cdot\nabla)V_a(x^a) \text{ and } (x^a\cdot\nabla)^2 V_a(x^a) \\ \text{are relatively compact with respect to } -\Delta^a \text{ in } L^2(\mathbf{X}^a) \text{ and there} \\ \text{exists } \rho > 1 \text{ such that} \\ \qquad\qquad |V_a(y)| \le C\langle y\rangle^{-\rho}, \quad \forall y \in \mathbf{X}^a.\end{array} \tag{1.2}$$

The main result of this paper can be stated as follows.

Theorem 1.1 *Let* $\dim \mathbf{X}^a \ge 2$, $\forall a \in \mathcal{A}$. *Let* $\mathcal{V}$ *be the class of all possible potentials for* N*-body systems:* $\mathcal{V} = \{\vec{V} = (V_a, a \in \mathcal{A}); \vec{V}$ *satisfies the condition (1.2)* $\}$. *For* $\vec{V} \in \mathcal{V}$, *let* $S_{00}(\vec{V})$ *be the free channel-free channel scattering operator associated to the* N*-body Schrödinger operator* $P = -\Delta + \sum_{a\in\mathcal{A}} V_a$. *Then for any* $R > 1$, *the mapping* $\mathcal{V} \ni \vec{V} \to S_{00}(\vec{V})E_{[R,\infty[}(-\Delta)$ *is injective. Here* $E_I(A)$ *denotes the spectral projection for a self-adjoint operator* A *onto the interval* I.

Remarks 1.1 (1). If $\dim \mathbf{X}^a = 1$, $P^a = -\Delta^a + V_a(x^a)$ is necessarily a two-body Schrödinger operator . Then we can apply the known results on inverse scattering for two-body problems (see [1, 5]) to determine the potential $V_a(x^a)$. See also a self-contained proof given in Sect. 4 in the case the potential is even. The argument used in the proof of Theorem 1.1 then allows us to recover all potentials V_a with $\dim \mathbf{X}^a \ge 2$.

(2). We indicate that to determine V_a, $\forall a \in \mathcal{A}$, it is sufficient to know $V(x) = \sum_{a\in\mathcal{A}} V_a(x^a)$ on $\mathbf{X}$. In fact for any $a \in \mathcal{A}$, put $W_a(x^a) = \sum_{b\subseteq a} V_b(x^b)$. Then $V(x) = W_a(x^a) + I_a(x)$. Let $\omega \in \mathbf{X}_a \setminus (\cup_{b\not\subseteq a}\mathbf{X}_b)$, $|\omega| = 1$. For any fixed x^a and $x = x^a + r\omega$, $r > 0$, one has $\forall b \subseteq a$, $\pi^b x = \pi^b x^a$ and

$$|\pi^b x| \ge r|\pi^b\omega| - |x^a| \ge cr - |x^a|, \quad c > 0, \forall b \not\subseteq a.$$

So under the condition (1.1), we have:

$$\lim_{r\to\infty} V(x^a + r\omega) = W_a(x^a). \tag{1.3}$$

By an induction on $\#a$, the number of clusters in a, we can recover all V_a. For $\#a = N-1$, the only $b \in \mathcal{A}$ with $b \subset a$ is $b = a_{\min}$. So in this case, we have $W_a(x^a) = \sum_{b\subseteq a} V_b(x^b) = V_a(x^a)$. Suppose now that all V_b with $\#b \ge k+1$ are known $(1 \le k \le N-2)$. For $\#a = k$, any $b \subset a$ has at least $k+1$ clusters. So $V_a(x^a)$ can be recovered from the relation $V_a(x^a) = W_a(x^a) - \sum_{b\subset a} V_b(x^b)$.

To reconstruct all V_a from $S_{00}(\vec{V})E_{[R,\infty[}(-\Delta)$, we shall study the high energy asymptotics of scattering matrix in a weak sense. We shall proceed in two steps:

(i). Let S (S^a, resp.) be the free channel-free channel scattering operator for the N-body Schrödinger operator P (for the sub-Hamiltonian P^a, resp.). Then, all S^a can be reconstructed from S at high energy limit: $\forall u \in L^2(\mathbf{X}^a)$, $S^a u = \lim_{n\to\infty} < S(uf_n), f_n >_{L^2(\mathbf{X}_a)}$, where f_n is an appropriate sequence in $L^2(\mathbf{X}_a)$, converging weakly to 0.

(ii). For any $\omega \in \mathbf{X} \setminus (\cup_{a\neq a_{\min}} \mathbf{X}_a)$, $|\omega| = 1$, $V^\omega(\cdot)$ defined by

$$V^\omega(y) = \int_{\mathbf{R}} V(y + t\omega)dt, \quad \omega \cdot y = 0, \tag{1.4}$$

can be reconstructed from the high energy asymptotics of S.

If V is a two-body short-range potential, (1.4) uniquely determines the potential if $\dim \mathbf{X} \geq 2$. For many body problems, the only condition $\dim \mathbf{X} \geq 2$ is not sufficient to recover V from (1.4). So we shall make use of (i) and apply (ii) to each S^a. Theorem 1.1 then follows from an induction on $\#a$, assuming $\dim \mathbf{X}^a \geq 2, \forall a$.

The step (i) will be done in Sect. 2. The result shows that the free channel -free channel scattering matrix contains sufficient informations and to recover the pair interactions for an N-body problem, it is sufficient to take a to be an associated cluster decomposition with $N-1$ clusters and to apply the known results on two-body inverse scattering. See [5]. (i) already allows us to prove the uniqueness of inverse scattering for physical N-body Schrödinger operator involving only pair interactions. To realize the step (ii), we first establish in Sect.3 a spectral representaion formula for the free channel scattering matrix $S(\lambda)$. As a consequence of this representation formula, we give in Sect. 4 the Born's formula for the scattering matrix at high energies for N-body Schrödinger operator , which is formally completely similar to the two-body case. The result obtained in Sect. 4 allows us to recover the potentials if they are even (without restriction on the dimension of $\mathbf{X}^a$). In Sect. 5, we study the high energy asymptotics for $S(\lambda)$ in a weak sense and finish the proof for Theorem 1.1. As to the methods used in this paper, there exists a commun point between the present work and the author's earlier one [9] on cross-sections: we use in both works the method of eikonal approximation and microlocal resolvent estimates at high energies (cf [8]) to estimate remainders.

In this paper, we shall frequently use the following notations. The norm and the scalar product in $L^2(\mathbf{X}_a)$ will be denoted by $\|\cdot\|_a$ and $<\cdot,\cdot>_a$, while those in $L^2(\mathbf{X})$ will be denoted by $\|\cdot\|$ and $<\cdot,\cdot>$, respectively. $\mathbf{S}^{d-1}$ will denote the unit sphere in $\mathbf{X}$ and the scalar product in $L^2(\mathbf{S}^{d-1})$ will be denoted by $(\cdot,\cdot)$. Finally, we write $\mathbf{S}^a = \mathbf{S}^{d-1} \cap \mathbf{X}^a$ and $\mathbf{S}_a = \mathbf{S}^{d-1} \cap \mathbf{X}_a$.

2 Recovering the Free Channel Wave Operators for Sub-Hamiltonians

Let $P = -\Delta + V(x)$, $P^a = -\Delta^a + \sum_{b\subseteq a} V_b(x^b)$ and $P_a = -\Delta_a + P^a$. Assume that the V_a's satisfy the condition (1.2). Let $U(t)$ ($U^a(t), U_0(t), U_0^a(t)$, resp.) be the unitary group generated by P ($P^a, -\Delta_x, -\Delta^a$, resp.). It is well known that under the assumption (1.2), the wave operators

$$W_\pm = s - \lim_{t\to\pm\infty} U(t)^* U_0(t) \quad \text{in } L^2(\mathbf{X}) \tag{2.1}$$

$$W_\pm^a = s - \lim_{t\to\pm\infty} U(t)^{a*} U_0^a(t) \quad \text{in } L^2(\mathbf{X}^a) \tag{2.2}$$

exist. Put $S = W_+^* W_-$ and $S^a = W_+^{a*} W_-^a$.

In this Section, we shall prove that $W^a_\pm$, $a \neq a_{\max}$, can all be recovered from $W_\pm$ at high energies. The main idea consists in comparing $W_\pm$ with $W^a_\pm \otimes 1_a$ with 1_a the indentity operator on $L^2(\mathbf{X}_a)$. In fact to recover $W^a_\pm$ from $W_\pm$, we just need to study the difference $W_\pm - W^a \otimes 1_a$ on functions of the form $\varphi \otimes f_n$, where $\varphi \in L^2(\mathbf{X}^a)$ is arbitrary, while $f_n \in L^2(\mathbf{X}_a)$ is a particular sequence.

