Inverse Problem Related to Channel Scattering Operators

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Abstract

In this paper, we study the inverse scattering related to channel scattering operator $S_{\alpha\beta}$ in many-body problems. If the incoming and outgoing channels are associated with a same, but arbitary cluster decomposition, a, of the N-particle system, we prove that all effective interactions between the mass-centers of the clusters in a can be recovered from the high energy asymptotics of scattering matrices $S_{\alpha\beta}(\lambda)$.

1 Introduction

It is known that in many-body problems, the free channel scattering operator determines uniquely all interactions of a generalized many-body Schrödinger operator. See [3] by time-dependent (or geometrical) method and [14] by stationary method. For inverse scattering with fixed energy, Novikov announced a result in 3-body problems by using Faddeev's method and assumptions ([8]) and Vasy [13] proved that the three-body free channel scattering matrix determines all two-body scattering matrices and that the uniqueness of inverse scattering in three-body problems with exponentially decaying potentials follows the known results in two-body problems without implicit assumption. More recently, Novikov [9] gives new proofs and improvements for some results of [14].

In this work, we shall study the inverse scattering problem related to a channel scattering operator with two arbitrary channels associated to a same cluster decomposition. If the incoming and outgoing channels are associated to a same two-cluster decomposition, this problem was studied by Enss-Weder in [3] (see also in [4] for long-range potentials). From the point of view of physics, corresponding to a k-cluster decomposition of an N-particle system, there is a natural k-particle system describing the relative motions of these k clusters, which consists of the mass-centers of clusters. Giving a scattering channel means fixing the state of the Subhamiltonian and the evolution of the whole system will then mainly be described by the intercluster motions. So in some sense, a scattering operator with k-cluster outgoing and incoming channel can be compared with the free channel scattering operator of the k-body system of mass-centers. Since it is known that the free channel scattering operator in an N-body problem determines uniquely all possible interactions, one expects that a scattering operator with arbitary incoming and outgoing scattering channels associated with a k-cluster decomposition should be sufficient to determine all possible *effective interactions* between the mass-centers. The result proved in this work justifies this intuition. (See Theorem 1.1 below.)

Now let us introduce some notation related to the scattering theory of N-body Schrödinger operators. In the current literature on the subject, one works more frequently with generalized N-body Schrödinger operators, which are notationally more concise and make more clear the underlying geometrical structure. See [1]. To make the present work as self-contained as possible, we introduce here the notation related to generalized N-body Schrödinger operators (see also the author's previous works [14, 15]). The usual N-body Schrödinger operator obtained after the removal of mass-center from the total energy operator

$$-\sum_{j=1}^{N} \frac{\hbar^2}{2m_j}\Delta_{x_j} + \sum_{i<j} V_{ij}(x_i - x_j), \quad x_j \in \mathbf{R}^n, \tag{1.1}$$

is a particular case which only involves pair interactions.

Let $\mathcal{A}$ be the set of all possible cluster decompositions of an N-body system labeled by $\{1, 2, \cdots, N\}$. Denote by $\#a$ the number of clusters of $a \in \mathcal{A}$. For $a, b \in \mathcal{A}$, we write $b \subset a$ if the cluster decomposition b is a refinement of a. The following two decompositions are of particular use: $a_{\max} = \{(1, 2, \cdots, N)\}$, $a_{\min} = \{(1), (2), \cdots, (N)\}$. For $a, b \in \mathcal{A}$, we define $a \cap b$ and $a \cup b$ as follows. The clusters in $a \cap b$ are the non-void intersections of those in a with those in b, while those in $a \cup b$ are obtained by the equivalence relation $i \sim j$ if i, j are in a same cluster of a *or* in a same cluster of b. Let $\mathbf{X}$ be a vectorial space of dimension d, $q(\cdot)$ a Euclidean structure on $\mathbf{X}$ (in the above example, $\mathbf{X} = \mathbf{R}^{n(N-1)}$ and $q(x) = (\sum_j 2m_j|x_j|^2)|_{\mathbf{X}}$, if $\hbar$ is set to 1). To each $a \in \mathcal{A}$, it is assigned a subspace $\mathbf{X}^a \subseteq \mathbf{X}$ with (i). $\mathbf{X}^{a_{\max}} = \mathbf{X}$, $\mathbf{X}^{a_{\min}} = \{0\}$; (ii). $\mathbf{X}^a \subset \mathbf{X}^b$ if $a \subset b$; (iii). $\mathbf{X}^{a\cup b} = \mathbf{X}^a + \mathbf{X}^b$, for all $a, b \in \mathcal{A}$. For each $a \in \mathcal{A}$, we denote by $\mathbf{X}_a$ the orthogonal complement (with respect to the Euclidean structure on $\mathbf{X}$) of $\mathbf{X}^a$ in $\mathbf{X}$: $\mathbf{X} = \mathbf{X}^a \oplus \mathbf{X}_a$. Accordingly, a generic point $x \in \mathbf{X}$ can be decomposed as: $x = x^a + x_a$. Sometimes, we also write it as $x = (x^a, x_a)$.

Let Δ be the Laplace-Beltrami operator on Euclidean space $(\mathbf{X}, q)$. The generalized N-body Schrödinger operators to be studied in this work are operators of the form:

$$P = -\Delta + \sum_{a\in\mathcal{A}} V_a(x^a). \tag{1.2}$$

Here $x^a = \pi^a x$ with π^a the orthogonal projection from $\mathbf{X}$ onto $\mathbf{X}^a$. The physical N-body Schrödinger operators can always be put into the above form ([1]). To simplify notation, we just assume $\mathbf{X} = \mathbf{R}^d$, $d \geq 2$ and q is the canonical structure over $\mathbf{X}$. This can be reached just by a change of coordinates.