Let $f \in \mathcal{S}(\mathbf{X}_a)$ such that its Fourier transform, $\hat{f}$, is of compact support and $\|f\| = 1$. For a fixed $\omega \in \mathbf{S}_a \setminus (\cup_{b \not\subseteq a} \mathbf{X}_b)$ ($\mathbf{S}_a = \mathbf{S}^{d-1} \cap \mathbf{X}_a$), put

$$f_n(x_a) = e^{in\omega \cdot x_a} f(x_a), n \in \mathbf{N}. \tag{2.3}$$

Then, $\hat{f}_n(\xi_a) = \hat{f}(\xi_a - n\omega)$. So for n sufficiently large, supp $\hat{f}_n$ is contained in $\mathbf{X}_a \setminus (\cup_{b \not\subseteq a} \mathbf{X}_b)$. Define $W^{(n)}_\pm, W^{a,(n)}_\pm : L^2(\mathbf{X}^a) \to L^2(\mathbf{X})$ by

$$W^{(n)}_\pm u = W_\pm(u \otimes f_n), \quad W^{a,(n)}_\pm u = (W^a_\pm u) \otimes f_n, \quad \forall u \in L^2(\mathbf{X}^a). \tag{2.4}$$

Remark that for $W^{(n)}_\pm$ to be well defined for large n, we just need to know $W_\pm$ at high energies. In fact, if we know $W_\pm$ on the range of $E_{[M,\infty[}(-\Delta_x)$, $W^{(n)}_\pm$ is already well defined for $n > M + R$ with $R > 0$ such that supp $\hat{f} \subseteq \{|\xi_a| \leq R\}$. It is clear that

$$\|W^{(n)}_\pm\| \leq 1 \text{ and } \|W^{a,(n)}_\pm\| \leq 1, \quad \forall n.$$

Theorem 2.1 *Under the assumption (1.2), one has:* $s\text{-}\lim_{n\to\infty}\{W^{(n)}_\pm - W^{a,(n)}_\pm\} = 0$ *in* $L^2(\mathbf{X})$, *that is to say, for any* $\varphi \in L^2(\mathbf{X}^a)$, $\lim_{n\to\infty}\{W^{(n)}_\pm - W^{a,(n)}_\pm\}\varphi = 0$ *in* $L^2(\mathbf{X})$. *In particular,* $W^a_\pm \varphi = \lim_{n\to\infty} < W^{(n)}_\pm \varphi, f_n >_a$, *for any* $\varphi \in L^2(\mathbf{X}^a)$.

Before giving the proof for Theorem 2.1, we need some preparations. To simplify notations, we only consider the $+$ case and omit the subscript $+$. By an argument of density, to prove Theorem 2.1, it is sufficient to prove

$$\lim_{n\to\infty}\{W^{(n)}_\pm - W^{a,(n)}_\pm\}\varphi = 0 \text{ in } L^2(\mathbf{X}), \tag{2.5}$$

for any $\varphi \in \mathcal{S}(\mathbf{X}^a)$ such that its Fourier transform, $\hat{\varphi}$, is of compact support with supp $\hat{\varphi} \subset \mathbf{X}^a \setminus (\cup_{b \subset a, b \neq a_{\min}} \mathbf{X}_b)$. For such a φ, put $\psi^a = W^a \varphi$. Let $U_a(t) = exp(-itP_a)$. One has:

$$W^{(n)}\varphi - W^{a,(n)}\varphi = W(\varphi \otimes f_n) - (W^a\varphi) \otimes f_n$$

$$= W((W^a)^{-1}\psi^a \otimes f_n) - \psi^a \otimes f_n = \lim_{t\to\infty} U(t)^* U_a(t)(\psi^a \otimes f_n) - \psi^a \otimes f_n.$$

To show (2.5), we need to introduce suitable cut-offs in the last time-dependent expression.

Lemma 2.2 *Let* χ *be a smooth function on* $\mathbf{R}$ *with* $0 \leq \chi \leq 1$ *and* $\chi(t) = 1$ *for* $t > 1$; 0 *for* $t < 1/2$. *Under the assumption (1.2) with* $\rho > 1$, *there exists* $M_0 > 0$ *such that*

$$\|\chi(\frac{|x^a|}{M\langle t\rangle}) U^a(t)\psi^a\| \leq C\langle t\rangle^{-\rho+1}, \quad t > 0, \tag{2.6}$$

uniformly in $M > M_0$.

Proof. Since supp $\hat{\varphi} \subset \mathbf{X}^a \setminus (\cup_{b\subset a, b\neq a_{\min}} \mathbf{X}_b)$ is compact, we can choose $J_a(x^a)$ a bounded smooth function such that $J_a(x^a) = 1$ for $\pm x^a$ in a small conic neighbourhood of supp $\hat{\varphi}$, supp $J_a \subset \{x^a, |x^b| > \delta |x^a|, \quad \forall b \subset a\}$ for some $\delta > 0$ and

$$|\partial^\alpha J_a(x^a)| \leq C_\alpha \langle x^a \rangle^{-|\alpha|}.$$

Then we have:

$$\psi^a = W^a \varphi = \lim_{t\to\infty} U(t)^{a*} U_0^a(t)\varphi = \lim_{t\to\infty} U(t)^{a*} J_a U_0^a(t)\varphi.$$

Writing $W_a = \sum_{b\subseteq a} V_b$, one has:

$$U^a(t)\psi^a = J_a(x^a)U_0^a(t)\varphi + i\int_t^\infty U^a(t-s)\{W_a J_a + [-\Delta_{x^a}, J_a]\}U_0^a(s)\varphi ds.$$

Since $\hat{\varphi}$ is of compact support, it is clear that for any $N > 0$ there is $M_0 > 0$ such that

$$\|\chi(\frac{|x^a|}{M\langle t\rangle})U_0^a(t)\varphi\| \leq C_N \langle t\rangle^{-N}, \quad t > 0,$$

uniformly in $M > M_0$. Since $J_a W_a = O(\langle x^a\rangle^{-\rho})$ and $[\Delta_{x^a}, J_a] = 0$ for $\pm x^a$ in a conic neighbourhood of supp$\hat{\varphi}$, using propagation estimates for the free propagator $U_0^a(t)$, one has:

$$\|W_a J_a U_0^a(s)\varphi\| \leq C\langle s\rangle^{-\rho},$$

$$\|[-\Delta_{x^a}, J_a]U_0^a(s)\varphi\| \leq C_N\langle s\rangle^{-N}, \quad \forall N > 0.$$

This proves the Lemma. $\square$

Lemma 2.2 is a kind of maximal velocity estimate. If we can prove $\langle x^a\rangle^s \psi^a \in L^2$ for some $s > \rho - 1$, it follows from the known results proved in [6].

Proof of Theorem 2.1. Recall first that by the choice of ω, the support of $\hat{f}_n(\cdot) = \hat{f}(\cdot - n\omega)$ is contained in the set $\{\xi_a; |\xi_a^b| > \delta n, \quad \forall b \not\subseteq a\}$ for some $\delta > 0$. So by the minimal velocity estimate, we have for some $c' > 0$ and all $b \not\subseteq a$

$$\|\chi_-(|x_a^b|/(c'nt))e^{it\Delta_{x_a}} f_n\| \leq C_N\langle t\rangle^{-N}, \tag{2.7}$$

uniformly in n. Here χ_- is a cut off function with $\chi_-(t) = 1$, if $t \leq 1$; 0 if $t > 2$. Put $\chi_+ = 1 - \chi_-$. By Lemma 2.2, one has for $n >> 1$,

$$W(\varphi \otimes f_n) = \lim_{t\to\infty} U(t)^* B_n(t) U_a(t)(\psi^a \otimes f_n),$$

where

$$B_n(t) = \chi_-(|x^a|/(\sqrt{n}\langle t\rangle))J_a(x_a, t)$$

with $J_a(x_a, t) = \prod_{b\not\subseteq a} \chi_+(|x_a^b|/(c_1 nt))$ for $t > 0$ and $c_1 = c'/2$. This gives

$$\begin{aligned} W(\varphi\otimes f_n) &= \chi_-(|x^a|/\sqrt{n})(\psi^a \otimes f_n) \\ &+ \int_0^\infty U(t)^*\{iI_a(x)B_n(t) + \frac{d}{dt}B_n(t) + i[-\Delta, B_n(t)]\}U_a(t)(\psi^a\otimes f_n)dt. \end{aligned} \tag{2.8}$$

Here $I_a(x) = \sum_{b\not\subseteq a} V_b(x^b)$. It is clear that $\lim_{n\to\infty}\{\chi_-(|x^a|/\sqrt{n})\psi^a \otimes f_n - \psi^a \otimes f_n\} = 0$ in $L^2(\mathbf{X})$. For each fixed $t > 0$, by the choice of $B_n(t)$, it can be easily checked that

$$\|\{iI_a(x)B_n(t) + \frac{d}{dt}B_n(t) + i[-\Delta, B_n(t)]\}U_a(t)(\psi^a \otimes f_n)\| \to 0, \text{ as } n \to \infty.$$

Therefore to prove

$$\lim_{n\to\infty}\int_0^\infty \|\{iI_a(x)B_n(t) + \frac{d}{dt}B_n(t) + i[-\Delta, B_n(t)]\}U_a(t)(\psi^a \otimes f_n)\|dt = 0,$$

it suffices to prove that

$$\|\{iI_a(x)B_n(t) + \frac{d}{dt}B_n(t) + i[-\Delta, B_n(t)]\}U_a(t)(\psi^a \otimes f_n)\| \le C\langle t\rangle^{-s}, \tag{2.9}$$

for some $s > 1$ uniformly in n. For $t > 1$, $x \in \text{supp } B_n(t)$ and $b \not\subseteq a$, we have: $|x^b| \ge c'nt - 2\sqrt{n}\langle t\rangle \ge c'nt/2$ for n large enough. This gives $|I_a(x)B_n(t)| \le C(nt)^{-\rho}$ uniformly in x and therefore

$$\|I_a(x)B_n(t)U_a(t)(\psi^a \otimes f_n)\| \le C\langle t\rangle^{-\rho},$$

uniformly in $n >> 1$. Write now

$$\begin{aligned}
&\frac{d}{dt}B_n(t) + i[-\Delta, B_n(t)] \\
=\ & (\frac{d}{dt}\chi_-(|x^a|/(\sqrt{n}\langle t\rangle)) + i[-\Delta, \chi_-(|x^a|/(\sqrt{n}\langle t\rangle))])J_a(x_a, t) \\
+\ & \chi_-(|x^a|/(\sqrt{n}\langle t\rangle))(\frac{d}{dt}J_a(x_a, t) + i[-\Delta, J_a(x_a, t)]) \\
\equiv\ & I_1(t) + I_2(t).
\end{aligned}$$

The derivatives of $\chi_-(|x^a|/(\sqrt{n}\langle t\rangle))$ are of the order $O(t^{-1})$ uniformly in x^a and n and are supported in the region $\{|x^a| \ge \sqrt{n}\langle t\rangle\}$. Consequently for $n >> 1$, we can apply Lemma 2.2 and conclude

$$\|I_1(t)U_a(t)(\psi^a \otimes f_n)\| \le C\langle t\rangle^{-\rho}, \quad \rho > 1,$$

uniformly in n. Remark that the support of the derivatives of $J_a(x_a, t)$ is contained in the set $\cup_{b\not\subseteq a}\{x_a; |\pi^b x_a| \le c'nt\}$. So we can apply (2.7) to conclude that

$$\|I_2(t)U_a(t)(\psi^a \otimes f_n)\| \le C_N\langle t\rangle^{-N},$$

for any $N > 0$, uniformly in n. This proves (2.9) and hence the limit (2.5). $\square$

Corollary 2.3 *$\forall a \in \mathcal{A}$, $a \ne a_{\max}$, let f_n be constructed as in Theorem 2.1. Put*

$$S^{(n)}\varphi =< S(\varphi \otimes f_n), f_n >_a, \forall \varphi \in L^2(\mathbf{X}^a). \tag{2.10}$$

Then $s-\lim_{n\to\infty} S^{(n)} = S^a$ on $L^2(\mathbf{X}^a)$.

Proof. Applying Theorem 2.1 for $W_\pm^{(n)}$, we have for any $\varphi \in L^2(\mathbf{X}^a)$: $S_n\varphi =< W_+^*(W_-(\varphi \otimes f_n)), f_n >_a =< W_+^*(W_-^a\varphi) \otimes f_n), f_n >_a + \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$. Put $\psi^a = W_-^a\varphi$. Then,

$$\|W_+^*(\psi^a \otimes f_n) - (W_+^{a*}\psi^a \otimes f_n\|$$

$$\leq \|\psi^a \otimes f_n - W_+(W_+^{a*}\psi^a) \otimes f_n)\| \to 0,$$

as $n \to \infty$. Here we used the equality $W_+^a W_+^{a*}\psi^a = \psi^a$ since ψ^a is in the continuous spectral subspace of P^a. This proves $\|S_n\varphi - S^a\varphi\| \to 0$ for any $\varphi \in L^2(\mathbf{X}^a)$. □

Corollary 2.3 shows that for all $a \in \mathcal{A}$, $a \neq a_{\max}$, the free channel scattering operator S^a for the sub-Hamiltonian P^a can be determined by S at high energies. For physical N-body systems P of the form $P = -\Delta + \sum_{1\leq i<j\leq N} V_{ij}(x^{ij})$ with mass center removed, there are only pair interactions. In particular, if we take $a \in \mathcal{A}$ to be the $(N-1)$-cluster decomposition in which the particles i and j belong to a same cluster, then, $P^a = -\Delta_{x^{ij}} + V_{ij}(x^{ij})$ is a two-body Schrödinger operator. In this case we can apply known results in inverse scattering for two-body Schrödinger operator to conclude that V_{ij} can be determined by S^a at high energies. See, for example, [1, 5] and the references therein and also a self-contained proof in Sect. 4 if the potentials are even. This means that for systems including only pair interactions between particles, Corollary 2.3 already shows that all potentials are uniquely determined by the free channel scattering operator S at high energies. If interactions between three or more particles are present, the inverse scattering becomes more complicated. In the following we shall study high energy asymptotics of the free channel scattering matrix to recover all potentials.

3 Representation of Scattering Matrix

In this Section, we shall give a spectral representation formula for the scattering matrix $S(\lambda)$ associated with the free channel - free channel scattering operator S. This kind of representations was also studied in [10]. Our method is based on microlocal resolvent estimates established in [8] and the high energy estimates obtained there play an important role in this work. Therefore let us first recall some results from [8].

Let P be a generalizd N-body Schrödinger operator : $P = -\Delta + \Sigma_{a\in\mathcal{A}} V_a(x^a)$. We write formally $V_a^0(x^a) = V_a(x^a)$ and $V_a^j(x^a) = (x^a \cdot \nabla^a)V_a^{j-1}(x^a)$, for $j = 1, 2, \cdots$. Assume that the following conditions are satisfied for some $n \in \mathbf{N}$ with $n \geq 2$:

$$\forall a \in \mathcal{A}, \forall 0 \leq j \leq n, V_a^j(\cdot) \text{ is relatively compact in } L^2(\mathbf{X}^a) \text{ with respect to } -\Delta^a. \tag{3.1}$$

For $\lambda \in \mathbf{R}$, put $d(\lambda) = dist(\lambda, \mathcal{T})$. $d(\lambda) = \lambda$ if $\lambda > 0$. We denote by $S_\pm(\lambda)$ the classes of λ-dependent symbols $b_\pm \in C^\infty(T^*\mathbf{X})$ satisfying

$$\exists \epsilon > 0, d' > 0 \text{ such that supp } b_\pm \subset \{(x,\xi); \pm x\cdot\xi > (-1+\epsilon)|x|d(\lambda)^{1/2} \text{ and } |\xi| > d'\} \tag{3.2}$$

$$\text{and } |\partial_x^\alpha \partial_\xi^\beta b_\pm(x,\xi)| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}, \forall\alpha,\ \beta \in \mathbf{N}^d,$$

uniformly in λ. We shall say that $b_+ \in S_+(\lambda), b_- \in S_-(\lambda)$ are a pair of symbols with the property of disjoint support, if supp$b_\pm \subset \{(x,\xi); \pm x\cdot\xi > \pm\delta_\pm|x|d(\lambda)^{1/2}\}$ for some $-1 \leq$

$\delta_- < \delta_+ \leq 1$. For a symbol $p(x,\xi)$, we denote by $p(x,D)$ the corresponding pseudodifferential operator defined by

$$p(x,D)u(x) = \frac{1}{(2\pi)^d}\int\int e^{i(x-y)\cdot\xi}p(x,\xi)u(y)dyd\xi.$$

For $\eta > 0$, we denote $\Omega_0(\eta) = \{x \in \mathbf{X}; |x^b| > \eta|x|, \forall b \in \mathcal{A}, b \neq a_{\min}\}$.

Theorem 3.1 (i). *Assume that the potentials V_a satisfy the conditions (3.1) for some $n \in \mathbf{N}$, $n \geq 2$. For $a \in \mathcal{A}$, let J_0 be a bounded function supported in $\Omega_0(\eta)$ for some $\eta > 0$. Let $b_\pm \in S_\pm(\lambda)$. Then, for any $l \in \mathbf{N}$ with $1 \leq l \leq n-1$, $\epsilon > 0$, $l - 1/2 < s < n-1$, one has:*

$$\|\langle x\rangle^{s-l}J_0(x)b_\mp(x,D)\frac{d^{l-1}}{d\lambda^{l-1}}R(\lambda \pm i0)\langle x\rangle^{-s}\| \leq C\langle\lambda\rangle^{-l/2},\ \forall\lambda \notin \mathcal{T}^\varepsilon. \tag{3.3}$$

Here $\mathcal{T}^\varepsilon$ denotes an $\varepsilon-$ neighbourhood of $\mathcal{T}$.
(ii). *Assume that V_a, $a \in \mathcal{A}$, satisfy the conditions (3.1) for some $n \in \mathbf{N}$ with $n \geq 3$. Let $b_\pm$, J_0 be the same as above. Assume in addition that (b_-, b_+) is a pair of symbols with the property of disjoint support. Then, for any $l \in \mathbf{N}$ with $1 \leq l \leq n-2$, $\epsilon > 0$, $s, r \in \mathbf{R}$ with $s_+ + r_+ < n - l - 1$, there exists $C > 0$, such that*

$$\|\langle x\rangle^{s}J_0(x)b_\mp(x,D)\frac{d^{l-1}}{d\lambda^{l-1}}R(\lambda \pm i0)b_\pm(x,D)J_0(x)\langle x\rangle^{r}\| \leq C\langle\lambda\rangle^{-l/2},\ \lambda \notin \mathcal{T}^\varepsilon. \tag{3.4}$$

For the proof of Theorem 3.1, we refer to [8] Theorem 2.1. In this paper, we shall only use (i) of Theorem 3.1 for $l = 1$ and (ii) of Theorem 3.1 for the operator $P_0 = -\Delta$. The following consequence of Theorem 3.1 will also be needed.