Denote $-\Delta^a$ ($-\Delta_a$, resp.) the Laplacian in x^a-variables (x_a- variables, resp.) and $D^a = -i\partial/\partial x^a$, $D_a = -i\partial/\partial x_a$. Put

$$P^a = -\Delta^a + \sum_{b\subseteq a} V_b(x^b), P_a = P^a - \Delta_a,$$

$$I_a(x) = \sum_{b\not\subseteq a} V_b(x^b).$$

P^a is the Subhamiltonian associated with the cluster decomposition a and I_a is the sum of all intercluster interactions. The detailed expression of P^a depends on the choice of coordinates on $\mathbf{X}^a$ and for those resulting from (1.1), see formulae given in [11] in some special coordinates. Let $\mathcal{T}$ denote the set of thresholds (including the eigenvalues) of P:

$$\mathcal{T} = \cup_a \sigma_{pp}(P^a).$$

Put

$$\Sigma_a = \mathbf{S}_a \setminus \cup_{b \not\subseteq a} \mathbf{X}_b. \tag{1.3}$$

Due to the geometry of an N-body system, one can check that $\Sigma_a = \mathbf{S}_a$ if $\# a = 2$. The norm and the scalar product in $L^2(\mathbf{X}_a)$, (in $L^2(\mathbf{S}_a)$, respectively), will be denoted by $\|\cdot\|_a$ and $<\cdot,\cdot>_a$ (by $|\cdot|_a, (\cdot,\cdot)_a$, respectively), while those in $L^2(\mathbf{X})$ will be denoted by $\|\cdot\|$ and $<\cdot,\cdot>$.

Let a be a non-trivial cluster decomposition (i.e., $a \in \mathcal{A}$ with $\# a \geq 2$). A scattering channel α stands for a collection of data $(a, E_\alpha, \varphi_\alpha)$, where $E_\alpha \in \sigma_{pp}(P^a)$ and φ_α is a normalized eigenfunction:

$$P^a \varphi_\alpha = E_\alpha \varphi_\alpha.$$

When $a = a_{\min}$ (i.e., $\# a = N$), one uses the convention that $P^a = 0$, $P_a = -\Delta$ and in this case, the only scattering channel is $\alpha = (a_{\min}, 0, 1)$ which is usually call the free channel. We shall say that α is a scattering channel with non-threshold energy, if

$$E_\alpha \in \sigma_{pp}(P^a) \setminus \cup_{b \subset a} \sigma_{pp}(P^b).$$

Let $\mathcal{J}_\alpha : L^2(\mathbf{X}_a) \to L^2(\mathbf{X})$ the channel identification:

$$(\mathcal{J}_\alpha f)(x) = \varphi_\alpha(x^a) f(x_a).$$

Let $\rho > 1$. Assume that $V_a, a \in \mathcal{A}$, are real functions satisfying

$$|\partial_y^\gamma V_a(y)| \leq C \langle y \rangle^{-\rho - |\gamma|}, \tag{1.4}$$

for all $y \in \mathbf{X}^a$ and $\gamma \in \mathbf{N}^{n^a}$ with $|\gamma| \leq \max\{d/2+1, 3\}$, where $\langle y \rangle = (1+|y|^2)^{1/2}$. Remark that the smoothness assumption on potentials is more than necessary and with a bit more technical works, mild local singularities can be included.

Under the assumption (1.4), it is well known that the channel wave operators

$$W_{\alpha,\pm} = \text{s-}\lim_{t \to \pm\infty} U(t)^* U_a(t) \mathcal{J}_\alpha$$

exist for any scattering channel α and are complete ([12]). Here $U(t)$ and $U_a(t)$ are unitary groups generated by P and P_a, respectively.

Now let $\alpha = (a, E_\alpha, \psi_\alpha)$ and $\beta = (b, E_\beta, \psi_\beta)$ be two given scattering channels. Let

$$S_{\alpha\beta} = W_{\beta,+}^* W_{\alpha,-}, \qquad \text{in} \quad L^2(\mathbf{X}),$$

be the scattering operator with incoming channel α and outgoing channel β. The purpose of this work is to study which informations on the potentials one can derive from $S_{\alpha\beta}$.

To state our result, we need to introduce the notion of effective interactions between the clusters for a given couple of scattering channels (see also [3, 4] in two-cluster case). Let $a \in \mathcal{A}$

be a k-cluster decomposition of an N-particle system: $a = (a_1, \cdots, a_k)$. The mass-centers of the clusters a_j form a k-particle system. To make clear the role of this new k-particle system, we introduce a generalized k-body Schrödinger operator as follows. For $b \in \mathcal{A}$, $a \subset b$, we denote

$$\mathcal{A}_{a,b} = \{c \in \mathcal{A}; a \cup c = b\}.$$

Remark that if $k = 2$, then the only $b \in \mathcal{A}$ with $b \supset a$ is $b = a_{\max}$. In this case, $\mathcal{A}_{a,b} = \{c \in \mathcal{A}; c \not\subseteq a\}$. In general case $k \geq 2$, let $b_j \in \mathcal{A}$ such that $a = b_0 \subset b_1 \subset \cdots \subset b_{k-1} = a_{\max}$ with $\# b_j = \# a - j = k - j$. Then we can check that

$$\{c \in \mathcal{A}; c \not\subseteq a\} = \cup_{j=1}^{k-1} \mathcal{A}_{a,b_j}, \quad \text{ and } \mathcal{A}_{a,b_j} \cap \mathcal{A}_{a,b_{j'}} = \emptyset, \text{ if } j \neq j'.$$

For $b \in \mathcal{A}, b \supset a$, we define $x_a^b = \pi^b \pi_a x$ and

$$V_{a,b}(x^b) = \sum_{c \in \mathcal{A}_{a,b}} V_c(x^c).$$

$V_{a,b}$ represents the interactions between those clusters in a which are included in a same cluster in b. Let $\alpha = (a, E_\alpha, \psi_\alpha)$, $\beta = (a, E_\beta, \psi_\beta)$ be two scattering channels associated to a. Set

$$V^b_{\alpha\beta}(x^b_a) = \int_{X^a} V_{a,b}(x^b) \psi_\alpha(x^a) \overline{\psi_\beta(x^a)} dx^a, \tag{1.5}$$

$$I_{\alpha\beta}(x_a) = \int_{X^a} I_a(x) \psi_\alpha(x^a) \overline{\psi_\beta(x^a)} dx^a. \tag{1.6}$$

$V^b_{\alpha\beta}(x^b_a)$ will be called effective potential between the mass-centers of those clusters of a which are included in a same cluster of b. When $b = a_{\max}$, the superscript will be omitted. Notice that when $\# a = 2$, the only cluster decomposition b with $b \supset a$ is $b = a_{\max}$ and in this case,

$$V_{a,b} = I_a, V^b_{\alpha\beta} = I_{\alpha\beta}.$$

Let the mass-centers of the clusters of a be labeled by $\{1', \cdots, k'\}$. Any $b \in \mathcal{A}$ with $b \supset a$ can be naturally identified with a cluster decomposition b' of $\{1', \cdots, k'\}$. Let $\mathcal{A}'$ be the set of all possible cluster decompositions for $\{1', \cdots, k'\}$ Introduce

$$\mathbf{Y} = \mathbf{X}_a, \quad \mathbf{Y}^{b'} = \mathbf{X}^b_a \equiv \pi^b \pi_a(\mathbf{X}), \quad \mathbf{Y}_{b'} = (\mathbf{Y}^{b'})^\perp = \mathbf{X}_b.$$

Then it is clear that

$$P' = -\Delta_y + \sum_{b' \in \mathcal{A}'} U_{b'}(y^{b'}) \tag{1.7}$$

is a generalized k-body Schrödinger operator , where $U_{b'}(y^{b'}) = V^b_{\alpha\beta}(x^b_a)$ and $y^{b'} = x^b_a$.