Corollary 3.2 *Let J_0 be chosen as in Theorem 3.1. Let $b_j \in S_+(\lambda) \cap S_-(\lambda), j = 1, 2$. Then one has:*

$$\|\langle x\rangle^{-1/2}J_0(x)b_1(x,D)R(\lambda \pm i0)J_0(x)b_2(x,D)\langle x\rangle^{-1/2}\| \leq C\langle\lambda\rangle^{-1/2},\ \forall\lambda \notin \mathcal{T}^\varepsilon. \tag{3.5}$$

To show (3.5), it suffices to apply (3.3) with $l = 1$ and an argument of interpolation.

Now let us study the spectral representation of the scattering matrices for

$$T = \frac{i}{2\pi}(S-1).$$

Let $F : L^2(\mathbf{X}) \to \mathbf{H} \equiv L^2(\mathbf{R}_+; L^2(\mathbf{S}^{d-1}))$ be defined by:

$$(Ff)(\lambda,\theta) = c(\lambda)\int_{\mathbf{R}^d} e^{-i\sqrt{\lambda}\,\theta\cdot x}f(x)dx, \quad \lambda > 0, \quad \theta \in \mathbf{S}^{d-1}, \tag{3.6}$$

where

$$c(\lambda) = 2^{-1/2}(2\pi)^{-d/2}\lambda^{(d-2)/4},$$

with $d = \dim\mathbf{X}$. Then we can verify that $\|Ff\|_{\mathbf{H}} = \|f\|$. By the Sobolev's lemma, F defines a family of maps, $F(\lambda), \lambda \in \mathbf{R}_+$, from $L^{2,s}(\mathbf{X}), s > 1/2$, to $L^2(\mathbf{S}^{d-1})$:

$$(F(\lambda)f)(\theta) = (Ff)(\lambda,\theta),$$

Here $L^{2,s}$ is the weighted L^2 space $L^{2,s}(\mathbf{X}) = L^2(\mathbf{X}, \langle x \rangle^{2s} dx)$.

Then $F(-\Delta)F^*$ can be represented as the multiplication by λ in $\mathbf{H}$ and FTF^* can be represented by a family of operators $\{T(\lambda) = F(\lambda)TF^*(\lambda); \quad \lambda \in \mathbf{R}_+\}$ mapping $L^2(\mathbf{S}^{d-1})$ to $L^2(\mathbf{S}^{d-1})$. Note that $T(\lambda)$ is a priori only defined a.e. in λ. To give a pointwise meaning to $T(\lambda)$, we introduce appropriate cut-offs to avoid bad directions in momentum space.

Let $Y = \mathbf{X} \setminus \cup_{b \neq a_{\min}} \mathbf{X}_b$ and $\Sigma_0 = Y \cap \mathbf{S}^{d-1}$. Let $\chi_j(\xi), j = 1, 2$, be a bounded symbol with its conic support contained in Y. Instead of looking for the spectral representation for T, we consider the localized operator $\chi_1(D)T\chi_2(D)$.

To apply Theorem 3.1, we take $j \in C_0^\infty(\mathbf{R})$ with $j(t) = 0$ if $t < 1/2$ and $j(t) = 1$ if $t \geq 1$. Put:

$$j_0(x) = \prod_{c \in \mathcal{A}, c \neq a_{\min}} j(\frac{|x^c|}{\delta |x|}),$$

and

$$J_0(x) = j_0(x) j(|x|) + (1 - j(|x|)). \tag{3.7}$$

As in [9], we can check that for $\delta > 0$ small enough, $J_0(x)$ is equal to 1 for x in a conic neighbourhood of $\mathrm{supp}\chi_j(\cdot)$. Consequently, one has:

$$|\hat{x} \cdot \hat{\xi}| \leq 1 - \varepsilon_1, \quad \varepsilon_1 > 0, \tag{3.8}$$

for (x, ξ) in the support of $\nabla J_0(x)\chi_k(\xi)$, $k = 1, 2$. Here $\hat{x} = x/|x|$ and $\hat{\xi} = \xi/|\xi|$.

Theorem 3.3 *Assume the condition (1.2) for some $\rho > 1$. For any $f_j \in \mathcal{S}(\mathbf{X}), j = 1, 2$, we denote: $f_j(\lambda, \theta) = (Ff_j)(\lambda, \theta)$. Let χ_j, J_0 be chosen as above. Then for any $f_j \in \mathcal{S}(\mathbf{X})$ such that $f_j(\lambda, \cdot) = 0$ for λnear 0, one has:*

$$< \chi_1(D)T\chi_2(D)f_2, f_1 >= \int (\tilde{T}(\lambda) f_b(\lambda, .), f_a(\lambda, .)) d\lambda, \tag{3.9}$$

where

$$\tilde{T}(\lambda) = F(\lambda)\{\chi_2(D)J_0(x)Q_1 - Q_2^* R(\lambda + i0)Q_1\}F(\lambda)^*, \tag{3.10}$$

for $\lambda > 0$. Here Q_k is defined by

$$Q_k = \{V(x)J_0(x) + [-\Delta, J_0]\}\chi_k(D). \tag{3.11}$$

In addition, the localized scattering matrix $\tilde{T}(\lambda) = F(\lambda)\chi_1(D)T\chi_2(D)F(\lambda)^*$ is strongly continuous for $\lambda > 0$: $\forall u \in L^2(\mathbf{S}^{d-1})$, $\lambda \to \tilde{T}(\lambda)u$ is continuous in $L^2(\mathbf{S}^{d-1})$.

We only sketch the proof of Theorem 3.3. (3.9) can be established just as Proposition 3.2 in [9]. The details are omitted. To see that $\tilde{T}(\lambda)$ defined by (3.10) is bounded on $L^2(\mathbf{S}^{d-1})$, we apply Theorem 3.1 and Corollary 3.2. Let $\chi(\cdot)$ be a smooth function on $\mathbf{R}$ with $\chi(t) = 1$ near $t = 1$ and 0 for t outside a small neighbourhood of 1. Then we can write: $J_0\chi(P/\lambda) = J_0\chi(-\Delta/\lambda) + R_1$ with $\|\langle x \rangle^r R_1 \langle x \rangle^s\| \leq C$ uniformly in λ for any $s + t \leq \rho$. (See Appendix in [9]). Now

$$\begin{aligned} & F(\lambda)Q_2^* R(\lambda + i0)Q_1 F(\lambda)^* \qquad (3.12) \\ = \ & F(\lambda)Q_2^* R(\lambda + i0)(1 - \chi(P/\lambda)^2)Q_1 F(\lambda)^* + F(\lambda)Q_2^* \chi(P/\lambda)R(\lambda + i0)\chi(P/\lambda)Q_1\}F(\lambda)^*. \end{aligned}$$

The first term on the right hand side of the above equality is evidently bounded on $L^2(\mathbf{S}^{d-1})$. Since

$$\chi(P/\lambda)Q_j = \chi(-\Delta/\lambda)[-\Delta, J_0]\chi_j(D) + O(\langle x\rangle^{-\rho}), \rho > 1,$$

and $\langle x\rangle[-\Delta, J_0]\chi_j(D)\chi(-\Delta/\lambda)$ is pseudodifferential operator with symbol in $S_+(\lambda)\cap S_-(\lambda)$ (if we choose χ with suppχ sufficiently near $\{1\}$), applying (3.3) and (3.5) (and the same results for the free resolvent), one sees that the second term in (3.12) is also bounded. This proves that $\tilde{T}(\lambda)$ is bounded on $L^2(\mathbf{S}^{d-1})$ (locally uniformly in $\lambda > 0$). To see the strong continuity of $\tilde{T}(\lambda)$, we just need to check it for u in a dense subset of $L^2(\mathbf{S}^{d-1})$, say, for $u \in C^\infty(\mathbf{S}^{d-1})$. We need the following lemma which follows from Theorem of Stationary Phase.

Lemma 3.4 *Let $\varphi \in C^\infty(\mathbf{S}^{d-1})$. Then for $x \neq 0$, $\lambda \to \infty$, one has:*

$$\begin{aligned}(F(\lambda)^*\varphi)(x) &= 2^{-1/2}(2\pi)^{-1/2}\lambda^{-1/4}|x|^{(1-d)/2} \\ &\times \{e^{i(\sqrt{\lambda}|x|+\frac{(d-1)\pi}{4})}(\varphi(\hat{x}) + r_+(x,\lambda)) + e^{-i(\sqrt{\lambda}|x|+\frac{(d-1)\pi}{4})}(\varphi(\hat{-x}) + r_-(x,\lambda))\}.\end{aligned} \tag{3.13}$$

Here $r_\pm$ are smooth functions having an asymptotic expansion of the form

$$r_\pm(x,\lambda) \sim \sum_{j=1}^{\infty} \lambda^{-j/2}|x|^{-j}a_{j,\pm}(x), \lambda^{1/2}|x| >> 1,$$

where $|a_{j,\pm}(x)| \leq C_j$ uniformly in x and $a_{j,\pm}(x) = 0$ for all $j \geq 1$ if $\varphi(\omega) = 0$ for ω near $\pm\hat{x}$.

Since now $[-\Delta, J_0] = 0$ for $\pm\hat{x}$ near the support of $\chi_2(\sqrt{\lambda}\cdot)u(\cdot)$, by Lemma 3.4 $[-\Delta, J_0]\chi_2(D)F(\lambda)^*u$ is rapidly decreasing in $|x|$ and $|\lambda|$. This shows that $\langle x\rangle^s Q_2\mathcal{J}_\beta F_\beta(\lambda)^*u$ is in $L^2(\mathbf{X})$ for some $s > 1/2$. Therefore, the continuity of $\lambda \to \tilde{T}(\lambda)u \in L^2(\mathbf{S}^{d-1})$ follows from that of $\lambda \to \langle x\rangle^{-s}R(\lambda \pm i0)\langle x\rangle^{-s}$ and of $\lambda \to \langle x\rangle^{s-1-\varepsilon}J_0(x)b_\mp(x,D)R(\lambda \pm i0)\langle x\rangle^{-s}$ for any $s > 1/2$ and $\varepsilon > 0$.

4 Born's Formula for N-body Problems

As a first application of Theorems 3.1 and 3.3, we show how one can easily obtain the high energy asymptotics of the scattering matrix $S(\lambda)$ in a week sense. The results in this Section allow us to recover the sum of all potentials if they are all even. Together with Corollary 2.3 (or by the argument of Remark 1.1), this enables us to prove the uniqueness of inverse scattering at high energies for generalized N-body problems.

Theorem 4.1 *Let $u_j \in C_0^\infty(\Sigma_0)$. Under the assumption (1.2), one has:*

$$(T(\lambda)u_1, u_2) = (K(\lambda)u_1, u_2) + O(\lambda^{-1}), \tag{4.1}$$

as $\lambda \to \infty$. Here $K(\lambda)$ is an operator whose distribution kernel is given by

$$K(\lambda; \theta, \omega) = c(\lambda)^2 \int e^{i\sqrt{\lambda}(\theta-\omega)\cdot x}V(x)dx$$

with $V(x) = \sum_{a\in\mathcal{A}} V_a(x^a)$ and $c(\lambda) = 2^{-1/2}(2\pi)^{-d/2}\lambda^{(d-2)/4}$.

(4.1) is a kind of Born's formula in weak sense for N-body systems and it is formally completely similar to the Born's formula in 2-body problems (see [4]). To see that (4.1) really gives the leading term of $(T(\lambda)u_1, u_2)$ as $\lambda \to \infty$, we indicate the following consequence of Theorem 4.1.

Corollary 4.2 *Under the assumptions of Theorem 4.1, suppose in addition that* $\mathbf{R}_+ \ni r \to V_a(r\omega^a)$ *is* C^1 *for any* $\omega^a \in \mathbf{S}^a$ *and*

$$|\hat{x}^a \cdot \nabla_{x^a} V_a(x^a)| \leq C\langle x^a\rangle^{-\rho}, \forall x^a, \tag{4.2}$$

for some $\rho > 1$*. One has for any* $u_j \in C_0^\infty(\Sigma_0)$*,*

$$(T(\lambda)u_1, u_2) = \frac{1}{4\pi\sqrt{\lambda}} \int_{\mathbf{X}} \frac{V(x) + V(-x)}{|x|^{d-1}} u_1(\hat{x})\overline{u_2(\hat{x})}dx + O(\lambda^{-1/2-1/(2(d+1))}), \tag{4.3}$$

as $\lambda \to \infty$*.*

Proof. We look at the leading term in (4.1):

$$(K(\lambda)u_1, u_2) =< V(x)F(\lambda)^*u_1, F(\lambda)^*u_2 > .$$

For $\sqrt{\lambda}|x| > \lambda^\varepsilon (\varepsilon > 0$ will be determined later), we can apply Lemma 3.4 to compute everything up to an error of the order $O(\lambda^{-\varepsilon})$. By the assumption (4.2), one can show that

$$\int_{|x|>\lambda^{-1/2+\varepsilon}} \frac{V(x)}{|x|^{d-1}} e^{i2\sqrt{\lambda}|x|} u_1(\hat{x})\overline{u_2(-\hat{x})}dx = O(\lambda^{-1/2}). \tag{4.4}$$

This gives, for $\varepsilon \leq 1/2$,

$$\begin{aligned}
&\int_{|x|>\lambda^{-1/2+\varepsilon}} V(x)F(\lambda)^*u_1\overline{F(\lambda)^*u_2}dx \\
= \;& \frac{1}{4\pi\sqrt{\lambda}} \int_{|x|>\lambda^{-1/2+\varepsilon}} \frac{V(x)}{|x|^{d-1}} \{u_1(\hat{x})\overline{u_2(\hat{x})} + u_1(-\hat{x})\overline{u_2(-\hat{x})}\}dx + O(\lambda^{-1/2-\varepsilon}) \\
= \;& \frac{1}{4\pi\sqrt{\lambda}} \int_{|x|>\lambda^{-1/2+\varepsilon}} \frac{V(x)+V(-x)}{|x|^{d-1}} u_1(\hat{x})\overline{u_2(\hat{x})}dx + O(\lambda^{-1/2-\varepsilon}) \\
= \;& \frac{1}{4\pi\sqrt{\lambda}} \int_{\mathbf{X}} \frac{V(x)+V(-x)}{|x|^{d-1}} u_1(\hat{x})\overline{u_2(\hat{x})}dx + O(\lambda^{-1/2-\varepsilon}).
\end{aligned}$$

In the last equality we have used the estimate:

$$\frac{1}{4\pi\sqrt{\lambda}} \int_{|x|\leq\lambda^{-1/2+\varepsilon}} \frac{V(x)+V(-x)}{|x|^{d-1}} u_1(\hat{x})\overline{u_2(\hat{x})}dx = O(\lambda^{-1+\varepsilon}) = O(\lambda^{-1/2+\varepsilon}),$$

which follows from the boundedness of V.

For $\sqrt{\lambda}|x| \leq \lambda^\varepsilon$, we use again the boundedness of potentials and the estimate: $|F(\lambda)^*u_j(x)| \leq C\lambda^{(d-2)/4}$ uniformly in x. This shows

$$|\int_{|x|\leq\lambda^{-1/2+\varepsilon}} V(x)F(\lambda)^*u_1\overline{F(\lambda)^*u_2}dx| \leq C\lambda^{(d-2)/2} \int_{|x|\leq\lambda^{-1/2+\varepsilon}} dx \leq C'\lambda^{-1+d\varepsilon}.$$

(4.3) is proved if we take $\varepsilon = \frac{1}{2(d+1)}$. $\square$

Remarks 4.1 (i). Without the additional assumption (4.2), one still has:

$$(T(\lambda)u_1, u_2) = \frac{1}{4\pi\sqrt{\lambda}} \int_{\mathbf{X}} \frac{V(x)+V(-x)}{|x|^{d-1}} u_1(\hat{x})\overline{u_2(\hat{x})}dx + o(\lambda^{-1/2}). \tag{4.5}$$

In fact, instead of (4.4), we have under the assumption (1.2) that

$$\lim_{\lambda\to\infty} \int_{|x|>\lambda^{-1/2+\varepsilon}} \frac{V(x)}{|x|^{d-1}} e^{i2\sqrt{\lambda}|x|} u_1(\hat{x})\overline{u_2(-\hat{x})}dx = 0.$$

(ii). If we take $u_1 = u_2$ with $u_1 \in C_0^\infty(\Sigma_0)$, we have:

$$(T(\lambda)u_1, u_1) = \frac{1}{4\pi\sqrt{\lambda}}\{\int_{\mathbf{X}} \frac{V(x)+V(-x)}{|x|^{d-1}} |u_1(\hat{x})|^2 dx + o(1)\}.$$

So in this case, if $V(x) \geq 0$ (or $V(x) \leq 0$) $\forall x \in \mathbf{X}$ and $V(x) \not\equiv 0$, the leading term in (4.1) is of the order $C(u_1)\lambda^{-1/2}$ with $C(u_1) \neq 0$.

The proof of Theorem 4.1 is based on the representation formula for $S(\lambda)$ given in Theorem 3.3. For given $u_j \in C_0^\infty(\Sigma_0)$, we take $\chi_j \in C_0^\infty(Y \setminus 0)$ such that $\chi_j(\xi) = 1$ for ξ in a small neighbourhood of $\{\sqrt{\lambda}\} \times supp u_j$ and that

$$|\partial_\xi^\alpha \chi_j(\xi)| \leq C_\alpha \lambda^{-|\alpha|/2}, \quad \forall \alpha \in \mathbf{N}^d.$$

Since $\chi_j(D)F(\lambda)^* u_j = F(\lambda)^* u_j$, we have:

$$(T(\lambda)u_1, u_2) = (F(\lambda)\chi_1(D)T\chi_2(D)F(\lambda)^* u_1, u_2) = (\tilde{T}(\lambda)u_1, u_2), \tag{4.6}$$

where $\tilde{T}(\lambda)$ is given by the equation (3.10). In particular, $\lambda \to (T(\lambda)u_1, u_2)$ is continuous for $\lambda > 0$.

Lemma 4.3 *Let Q_j be defined by (3.11).Then one has:*

$$(F(\lambda)Q_2^* R(\lambda + i0)Q_1 F(\lambda)^* u_1, u_2) = O(\lambda^{-1}),$$

as $\lambda \to \infty$.