The main result of this work is the following

Theorem 1.1 *Let $a \in \mathcal{A}$ with $\# a \geq 2$. Under the condition (1.4), assume in addition that*

$$\dim \mathbf{X}^b_a \geq 2, \forall b \in \mathcal{A}, b \supset a. \tag{1.8}$$

and that the eigenfunctions ψ_α, ψ_β have some decay in $\mathbf{X}^a$:

$$\psi_\gamma \in L^{2,\varepsilon_\gamma}(\mathbf{X}^a), \gamma = \alpha, \beta, \tag{1.9}$$

with $\varepsilon_\alpha \geq 0, \varepsilon_\beta \geq 0$ and $\varepsilon_\alpha + \varepsilon_\beta > 1$, where $L^{2,s}$ is the weighted L^2-space: $L^{2,s}(\mathbf{X}; dx) = L^2(\mathbf{X}; \langle x \rangle^{2s} dx)$. Then, the scattering operator $S_{\alpha\beta}$ from the channel α to the channel β determines uniquely all effective potentials $V^b_{\alpha\beta}$, $b \supset a$, defined in (1.5).

Remarks 1.1. (a). Theorem 1.1 says that the inverse scattering informations obtained from the channel scattering operator $S_{\alpha\beta}$ can be compared with those obtained from the free channel scattering operator for P' on $\mathbf{Y}$. This result is physically intuitive. Notice also that if α, β are the free channel, Theorem 1.1 is reduced to the known result proved in [3] and [14]. In fact, in this case, the assumption (1.9) is always satisfied ($\mathbf{X}^a = \{0\}$ is compact) and $V^b_{\alpha\beta}$ is just V_b for all $b \in \mathcal{A}$.

(b). The case $\#a = 2$ was studied in [3, 4] using time-dependent method under the assumption on the eigenfunctions that

$$(1+R)^\sigma \{\int_{|x^a|\geq R} |\psi_\gamma(x^a)|^2 dx^a\}^{1/2} \in L^1(\mathbf{R}; dR),$$

for some $\sigma \geq 0$ and $\gamma = \alpha, \beta$. This condition can be compared to Enss' formulation of short-range condition in potential scattering. Our result implies that if one of the eigenfunction is "short-range", the other can be arbitrary. In particular, if one of the scattering channel is of non-threshold energy, the other can be arbitary.

(c). The result of Theorem 1.1 in the case $\#a \geq 3$, $a \neq a_{\min}$, is new. This happens, for example, when $N = 4$ and $a = \{(12), (3), (4)\}$.

The proof of Theorem 1.1 is based on the stationary methods developed in [15], where the high energy asymptotics of scattering matrices in weak sense were studied. In this work, we shall still look at the weak high energy asymptotics of scattering matrices, but for test functions with smaller and smaller support as the energy λ tends to infinity and prove that

$$(T_{\alpha\beta}(\lambda)h_1(\cdot,\lambda), h_2(\cdot,\lambda))_a = \frac{1}{16\pi^2\lambda} < I^\omega_{\alpha\beta} h^\omega_1, h^\omega_2 >_{L^2(\Pi_\omega)} + O(\lambda^{-1-\delta}), \delta > 0, \tag{1.10}$$

as $\lambda \to \infty$. Here $T_{\alpha\beta}(\lambda) = \frac{i}{2\pi}(\delta_{\alpha\beta} - S_{\alpha\beta}(\lambda))$, $S_{\alpha\beta}(\lambda)$ being the scattering matrices related to $S_{\alpha\beta}$, $h_j(\cdot,\lambda) \in C_0^\infty(\Sigma_a), j = 1, 2$, are contructed from suitable test functions h_j using spectral representation for P^a and for a function f, we denoted

$$f^\omega(y) = \int_{\mathbf{R}} f(y + t\omega) dt,$$

whenever it is well defined. See Theorem 3.1 for more precisions. Due to the λ-dependence of the support of test functions, some arguments of [15] (in particular, those based on stationary phase method) are no longer valid in this work. The consequence of (1.10) in inverse scattering is that in the case $\#a = 2$, it allows to determine $I^\omega_{\alpha\beta}$ for all $\omega \in \mathbf{S}_a$ and hence $I_{\alpha\beta}$ by Radon transform. In the case $\#a > 2$, in order to recover all effective potentials $V^b_{\alpha\beta}, b \supset a$, we need to study in details the case where some centers of mass of clusters in a spread out faster faster than the others. To do this, our idea is to consider P^a as a Subhamiltonian of P^b and study the relative scattering for P^b in the channels α and β and establish the high energy asymptotics (1.10) for all relative scattering matrices (Theorem 3.1). The $V^b_{\alpha\beta}$'s are then recontructed by an induction on $\#b - \#a$.

The organization of this work is as follows: We shall prove in Section 2 that for all $b \supset a$, the channel wave operators relative to P^b can be reconstructed from those relative to P. In Section 3, we study the high energy asymptotics of scattering matrices. The method used here is stationary: we use the microlocal resolvent estimates to give the representation of scattering matrices and study the high energy asymptotics of the latter in week sense. Finally in Section 4, we reconstruct all effective potentials by an induction on $\#a - \#b$. In the case $\#b = \#a - 1$, P^a is a two-cluster Subhamiltonian of P^b and the method of inverse scattering for two-body problems can be applied. In the general case $\#a - \#b \geq 2$, we take a sequence of cluster decompositions $a = b_1 \subset b_2 \subset \cdots \subset b_k = b$ where $\#b_{j+1} = \#b_j - 1$. Therefore P^{b_j} can be regarded as a two-cluster Subhamiltonian of $P^{b_{j+1}}$ and the results of Sections 2 and 4 enable us to use an inductive argument to obtain all effective potentials $V^{b_j}_{\alpha\beta}$, $j = 2, \cdots, k$, in particular, $V^b_{\alpha\beta}$, $b \supset a$.