Proof. Recall that $Q_j = \{V(x)J_0(x) + [-\Delta, J_0(x)]\}\chi_j(D)$ and by the choice of J_0, $\nabla J_0(x)\chi_j(\xi)$ is supported in the region $\{|x\cdot\xi| \leq (1-\delta)\sqrt{\lambda}|x|\}$ for some $\delta > 0$ and $[-\Delta, J_0] = 0$ for $\pm\hat{x}$ in a small neighbourhood of $u_j(\cdot)$, $j = 1, 2$. Making use of (3.3) (with $l = 1$), Lemma 3.4 and the short range assumption on V_a, one has

$$|(F(\lambda)Q_2^* R(\lambda + i0)Q_1 F(\lambda)^* u_1, u_2)| \leq C\lambda^{-1/2} \|\langle x\rangle^{-s} F(\lambda)^*\|^2_{\mathcal{L}(L^2(\mathbf{S}^{d-1}); L^2(\mathbf{R}^d))} + O(\lambda^{-\infty}).$$

Here $s = \rho/2 > 1/2$. To estimate $\|\langle x\rangle^{-s} F(\lambda)^*\|^2_{\mathcal{L}(L^2(\mathbf{S}^{d-1}); L^2(\mathbf{R}^d))}$, we use the Stone's formula:

$$\langle x\rangle^{-s} F(\lambda)^* F(\lambda)\langle x\rangle^{-s} = \frac{1}{2\pi i}\langle x\rangle^{-s}\{R_0(\lambda + i0) - R_0(\lambda - i0)\}\langle x\rangle^{-s},$$

where $R_0(z) = (-\Delta - z)^{-1}$. It follows from the high energy estimate for the free resolvent that

$$\|\langle x\rangle^{-s}F(\lambda)^*\|^2_{\mathcal{L}(L^2(\mathbf{S}^{d-1});L^2(\mathbf{R}^d))} = \|\langle x\rangle^{-s}F(\lambda)^*F(\lambda)\langle x\rangle^{-s}\|$$
$$\leq \frac{1}{\pi}\|\langle x\rangle^{-s}R_0(\lambda+i0)\langle x\rangle^{-s}\| \leq C\lambda^{-1/2},$$

for $\lambda > 1$. This proves the lemma. □

Proof of Theorem 4.1. Lemma 4.3 gives

$$(T(\lambda)u_1, u_2) = (F(\lambda)\chi_2(D)J_0(x)Q_1F(\lambda)^*u_1, u_2) + O(\lambda^{-1}), \text{ as } \lambda \to \infty. \tag{4.7}$$

By the choice of χ_j, one has:

$$\begin{aligned}
&(F(\lambda)\chi_2(D)J_0(x)Q_1F(\lambda)^*u_1, u_2)\\
&= (F(\lambda)\{J_0(x)^2V(x) + J_0[-\Delta, J_0]\}F(\lambda)^*u_1, u_2)\\
&= (F(\lambda)V(x)F(\lambda)^*u_1, u_2) + (F(\lambda)\{(J_0(x)^2-1)V(x) + J_0[-\Delta, J_0]\}F(\lambda)^*u_1, u_2).
\end{aligned}$$

Since now $1 - J_0^2$ and $[-\Delta, J_0]$ vanish for $|x| \leq 1/2$ or $\pm\hat{x}$ in a neighbourhood of supp u_j, we can again apply Lemma 3.4 to show that the contributions from these terms are $O(\lambda^{-\infty})$. Consequently, we obtain:

$$(T(\lambda)u_1, u_2) = (F(\lambda)V(x)F(\lambda)^*u_1, u_2) + O(\lambda^{-1}).$$

This finishes the proof of Theorem 4.1. □

To reconstruct the sum of potentials from the high energy asymptotics of the scattering matrix, we need a parameter dependent version of Theorem 4.1.

Theorem 4.4 *Let $u_j \in C_0^\infty(\Sigma_0)$. For any $x_0 \in \mathbf{X}$, put*

$$u_j(\omega; \lambda, x_0) = e^{i\sqrt{\lambda}x_0\cdot\omega}u_j(\omega).$$

Under the condition of Theorem 4.1, one has:

$$\begin{aligned}
&(T(\lambda)u_1(\cdot;\lambda,x_0)), u_2(\cdot;\lambda,x_0)\\
&= \frac{1}{4\pi\sqrt{\lambda}}\int_{\mathbf{X}}\frac{V(x+x_0)+V(-x-x_0)}{|x|^{d-1}}u_1(\hat{x})\overline{u_2(\hat{x})}dx + o(\lambda^{-1/2}),
\end{aligned} \tag{4.8}$$

as $\lambda \to \infty$ locally uniformly in $x_0 \in \mathbf{X}$.

Proof. It follows the same line as the proof of (4.5), taking notice that $(F(\lambda)^*u_j(\cdot;\lambda,x_0))(x) = (F(\lambda)^*u_j(\cdot))(x-x_0)$. □

For each $\omega_0 \in \Sigma_0$, let u_{m,ω_0} be a family of smooth functions on $\mathbf{S}^{d-1}$, supported in $\{|\omega-\omega_0| \leq 1/m\}$ and $\|u_{m,\omega_0}\| = 1$. Applying Theorem 4.4 with $u_1 = u_2 = u_{m,\omega_0}$, we can derive that

$$\lim_{m\to\infty}\lim_{\lambda\to\infty} 4\pi\sqrt{\lambda}(T(\lambda)u_{m,\omega_0}(\cdot;\lambda,x_0), u_{m,\omega_0}(\cdot;\lambda,x_0)) = F(x_0,\omega_0) \tag{4.9}$$

$a.e.$ in $\omega_0 \in \Sigma_0$ and $x_0 \in \mathbf{X}$. Here F is defined by

$$F(x_0, \omega_0) = \int_{\mathbf{R}_+} (V(r\omega_0 + x_0) + V(-r\omega_0 - x_0))dr \tag{4.10}$$

for any $\omega_0 \in \Sigma_0$ and $x_0 \in \mathbf{X}$.

Take $x_0 = s\omega_0$ with $\omega_0 \in \Sigma_0$. One has:

$$F(x_0, \omega_0) = \int_s^\infty (V(r\omega_0) + V(-r\omega_0)dr.$$

Since $\cup_{a \neq a_{\min}} \mathbf{X}_a$ is of measure 0 in $\mathbf{X}$, it follows that

$$\frac{d}{ds} F(s\omega_0, \omega_0) = -(V(x_0) + V(-x_0)), \quad a.e. \text{ in } x_0 \in \mathbf{X}. \tag{4.11}$$

(4.11) uniquely determines $V(x) + V(-x)$, because V is a measurable function on $\mathbf{X}$. Therefore, we have proved the following

Theorem 4.5 *Let $V(x) = \sum_{a \in \mathcal{A}} V_a(x^a)$. Under the condtions of Theorem 4.1 , assume in addition that V_a, $\forall a \in \mathcal{A}$, is even: $V_a(x^a) = V_a(-x^a)$. Then $V(\cdot)$ (and therefore, by the argument of Remarks 1.1 (2), all $V_a(\cdot)$) can be reconstructed from the high energy asymptotics of the free channel - free channel scattering matrix $S(\lambda)$.*

5 Uniqueness of Inverse Scattering at High Energies

To reconstruct all potentials from the scattering matrix $S(\lambda)$ at high energies without the assumption of evenness of V, we study the high energy asymptoics of $(T(\lambda)u_1, u_2)$ with u_j some test functions concentrated near a point $\omega \in \Sigma_0$. Roughly speaking, Theorem 4.1 gives

$$(T(\lambda)u_1, u_2) \sim c(\lambda)^2 \int_{\mathbf{S}^{d-1}_{\omega'}} \int_{\mathbf{S}^{d-1}_{\theta}} \hat{V}(\sqrt{\lambda}(\theta - \omega'))u_1(\omega')\overline{u_2(\theta)}d\omega' d\theta.$$

So if we could take u_1, u_2 appropriately normalized and sufficiently concentrated near a direction such that $\sqrt{\lambda}(\theta - \omega') \to \xi$, $\forall \omega' \in \text{supp}u_1$, $\theta \in \text{supp}u_2$, one would expect then $(T(\lambda)u_1, u_2) \to \hat{V}(\xi)$. This however is technically difficult to realize, because too much concentration of the support prevents us from obtaining good estimates on remainders. For this reason, we choose test functions in the following way: Let $f \in \mathcal{S}(\mathbf{X})$ with supp $\hat{f}$ compact. Let $\omega \in \Sigma_0$ be fixed. Put

$$e_\lambda = e^{i\sqrt{\lambda}x\cdot\omega}, \quad f_\lambda = F(\lambda)(e_\lambda f). \tag{5.1}$$

Then $f_\lambda(\omega') = c(\lambda)\hat{f}(\sqrt{\lambda}(\omega' - \omega))$ is supported in $\{|\omega' - \omega| \leq M/\sqrt{\lambda}\}$ and $\|f_\lambda\|_{L^2(\mathbf{S}^{d-1})} = O(\lambda^{-1/4})$. By the Stone's formula, we have the following identity,

$$F(\lambda)^* f_\lambda = \frac{1}{2i\pi}\{R_0(\lambda + i0) - R_0(\lambda - i0)\}(e_\lambda f). \tag{5.2}$$

We denote $\Pi_\omega = \{x \in \mathbf{X}; \quad x \cdot \omega = 0\}$. For f a function defined on $\mathbf{X}$ which decays as $O(\langle x \cdot \omega\rangle^{-s}), s > 1$, we put

$$f^\omega(y) = \int_{\mathbf{R}} f(y + t\omega)dt, \quad y \in \Pi_\omega.$$

Theorem 5.1 *For $f_j \in \mathcal{S}(\mathbf{X})$ with supp $\hat{f}_j$ compact. Define $f_{j,\lambda}$ by (5.1) with f replaced by f_j. Under the condition (1.2), one has:*

$$(T(\lambda)f_{1,\lambda}, f_{2,\lambda}) = \frac{1}{16\pi^2\lambda} < V^\omega f_1^\omega, f_2^\omega >_{L^2(\Pi_\omega)} + O(\lambda^{-1-\eta}), \qquad as\ \lambda \to \infty. \tag{5.3}$$

Here $\eta = 1/2$ if $\rho > 2$ and $\eta = (\rho-1)/2 + \varepsilon$ for any $\varepsilon > 0$ if $1 < \rho \leq 2$.