The results of this work are announced in [15].

2 Relative Scattering for a Subhamiltonian

Let $a, b \in \mathcal{A}$ with $a \subset b$. One has:$\mathbf{X}^a \subset \mathbf{X}^b$ and $\mathbf{X} = \mathbf{X}^a \oplus \mathbf{X}^b_a \oplus \mathbf{X}_b$, where $\mathbf{X}^a_b = \pi_a \pi^b \mathbf{X} = \pi^b \pi_a \mathbf{X}$. P^a can thus be regarded as a Subhamiltonian of P^b,

$$P^b = P^a + \sum_{c \subseteq b, c \not\subseteq a} V_c(x^c) - \Delta_{x^b_a},$$

We shall write in the following

$$P^b_a = P^a - \Delta_{x^b_a}, I^b_a = I_a - I_b = \sum_{c \not\subseteq a, c \subseteq b} V_c(x^c),$$

$$U^b(t) = e^{-itP^b}, U^b_a(t) = e^{-itP^b_a}, U^b_{a,0}(t) = e^{it\Delta_{x^b_a}}.$$

For a given scattering channel $\alpha = (a, E_\alpha, \psi_\alpha)$, we can consider the scattering in the channel α relative to the Subhamiltonian P^b. Denote still by $\mathcal{J}_\alpha$: $L^2(\mathbf{X}^b_a) \to L^2(\mathbf{X}^b)$ the channel identification $J_\alpha f(x^b) = \psi_\alpha(x^a) f(x^b_a)$, $f \in L^2(\mathbf{X}^b_a)$. Define the wave operators

$$\Omega^b_{\alpha,\pm} = \text{s-}\lim_{t \to \pm\infty} U^b(t)^* U^b_a(t) J_\alpha, \text{ in } L^2(\mathbf{X}^b). \tag{2.1}$$

Under the assumption (1.4), these wave operators exist. The following result shows that $\Omega^b_{\alpha,\pm}$ can be recovered from $\Omega_{\alpha,\pm}$.

Theorem 2.1 *Let $\omega \in \mathbf{S}_b \setminus (\cup_{c \not\subseteq b} \mathbf{X}_c)$ and $f \in \mathcal{S}(\mathbf{X}_b)$, $\hat{f} \in C^\infty_0$ and $\|f\|_{L^2} = 1$. Denote $f_n(x_b) = e^{in\omega \cdot x_b} f(x_b)$. Define $\Omega^{(n)}_{\alpha,\pm} : L^2(\mathbf{X}^b_a) \to L^2(\mathbf{X}^b)$ by*

$$(\Omega^{(n)}_{\alpha,\pm}\varphi)(x^b) = <\Omega_{\alpha,\pm}(\varphi \otimes f_n), f_n>_b, \quad \varphi \in L^2(\mathbf{X}^b_a). \tag{2.2}$$

Then

$$\text{s-}\lim \Omega^{(n)}_{\alpha,\pm} = \Omega^b_{\alpha,\pm} \tag{2.3}$$

as operators from $L^2(\mathbf{X}^b_a)$ to $L^2(\mathbf{X}^b)$.

Proof. Theorem 2.1 can be compared with Theorem 2 in [14]. But the proof of Theorem 2 can only be carried over under the additional assumption that $\psi_\alpha \in L^{2,\varepsilon_\alpha}(\mathbf{X}^a)$ with $\varepsilon_\alpha > 0$. Here we give a direct proof for Theorem 2.1 in general case. We treat only the outgoing wave operators and write henceforth $\Omega^{(n)}_{\alpha,+}$ as $\Omega^{(n)}_{\alpha}$ and $\Omega^{b}_{\alpha,+}$ as Ω^{b}_{α}. Since $\|\Omega^{(n)}_{\alpha}\| \leq 1$ for all n, it suffices to prove (2.3) on a dense subset of $L^2(\mathbf{X}^b_a)$, say $\mathcal{D} = \{\varphi \in \mathcal{S}(\mathbf{X}^b_a); \hat{\varphi} \in C_0^\infty$ and $\mathrm{supp}\ \hat{\varphi} \cap (\cup_{c\subseteq b, c\not\subseteq a}\mathbf{X}_c) = \emptyset\}$. One has:

$$\Omega^{(n)}_{\alpha}\varphi - \psi_\alpha\varphi = i\int_0^\infty < U(t)^* I_a U_a(t)\mathcal{J}_\alpha(\varphi\otimes f_n), f_n >_b dt. \tag{2.4}$$

The integral in (2.4) is convergent uniformly with respect to n. To see this, we introduce a cut-off on $\mathbf{R}$: $\chi_- \in C^\infty$, $0 \leq \chi_- \leq 1$, such that $\chi_-(r) = 1$ for $r \leq 1$, 0 for $r \geq 2$. Let $\chi_+ = 1-\chi_-$. Since the support of the Fourier transform of φf_n is away from $\cup_{c\not\subseteq a}\mathbf{X}_c$ uniformly with respect to n, we can find some $\delta > 0$ such that for any $c \not\subseteq a$,

$$\|\chi_-(|x^c|/(\delta t))U_a(t)\mathcal{J}_\alpha(\varphi\otimes f_n)\| \leq C_N < t >^{-N}, \text{ for any } N, t \geq 1, \tag{2.5}$$

uniformly in n. Notice that $V_c(x^c)\chi_+(|x^c|/(\delta t)) = O(t^{-\rho})$ by the assumption (1.4). This gives

$$\|I_a U_a(t)\mathcal{J}_\alpha(\varphi\otimes f_n)\| \leq C < t >^{-\rho}, \text{ for } t \geq 1, \text{ uniformly in } n. \tag{2.6}$$

So the integral in (2.4) is uniformly convergent with respect to n.

Now we propose to compute the limit of the integrand in (2.4) as $n \to \infty$ for each fixed t. We denote

$$u(x) = \psi_\alpha(x^a)\varphi(x^b_a)f(x_b), I_a = I_b + I^b_a.$$

For the piece corresponding to I_b, we have:

$$\begin{aligned} &\| < U(t)^* I_b U_a(t)\mathcal{J}_\alpha(\varphi\otimes f_n), f_n >_b \|_{L^2(\mathbf{X}^b)} \\ &\leq \|I_b e^{-it(D_a+n\omega)^2}u\| = \|I_b(x+2tn\omega)e^{-itD_a^2}u\|. \end{aligned}$$

For any $c \not\subseteq b$, $\pi^c\omega \neq 0$. So

$$\lim_{n\to\infty} I_b(x+2tn\omega) = 0$$

pointwisely in x and $t > 0$. The fact that $u \in L^2$ gives by Dominated Convergence Theorem that

$$\lim_{n\to\infty} < U(t)^* I_b U_a(t)\mathcal{J}_\alpha(\varphi\otimes f_n), f_n >_b = 0 \tag{2.7}$$

in $L^2(\mathbf{X}^b)$ for any $t > 0$. For the piece corresponding to I^b_a in (2.4), taking notice that $\pi^c\omega = 0$ for any $c \subseteq b$, we can write it as

$$\begin{aligned} & < e^{it((D+n\omega)^2+V)}e^{-it(D_b+n\omega)^2} I^b_a U^b_a(t)u, f >_b \\ = \ & U^b(t)^* I^b_a U^b_a(t)(\psi_\alpha\otimes\varphi) + < (e^{it((D+n\omega)^2+V)}e^{-it(D_b+n\omega)^2} - U^b(t)^*) I^b_a U^b_a(t)u, f >_b, \end{aligned}$$

and

$$\begin{aligned} & < (e^{it((D+n\omega)^2+V)}e^{-it(D_b+n\omega)^2} - U^b(t)^*) I^b_a U^b_a(t)u, f >_b \\ = & \int_0^t < (e^{is((D+n\omega)^2+V)} I_b e^{-is(D_b+n\omega)^2} U^b(s-t) I^b_a U^b_a(t)u, f >_b ds. \end{aligned}$$

As in the proof of (2.7), one can show that the above integral tends to 0 as $n \to \infty$. This shows

$$\lim_{n\to\infty} < e^{it((D+n\omega)^2+V)}e^{-it(D_b+n\omega)^2} I^b_a U^b_a(t)u, f >_b = U^b(t)^* I^b_a U^b_a(t)(\psi_\alpha\otimes\varphi)$$

in $L^2(\mathbf{X}^b)$.

Since the integral in (2.4) is uniformly convergent with respect to n, we can take the limit under integration and obtain from (2.4) that for any $\varphi \in \mathcal{D}$,

$$\lim_{n\to\infty} \Omega_\alpha^{(n)}\varphi = \psi_\alpha\varphi + i\int_0^\infty U^b(t)^* I_a^b U_a^b(t)\mathcal{J}_\alpha\varphi dt = \Omega_\alpha^b\varphi,$$

in $L^2(\mathbf{X}^b)$. □

A consequence of Theorem 2.1 needed in the proof of Theorem 1.1 is the following

Corollary 2.2 *Let α, β be two arbitrary scattering channels. Set $S_{\alpha\beta}^b = (\Omega_{\beta,+}^b)^*\Omega_{\alpha,-}^b$ and $S_{\alpha\beta}^{(n)}\varphi =< S_{\alpha\beta}(\varphi\otimes f_n), f_n >_a$, $\varphi \in L^2(\mathbf{X}_a^b)$. Then*

$$\text{s-}\lim_{n\to\infty} S_{\alpha\beta}^{(n)} = S_{\alpha\beta}^b,$$

as operators from $L^2(\mathbf{X}_a^b)$ to $L^2(\mathbf{X}^b)$.

Corollary 2.2 is useful in the study of inverse scattering for $S_{\alpha\beta}$ at high energy. An interesting open question is to see whether one can obtain the scattering matrices related to $S_{\alpha\beta}^b$ from those related to $S_{\alpha\beta}$ at a fixed energy λ. This kind of result should be useful to studying the inverse problem with fixed energy. See [8, 9, 13] for the case where α, β are the free channel in three-body problems.

3 High Energy Asymptotics

In this Section, we look for the leading term of the high energy asymptotics of scattering matrices associated to $S_{\alpha\beta}^b$, the scattering operator for the operators (P_a^b, P^b) with the incoming channel α and the outgoing channel β. Let us first give a formula for the representation of the scattering matrices. To simplify notations, we just write down the results in the case $b = a_{\max}$ and the superscript b will be omitted. The other cases can be treated completely similarly.

Put

$$T_{\alpha\beta} = \frac{i}{2\pi}\{S_{\alpha\beta} - \delta_{\alpha\beta}\}.$$

Let $F_\alpha : L^2(\mathbf{X}_a) \to \mathbf{H}_\alpha \equiv L^2(\mathbf{I}_\alpha; L^2(\mathbf{S}_a))$, $\mathbf{I}_\alpha =]E_\alpha, +\infty[$, be defined by:

$$(F_\alpha f)(\lambda,\theta) = c_\alpha(\lambda)\int e^{-i\sqrt{(\lambda-E_\alpha)}\,\theta\cdot x_a} f(x_a)dx_a, \quad (\lambda,\theta)\in \mathbf{I}_\alpha\times\mathbf{S}_a, \tag{3.1}$$

where

$$c_\alpha(\lambda) = 2^{-1/2}(2\pi)^{-n_a/2}(\lambda - E_\alpha)^{(n_a-2)/4},$$

with $n_a = \dim \mathbf{X}_a$. We can verify that $\|F_\alpha f\|_{\mathbf{H}_\alpha} = \|f\|_a$. Put $\mathcal{F}_\alpha = F_\alpha\mathcal{J}_\alpha^*$. By the Sobolev's lemma, F_α defines a family of maps, $F_\alpha(\lambda), \lambda\in\mathbf{I}_\alpha$, from $L^{2,s}(\mathbf{X}_a), s > 1/2$, to $L^2(\mathbf{S}_a)$:

$$(F_\alpha(\lambda)f)(\theta) = (F_\alpha f)(\lambda,\theta),$$

The spectral representation for the Subhamiltonian P_a with scattering channel α is now defined by

$$\mathcal{F}_\alpha(\lambda) = F_\alpha(\lambda)\mathcal{J}_\alpha^*. \tag{3.2}$$

One has

$$\mathcal{F}_\alpha(\lambda)P_a\mathcal{F}(\lambda)^* = \lambda$$

in the sense of non-bounded operators in $\mathbf{H}_\alpha$. Similarly, we can construct a spectral representation $\mathcal{F}_\beta$ for the Subhamiltonian P_a with scattering channel β.