Remark 5.1 In [2], Enss also used test functions of the form $e_\lambda f$ to prove the uniqueness of inverse scattering in two-body problems by time-dependent method.

To prove Theorem 5.1, we again use the representation formula for the scattering matrix. But this time, we have to use λ-dependent cut-offs. Let χ be a smooth function on $\mathbf{X}$ which equal 1 near ω and 0 outside a slightly larger neighbourhood of ω in $\mathbf{X}$. Put $\chi_\lambda(\xi) = \chi(\frac{\xi}{\sqrt{\lambda}})$. Then,

$$\chi_\lambda(D)(e_\lambda f_j) = e_\lambda f_j,$$

for λ large enough. Let J_0 be constructed as in Sect. 3. Put $J_\lambda(x) = J_0(\frac{x}{\sqrt{\lambda}})$. Then if the $\delta > 0$ used in the definition of J_0 is chosen sufficiently small, $J_\lambda(x) = 1$ for $\pm x$ in a conic neighbourhood of $\chi_\lambda(\cdot)$ and since $\chi_\lambda(\xi)$ is supported near $\sqrt{\lambda}\omega$, we have: $|x\cdot\xi| < (1-\delta)\sqrt{\lambda}|x|$, $\delta > 0$, for (x,ξ) in the support of $(1 - J_\lambda(x))\chi_\lambda(\xi)$. Applying Theorem 3.3, we obtain

$$(T(\lambda)f_{1,\lambda}, f_{2,\lambda}) = (\tilde{T}(\lambda)f_{1,\lambda}, f_{2,\lambda}), \qquad \text{for } \lambda > 0, \tag{5.4}$$

where

$$\tilde{T}(\lambda) = F(\lambda)\{\chi_\lambda(D)J_\lambda(x)Q_\lambda - Q_\lambda^* R(\lambda + i0)Q_\lambda\}F(\lambda)^*, \tag{5.5}$$

and

$$Q_\lambda = \{V(x)J_\lambda(x) + [-\Delta, J_\lambda]\}\chi_\lambda(D).$$

Lemma 5.2 *Under the conditions of Theorem 5.1, one has:*

$$\|\langle x\rangle^{\rho/2} Q_\lambda F(\lambda)^* f_{k,\lambda}\| \leq C\lambda^{-1/2},$$

for $k = 1, 2$ and $\lambda >> 1$.

Proof. By the decay assumption on the potentials and the choice of J_λ, one sees that

$$VJ_\lambda = O(|x|^{-\rho}) + V(x)j_\lambda(1 - j_0(x)).$$

where $j_\lambda(x) = j(|x|/\sqrt{\lambda})$. See (3.7) for the definition of j_0 and j. $(1-j(|x|))(1-j_0(x))\chi_\lambda(D)$ and $\langle x\rangle[-\Delta, J_\lambda]\chi_\lambda(D)$ are pseudodifferential operators with symbol in the $S_-(\lambda)\cap S_+(\lambda)$ and are bounded uniformly in λ. We can therefore apply the microlocal resolvent estimates ((i) of Theorem 3.1) to the free resolvent $R_0(\lambda \pm i0)$ and obtain from (5.2) that

$$\|\langle x\rangle^{\rho/2} Q_\lambda F(\lambda)^* f_{k,\lambda}\| \leq C\{\|\langle x\rangle^{-\rho/2} F(\lambda)^* f_{k,\lambda}\|$$

$$+\|\langle x\rangle^{\rho/2}\{(1-j(|x|))(1-j_0(x)) + [-\Delta, J_\lambda]\}\chi_\lambda(D)F(\lambda)^* f_{k,\lambda}\|\} \leq C'\lambda^{-1/2}\|\langle x\rangle^{1+\rho/2} f_k\|.$$

This proves the Lemma. □

Since $\rho > 1$, Lemma 5.2 gives:

$$(T(\lambda)f_{1,\lambda}, f_{2,\lambda}) = (F(\lambda)\chi_\lambda(D)J_\lambda(x)Q_\lambda F(\lambda)^* f_{1,\lambda}, f_{2,\lambda}) + O(\lambda^{-3/2}).$$

Taking notice that $1 - J_\lambda$ and $[-\Delta, J_\lambda]$ are supported in $\{|x| > c\lambda^{1/2}\}$ for some $c > 0$ and that on the support of $(1 - J_\lambda(x))\chi_\lambda(\xi)$ and $\nabla J_\lambda(x)\chi_\lambda(\xi)$, $|x \cdot \xi|/le(1-\delta)/sqrt\lambda|x|$ for some $\delta > 0$, we can apply (5.2) and the microlocal resolvent estimates for $R_0(\lambda \pm i0)$ to show that for any $N > 0$,

$$\|\langle x\rangle^N (1 - J_\lambda)\chi_\lambda(D)F(\lambda)^* f_{2,\lambda}\| \le C_N \lambda^{-N},$$

and

$$\|\langle x\rangle^N [-\Delta, J_\lambda]\chi_\lambda(D)F(\lambda)^* f_{1,\lambda}\| \le C_N \lambda^{-N}.$$

Write VJ_λ as $VJ_\lambda = Vj_0(x)(1 - j_\lambda) + Vj_\lambda$. By the decay assumption on the potentials, $Vj_0(x)(1 - j_\lambda) = O((|x| + \sqrt{\lambda})^{-\rho})$. As a consequence, making use of (5.2), one has:

$$|(F(\lambda)Vj_0(1 - j_\lambda)F(\lambda)^* f_{1,\lambda}, f_{2,\lambda})| \le C\lambda^{-1-\frac{\rho-2s}{2}}$$

for any $s > 1/2$. Finally we obtain

$$(T(\lambda)f_{1,\lambda}, f_{2,\lambda}) = (F(\lambda)V(x)j_\lambda(x)F(\lambda)^* f_{1,\lambda}, f_{2,\lambda}) + O(\lambda^{-1-\eta}), \tag{5.6}$$

where $\eta > 0$ is defined in Theorem 5.1.

It remains to compute the asymptotics of

$$I(\lambda) \equiv (F(\lambda)V(x)j_\lambda(x)F(\lambda)^* f_{1,\lambda}, f_{2,\lambda})$$

as $\lambda \to \infty$. To do this, we use the method of eikonal approximation, which is similar to that used in [9]. Since f_j is rapidly decreasing, one obtain from (5.2)

$$I(\lambda) = \frac{1}{4\pi^2} < Vj_\lambda\chi_\lambda(D)(u_1^+ - u_1^-), u_2^+ - u_2^- > +O(\lambda^{-\infty}),$$

where

$$u_j^\pm = R_0(\lambda \pm i0)(e_\lambda j_\lambda f_j).$$

To construct the eikonal approximation for $u_k^\pm$, we remark that by a heuristic computation,

$$\begin{aligned} u_k^\pm &= \lim_{\varepsilon\to 0}\int_0^{\pm\infty} e^{-\varepsilon|t| - itP_0 + i\lambda t}(e_\lambda j_\lambda f_k)dt \\ &= ie_\lambda \lim_{\varepsilon\to 0}\int_0^{\pm\infty} e^{-\varepsilon|t| - itP_0 + i2t\sqrt{\lambda}\omega\cdot D}(j_\lambda f_k)dt \\ &= \frac{ie_\lambda}{2\sqrt{\lambda}}\int_0^{\pm\infty} e^{-itP_0/(2\sqrt{\lambda})}(j_\lambda f_k)(x - t\omega)dt. \end{aligned}$$

Since formally $e^{-itP_0/(2\sqrt{\lambda})} \sim 1$ as $\lambda \to \infty$, we define:

$$g_k^\pm = \frac{1}{2\sqrt{\lambda}} j_{4\lambda}(x)\int_0^{\pm\infty}(j_\lambda f_k)(x - t\omega)dt. \tag{5.7}$$

Since $j_{4\lambda}(x) \equiv j(|x|/\sqrt{4\lambda})$ is equal to 1 on supp j_λ, we can check that:

$$(-\Delta - \lambda)(e_\lambda g_k^\pm) = e_\lambda(-ij_\lambda f_k - r_k^\pm),$$

where
$$r_k^\pm = \Delta g_k^\pm + i(\omega\cdot\nabla j_{4\lambda})\int_0^{\pm\infty}(j_\lambda f_k)(x-t\omega)dt.$$
This gives a formula for approximating $u_k^\pm$:
$$u_k^\pm = ie_\lambda g_k^\pm + iR_0(\lambda\pm i0)r_k^\pm.$$

Let $\Pi_\omega = \{x\in\mathbf{X};\quad \omega\cdot x = 0\}$. For any $x\in\mathbf{X}$, denote by y its projection onto Π_ω: $y = x-(\omega\cdot x)\omega$. Since $f_k\in\mathcal{S}(\mathbf{X})$, it is easy to check that $\forall N>0, \alpha\in\mathbf{N}^d$,
$$|\partial^\alpha g_k^\pm(x)| \le C_{N\alpha}\lambda^{-1/2}\langle y\rangle^{-N},\quad \forall x\in\mathbf{X}, \lambda>1. \tag{5.8}$$
If $\pm\omega\cdot\hat{x} < \pm(1-\varepsilon),\quad \varepsilon>0$, then, $|x-t\omega| > c(\varepsilon)(|x|+|t|)$ for $\pm t>0$ with $c(\varepsilon)>0$. In this case, we have:
$$|\partial^\alpha g_k^\pm(x)| \le C_{N\alpha\varepsilon}\lambda^{-1/2}\langle x\rangle^{-N},\quad \forall x\in\mathbf{X} \text{ with } \pm\,\omega\cdot\hat{x} < \pm(1-\varepsilon), \lambda>1, \tag{5.9}$$
for any $\varepsilon>0$. Finally remark that supp $g_k^\pm \subset \{x; |x|\le 4\sqrt{\lambda}\}$.