Denote now $\mathbf{I}_{\alpha\beta} =]\max\{E_\alpha, E_\beta\}, +\infty[$. Then $F_\alpha T_{\alpha\beta} F_\beta^*$ can be represented by a family of operators

$$\{T_{\alpha\beta}(\lambda) = F_\alpha(\lambda)T_{\alpha\beta}F_\beta(\lambda)^*; \lambda \in \mathbf{I}_{\alpha\beta}\}$$

which act on $L^2(\mathbf{S}_a)$. Similarly, we can also construct from $S^b_{\alpha,\beta}$, $b \supset a$, a family of operators $T^b_{\alpha\beta}(\lambda)$, which now act on $L^2(\mathbf{S}^b_a)$. Notice that $T_{\alpha\beta}(\lambda)$ is a priori only defined almost everywhere in λ. For the representation of scattering matrices, see [15, 17].

Take $f_j \in \mathcal{S}(\mathbf{X}^b_a)$ such that $\hat{f}_j$ is of compact support. Let $\omega \in \mathbf{S}^b_a \setminus (\cup_{c\not\subseteq a}\mathbf{X}_c)$ be a fixed direction. Set

$$f_{1,\alpha}(\theta,\lambda) = F_\alpha(\lambda)(e^{i\sqrt{\lambda-E_\alpha}\omega\cdot x^b_a}f_1)(\theta),$$

$$f_{2,\beta}(\theta,\lambda) = F_\beta(\lambda)(e^{i\sqrt{\lambda-E_\alpha}\omega\cdot x^b_a}f_1)(\theta), \theta \in \mathbf{S}^b_a.$$

Theorem 3.1 *Under the assumptions (1.4) and (1.9), one has:*

$$(T^b_{\alpha\beta}(\lambda)f_{2,\beta}(\cdot,\lambda), f_{1,\alpha}(\cdot,\lambda))_a = \frac{1}{16\pi^2\lambda} < I^{b,\omega}_{\alpha\beta}f^\omega_2, f^\omega_1 >_{L^2(\Pi^b_\omega)} + O(\lambda^{-1-\delta_0}), \tag{3.3}$$

as $\lambda \to \infty$. Here we denoted for a function f

$$f^\omega(y) = \int_{\mathbf{R}} f(y+t\omega)dt,$$

whenever it is well defined and

$$I^b_{\alpha\beta}(x^b_a) = \int_{X^a} I^b_a(x^b)\overline{\psi_\alpha(x^a)}\psi_\beta(x^a)dx^a, \quad I^b_a = I_a - I_b,$$

and $\Pi^b_\omega = \{x^b_a \in \mathbf{X}^b_a; x^b_a\cdot\omega = 0\}$, $\delta_0 = \min\{1/2, (\varepsilon_\alpha+\varepsilon_\beta-1)/2, (\rho-1-\varepsilon)/2\}$ *for any* $0 < \varepsilon < \rho - 1$.

For notational reasons, we shall only give the proof of Theorem 3.1 for $b = a_{\max}$ and omit henceforth the superscript b.

First, we give a representation formula for $(T_{\alpha\beta}(\lambda)f_{2,\beta}(\cdot,\lambda), f_{1,\alpha}(\cdot,\lambda))_a$. Note that the support of $f_{j,\gamma}(\cdot,\lambda)$ is in a neighbourhood of ω of radius $O(\lambda^{-1/2})$. So we can find $\chi \in C_0^\infty(\mathbf{X}_a)$ with support in $\mathbf{X}_a \setminus (\cup_{c\not\subseteq a}\mathbf{X}_c)$ such that

$$(T_{\alpha\beta}(\lambda)f_{2,\beta}(\cdot,\lambda), f_{1,\alpha}(\cdot,\lambda))_a = (T^\chi_{\alpha\beta}(\lambda)f_{2,\beta}(\cdot,\lambda), f_{1,\alpha}(\cdot,\lambda))_a,$$

where

$$T^\chi_{\alpha\beta}(\lambda) = F_\alpha(\lambda)\chi(D_a)T_{\alpha\beta}\chi(D_a)F_\beta(\lambda)^*.$$

χ is λ-dependent, but can be chosen so that

$$|\partial^\gamma\chi| = O(\lambda^{-|\gamma|/2})$$

uniformly in ξ_a and $\lambda > 1$.

To give a representation formula for $(T_{\alpha\beta}(\lambda) f_{2,\beta}(\cdot,\lambda), f_{1,\alpha}(\cdot,\lambda))_a$, we take $j \in C_0^\infty(\mathbf{R})$ with $j(t) = 0$ if $t < 1/2$ and $j(t) = 1$ if $t \geq 1$. Put:

$$j_a(x) = \prod_{c\in\mathcal{A}, c\not\subseteq a} j(\frac{|x^c|}{\eta|x|}),$$

and

$$J_a(x) = j_a(x) j(|x|/\mu^{1/2}) + (1 - j(|x|/\mu^{1/2})), \mu > 1. \tag{3.4}$$

The parameter μ is introduced for technical reasons. As in [14], we can check that for $\eta > 0$ small enough, $J_a(x)$ is equal to 1 for x in a conic neighbourhood of $\pm\omega \in \mathbf{S}_a \subset \mathbf{S}^{d-1}$. Consequently, one has:

$$|\hat{x}\cdot\hat{\xi}| \leq 1 - \varepsilon_1, \quad \varepsilon_1 > 0, \tag{3.5}$$

for (x, ξ) in the support of $\nabla J_a(x)\chi(\xi_a)$ and λ sufficiently large. Here $\hat{x} = x/|x|$ and $\hat{\xi_a} = \xi_a/|\xi_a|$. With these notations, we have for any $\mu > 1$:

$$T^\chi_{\alpha\beta}(\lambda) = \mathcal{F}_\alpha(\lambda)\{\chi(D_a)J_a(x) - Q_a^* R(\lambda + i0)\}Q_a\mathcal{F}_\beta(\lambda)^*, \text{ for } \lambda \in \mathbf{I}_{\alpha\beta}. \tag{3.6}$$

Here Q_a is defined by

$$Q_a = \{I_a(x)J_a(x) + [-\Delta, J_a]\}\chi(D_a). \tag{3.7}$$

By Theorem 2.5 in [16], $T^\chi_{\alpha\beta}(\lambda)$ is strongly continuous for in λ. See also Yafaev [17] for other kind of representations for scattering matrices. From now on, we set $\mu = \lambda$ in J_a.

Proposition 3.2 *Under the assumptions (1.4) and (1.9), one has*

$$(\mathcal{F}_\alpha(\lambda)Q_a^* R(\lambda + i0)Q_a\mathcal{F}_\beta(\lambda)^* f_{2,\beta}(\cdot,\lambda), f_{1,\alpha}(\cdot,\lambda))_a = O(\lambda^{-1-\delta}). \tag{3.8}$$

Here $\delta > 0$ is given by

$$\delta = \min\{1/2, (\varepsilon_\alpha + \varepsilon_\beta - 1)/2, (\rho + \varepsilon_\alpha - 1)/2, (\rho + \varepsilon_\beta - 1)/2\}.$$

Proof. Recall that in [16], we prove that for $\varphi_\gamma \in C_0^\infty(\Sigma_a), \gamma = \alpha, \beta$,

$$(\mathcal{F}_\alpha(\lambda)Q_a^* R(\lambda + i0)Q_a\mathcal{F}_\beta(\lambda)^*\varphi_\beta, \varphi_\alpha)_a = O(\lambda^{-1/2-\delta}).$$

See Proposition 3.2 of [16]. In the present case, $\varphi_\gamma = f_{j,\gamma}(\cdot,\lambda)$ is λ-dependent and

$$\|f_{j,\gamma}(\cdot,\lambda)\|_{L^2(\mathbf{S}_a)} = O(\lambda^{-1/4})$$

for $\gamma = \alpha$, β and $j = 1$ or 2. We can mimic the arguments of [16] to prove (3.8) and the details are left to interested readers. □

Now let us look at the term

$$(\mathcal{F}_\alpha(\lambda)\chi(D_a)J_a(x)Q_a\mathcal{F}_\beta(\lambda)^* f_{2,\beta}(\cdot,\lambda), f_{1,\alpha}(\cdot,\lambda))_a.$$

Denote $\lambda_\gamma = \lambda - E_\gamma$, $e_\gamma = e^{i\sqrt{\lambda_\gamma}x_a\cdot\omega}\psi_\gamma(x^a)$ and

$$g_\beta(x,\lambda) = 2\pi(\mathcal{F}_\beta(\lambda)^* f_{2,\beta}(\cdot,\lambda))(x), \quad g_\alpha(x,\lambda) = 2\pi(\mathcal{F}_\alpha(\lambda)^* f_{1,\alpha}(\cdot,\lambda))(x).$$

Lemma 3.3 *Let $\chi_\pm$ be a bounded smooth function on* $\mathbf{R}$ *introduced in the proof of Theorem 2.1. Then for $\gamma = \alpha$ or β, $s > 1/2$, there exist C, λ_0 depending on supp $\hat{f}$ such that*

$$\begin{aligned}
\|\langle x_a\rangle^{s-1}\chi_-(|x_a\cdot\omega|/\langle x_a\rangle)g_\gamma(\cdot,\lambda)\| &\le C\lambda^{-1/2}\|\langle x_a\rangle^s f\|, && (3.9)\\
\|\langle x\rangle^{\varepsilon_\gamma-1/2}\chi_+(|x^a|/(\varepsilon\langle x\rangle))g_\gamma(\cdot,\lambda)\| &\le C\lambda^{-1/2}\|\langle x_a\rangle^s f\|, && (3.10)
\end{aligned}$$

uniformly in $\lambda > \lambda_0$ and $R > 1$.

Proof. Notice that the support of the Fourier transform of $e^{i\sqrt{\lambda_\gamma}x_a\cdot\omega}f(x_a)$ is contained in a region $\Omega = \{|\xi_a - \sqrt{\lambda}\omega| \le C_1\}$ and if λ is big enough, $|x_a\cdot\xi_a| \le (1-\eta)|x_a||\xi_a|$ for $\xi_a \in \Omega$ and x_a such that $|x_a\cdot\omega| \le (1-\varepsilon)|x_a|$. The first estimate of Lemma 3.3 then follows from microlocal resolvent estimates for $-\Delta_a$. See for example [15].

To prove the second one, we just apply the estimate

$$\|\langle x\rangle^{\varepsilon_\gamma-1/2}\chi_+(|x^a|/(\varepsilon\langle x\rangle))\mathcal{F}_\gamma(\lambda)^*\|_{\mathcal{L}(L^2(\mathbf{S}_a),L^2(\mathbf{X}))} \le C\lambda^{-1/4}.$$

which can be derived from the resolvent estimates of Agmon-Hörmander (cf. [6]). See also (2.20) in [15] where the estimate with $\varepsilon_\gamma = 0$ is proved. (3.10) then follows from the above estimate and the following well known one: for any $s > 1/2$, there is $C > 0$, such that

$$\|F_\gamma(\lambda)\langle x_a\rangle^{-s}\|_{\mathcal{L}(L^2(\mathbf{X}_a),L^2(\mathbf{S}_a))} \le C\lambda^{-1/4}, \quad \lambda > 1.$$

□

Proposition 3.4 *Under the assumptions (1.4) and (1.9), one has*

$$< J_a[-\Delta, J_a]g_\beta, g_\alpha >= O(\lambda^{-1-(\varepsilon_\alpha+\varepsilon_\beta-1)/2}),$$

as $\lambda \to \infty$.