Proposition 5.3 *With the above notations, we have for any $1/2<s<\rho/2$,*
$$\|\langle x\rangle^{-s} g_k^\pm\| \le C\lambda^{-1/2} \tag{5.10}$$
and
$$\|\langle x\rangle^{s} V(x)\chi_\lambda(D)R_0(\lambda\pm i0)r_k^\pm\| \le C_\varepsilon\lambda^{-1+\zeta},\quad \lambda>1, \tag{5.11}$$
where $\zeta=0$ if $\rho>s+3/2$ and $\zeta = (-\rho+s+3/2)/2+\varepsilon$for any $\varepsilon>0$ if $\rho\le s+3/2$.

Proof. (5.10) is clear and we only give the proof of (5.11) for r_k^+. Let J_0 be constructed as in Sect. 3. Then
$$V\chi_\lambda(D) = (VJ_0 + V(1-J_0))\chi_\lambda(D) = O(\langle x\rangle^{-\rho}) + O(1)(1-J_0)\chi_\lambda(D). \tag{5.12}$$
By the choice of J_0, the support of $(1-J_0(x))\chi_\lambda(\xi)$ is contained in $\{(x,\xi); |x\cdot\xi| \le (1-\varepsilon_0)\sqrt{\lambda}|x|\}$ for some $\varepsilon_0>0$. Introduce now a partition of unity: $\psi_-(\hat{x}\cdot\omega)+\psi_+(\hat{x}\cdot\omega)\equiv 1$, $\forall x$, such that $\psi_-(t)=1$ for $t\le 1-2\varepsilon$; 0 for $t>1-\varepsilon$ where $0<\varepsilon<<\varepsilon_0$. According to (5.9), $\psi_-(\hat{x}\cdot\omega)r_k^+ = O(\lambda^{-1/2}\langle x\rangle^{-N})$ for any $N>0$. By (i) of Theorem 3.1 for $R_0(\lambda+i0)$, we obtain immediately from (5.12) that
$$\|\langle x\rangle^s V(x)\chi_\lambda(D)R_0(\lambda+i0)\psi_- r_k^+\| \le C\lambda^{-1},\quad \lambda>1.$$
Now on the support of $\chi_\lambda(\xi)\psi_+(\hat{x}\cdot\omega)$, we have $\hat{x}\cdot\xi > (1-4\varepsilon)\sqrt{\lambda}$, if the support of χ is chosen to be sufficiently near ω. Therefore we can apply (i) of Theorem 3.1 to estimate $\langle x\rangle^s VJ_0\chi_\lambda(D)R_0(\lambda+i0)\psi_+ r_k^+$ and (ii) of Theorem 3.1 to estimate $\langle x\rangle^s(1-J_0)\chi_\lambda(D)R_0(\lambda+i0)\psi_+ r_k^+$ to conclude that
$$\|\langle x\rangle^s V\chi_\lambda(D)R_0(\lambda+i0)\psi_+ r_k^+\| \le C\lambda^{-1} + C\lambda^{-1/2}\|\langle x\rangle^{-\rho+s+1}r_k^+\|,\quad \lambda>1.$$
Remark that supp r_k^+ is contained in $\{|x|\le C_1\sqrt{\lambda}\}$. It is elementary to estimate that $\|\langle x\rangle^{-\rho+s+1}r_k^+\| \le C\lambda^{-1/2}$ if $\rho-1-s>1/2$ and $\|\langle x\rangle^{-\rho+s+1}r_k^+\| \le C\lambda^{-1/2+(-\rho+s+3/2)/2+\varepsilon}$ for any $\varepsilon>0$, if $\rho-1-s\le 1/2$. (5.11) for r_k^+ follows. $\square$

Proof of Theorem 5.1. Applying Proposition 5.3 with $s = (1+\varepsilon)/2$, $\varepsilon > 0$ sufficiently small, we derive that

$$I(\lambda) = \frac{1}{16\pi^2\lambda} < Vj_\lambda \chi_\lambda(D)(e_\lambda g_1), e_\lambda g_2 > + O(\lambda^{-1-\eta}),$$

where $\eta > 0$ is defined in Theorem 5.1 and

$$g_k(x) = j_{4\lambda}(x) \int_{-\infty}^{+\infty} (j_\lambda f_k)(x - t\omega) dt.$$

Now notice that $\chi_\lambda(D)(e_\lambda g_1) = e_\lambda \chi(\frac{1}{\sqrt{\lambda}} D + \omega) g_1$. Since $\omega \in \Sigma_0$ and $\chi(\omega) = 1$, by the Taylor expansion for $\chi(\frac{1}{\sqrt{\lambda}}\xi + \omega)$ around ω, we can estimate:

$$\| \langle x \rangle^{\rho/2} V(x)(\chi_\lambda(D) - 1)(e_\lambda g_1) \| \leq C\lambda^{-1/2}.$$

Now (5.3) follows easily. $\square$

Now we can give the proof of Theorem 1.1 by making use of Corollary 2.3 and Theorem 5.1.

Proof of Theorem 1.1. According to Corollary 2.3, S^a, $\forall a \in \mathcal{A}, a \neq a_{\max}$, is uniquely determined by S at high energies. Applying Theorem 5.1 to S^a, we can recover

$$W_a^\omega(y^a) = \int_{\mathbf{R}} W_a(y^a + t\omega) dt, \tag{5.13}$$

for $\omega \in \mathbf{S}^a \setminus (\cup_{b \subset a, b \neq a_{\min}} \mathbf{X}_b)$, $y^a \in \mathbf{X}^a$ with $\omega \cdot y^a = 0$. Here $W_a = \sum_{b \subseteq a} V_b$. If $\#a = N-1$, the only $b \in \mathcal{A}$ with $b \subset a$ is $b = a_{\min}$. In this case, $\mathbf{S}^a \setminus (\cup_{b \subset a, b \neq a_{\min}} \mathbf{X}_b) = \mathbf{S}^a$ and $W_a(x^a) = V_a(x^a)$. Assume $\dim \mathbf{X}^a \geq 2$. In the sense of temperate distributions over $\mathbf{X}^a$, the Fourier transform $\hat{V}_a$ of V_a can be recovered from $V_a^\omega(y^a) = \int V_a(y^a + t\omega)dt, \forall \omega \in \mathbf{S}^a$, by the formula:

$$\hat{V}_a(\xi) = \int e^{-i\xi \cdot y^a} V_a^\omega(y^a) dy^a$$

for $\xi \in \Pi_\omega$. In particular, since $\cup_{\omega \in \mathbf{S}^a} \Pi_\omega = \mathbf{X}^a$, if $V_a^\omega(y) = 0$ for any $\omega \in \mathbf{S}^a$ and $y \in \Pi_\omega$, then, $\hat{V}_a \equiv 0$, which implies $V_a \equiv 0$.

For any a with $\#a \leq N-2$, we use an induction on $\#a$. If all $V_b, b \subset a$, is already recovered, then, (5.13) determines $V_a^\omega(y^a)$ which is now well defined for all $\omega \in \mathbf{S}^a$ due to the short-range assumption on V_a. V_a can be again recovered from $V_a^\omega, \omega \in \mathbf{S}^a$ by the above argument. $\square$

Remark 5.2 The condition $\dim \mathbf{X}^a \geq 2$ for any $a \neq a_{\min}$ is necessary in order to recover $V = \sum_a V_a$ from $V^\omega, \omega \in \Sigma_0$. In fact, consider a genelarized N-body potential of the form

$$V(x) = \sum_{k=1}^{M} V_k(x_k),$$

where $x_k \in \mathbf{R}$, $\forall k$. Suppose that V_k satisfies the condition (1.2) and

$$\int_{\mathbf{R}} V_k(t) dt = 0, \quad \forall k.$$

Then $V^\omega(y) = 0$ for any $\omega \in \Sigma_0, \quad y \in \Pi_\omega$. In this case, the V_k's are not uniquely determined by V^ω, $\omega \in \Sigma_0$.

In this paper, we only studied the high energy asymptotics for the free channel - free channel scattering matrix. By the microlocal resolvent estimates established in [8], we can also study the high energy asymptotics in a weak sense for scattering matrices $S_{\alpha\beta}(\lambda)$ with arbitrary scattering channels α and β. In these cases, the representation formulae for $S_{\alpha\beta}(\lambda)$ can only be established for sufficiently large λ depending on the support of cut-offs in momentum space.

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