Proof. We begin by noticing that the support of ∇J_a is contained in $\{|x| \ge C\sqrt{\lambda}\}\cap\{|x_a\cdot\omega| \le (1-\varepsilon_0)|x|\}$ for some $\varepsilon_0 > 0$ and $\nabla J_a(x) = O(\lambda^{-\tau/2}\langle x\rangle^{-(1-\tau)})$ for any $\tau \in [0,1]$. Let $\chi_\pm$ be chosen as in the proof of Theorem 2.1. By (3.10),

$$| < \chi_+(|x^a|/(\varepsilon\langle x\rangle))J_a[-\Delta, J_a]g_\beta, g_\alpha > | \le C\lambda^{-1-(\varepsilon_\alpha+\varepsilon_\beta-1)/2}.$$

On the support of $\chi_-(|x^a|/(\varepsilon\langle x\rangle))\nabla J_a(x)$,

$$|x^a| \le \varepsilon(1-2\varepsilon)\langle x\rangle, |x\cdot\omega| \le (1-\varepsilon_0)|x|.$$

Taking $\varepsilon > 0$ sufficiently small, we get on this support

$$|x_a\cdot\omega| \le (1-\varepsilon_0/2)|x_a|.$$

So we can apply (3.9) (with $s > 2N$) to this piece and obtain

$$< \chi_-(|x^a|/(\varepsilon\langle x\rangle))J_a[-\Delta, J_a]g_\beta, g_\alpha > | \le C\lambda^{-N}.$$

This proves Proposition 3.4. □

Similarly, one can also prove that

$$< (1-J_a)I_a g_\beta, g_\alpha >= O(\lambda^{-1-(\varepsilon_\alpha+\varepsilon_\beta-1)/2}).$$

From Propositions 3.2 and 3.4, it results that

$$(T_{\alpha\beta}(\lambda)f_{2,\beta}(\cdot,\lambda), f_{1,\alpha}(\cdot,\lambda))_a = \frac{1}{4\pi^2} < I_a g_\beta, g_\alpha > +O(\lambda^{-1-\delta}). \tag{3.11}$$

Now to finish the proof of Theorem 3.1, we need to pick out the leading term in $< I_a g_\beta, g_\alpha >$ as $\lambda \to \infty$. Notice that

$$\begin{aligned} g_\gamma(x,\lambda) &= \lim_{\varepsilon\to 0}\int_{\mathbf{R}} e^{-it(H_{0,a}-\lambda_\gamma)-\varepsilon|t|}(e_\gamma f)dt \\ &= e_\gamma \lim_{\varepsilon\to 0}\int_{\mathbf{R}} e^{-it(H_{0,a}+2\sqrt{\lambda_\gamma}\omega\cdot D_a)-\varepsilon|t|} f(x_a)dt \\ &= \frac{e_\gamma}{2\sqrt{\lambda_\gamma}}\int_{\mathbf{R}} e^{i\frac{t}{2\sqrt{\lambda_\gamma}}H_{0,a}} f(x_a+t\omega)dt \end{aligned} \tag{3.12}$$

Here γ equals α or β and f can be f_1 or f_2. This motivates us to take

$$h_\gamma(x,\lambda) = \frac{e_\gamma}{2\sqrt{\lambda_\gamma}}\int_{\mathbf{R}} f(x_a+t\omega)dt$$

as an approximation for g_γ. Put $r_\gamma = g_\gamma - h_\gamma$.

Lemma 3.5 *If $s > 1/2$,*

$$\|\langle x_a\cdot\omega\rangle^{-s} r_\gamma\| \le C_s/\sqrt{\lambda}, \tag{3.13}$$

and if $s > 3/2$,

$$\|\langle x_a\cdot\omega\rangle^{-s} r_\gamma\| \le C_s/\lambda. \tag{3.14}$$

For any $\varepsilon > 0$, $N > 0$, there is $C > 0$ such that

$$\|\langle x_a\cdot\omega\rangle^{N}\chi_-(\frac{|x_a\cdot\omega|}{\langle x_a\rangle}) r_\gamma\| \le C/\lambda. \tag{3.15}$$

Proof. (3.13) is clear, since it is true separately for g_γ and for r_γ. To prove (3.14), we write

$$r_\gamma = \frac{ie_\gamma}{4\lambda_\gamma}[(\frac{H_{0,a}}{2\sqrt{\lambda_\gamma}}+\omega\cdot D_a+i0)^{-1}(\omega\cdot D_a+i0)^{-1}$$

$$-(\frac{H_{0,a}}{2\sqrt{\lambda_\gamma}}+\omega\cdot D_a-i0)^{-1}(\omega\cdot D_a-i0)^{-1}]H_{0,a}\chi(D_a)f,$$

where $\chi\in C_0^\infty$, $\chi(\xi_a)=1$ near the support of $\hat{f}$. Remark that $\omega\cdot x_a$ is a conjugate operator in the sense of Mourre both for $\frac{H_{0,a}}{2\sqrt{\lambda_\gamma}}+\omega\cdot D_a$ and for $\omega\cdot D_a$ when $\lambda >> 1$. (3.14) follows from Mourre's theory. (3.15) can be most easily seen by writing down the Fourier transform of r_γ in time-dependent version and by applying the non-stationary phase method for $\lambda >> 1$. □

Proof of Theorem 3.1. We rewrite (3.11) in the form

$$\begin{aligned}
&(T_{\alpha\beta}(\lambda)f_{2,\beta}(\cdot,\lambda), f_{1,\alpha}(\cdot,\lambda))_a \\
&= \frac{1}{4\pi^2}\{< I_a h_\beta, h_\alpha > + < I_a r_\beta, h_\alpha > + < I_a h_\beta, r_\alpha > + < I_a r_\beta, r_\alpha >\} + O(\lambda^{-1-\delta}).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
< I_a h_\beta, h_\alpha > &= \frac{1}{4\lambda} < e^{i\frac{E_\alpha - E_\beta}{\sqrt{\lambda_\alpha}+\sqrt{\lambda_\beta}} x_a\cdot\omega} I_{\alpha\beta} f_2^\omega, f_1^\omega >_a + O(\lambda^{-3/2}) \\
&= \frac{1}{4\lambda} < I^\omega_{\alpha\beta} f_2^\omega, f_1^\omega >_{L^2(\Pi^\omega)} + O(\lambda^{-1}R(t)), t\to\infty,
\end{aligned}$$

where $R(t) = t^{-\min\{1/2,(\rho-1)/2\}}$ if $\rho\neq 2$ and $R(t) = t^{-1/2}\ln t$ if $\rho = 2$. This gives the leading term in Theorem 3.1. To estimate $< I_a r_\beta, h_\alpha >$, let $\chi_\pm$ be chosen as before. Then by Lemma 3.5,

$$< \chi_-(\frac{|x_a\cdot\omega|}{\langle x_a\rangle}) I_a r_\beta, h_\alpha >= O(\lambda^{-3/2}).$$

On the support of $\chi_-(\frac{|x^a|}{\varepsilon\langle x\rangle})\chi_+(\frac{|x_a\cdot\omega|}{\langle x_a\rangle})$, one has, for any $c\in\mathcal{A}, c\not\subseteq a$, $|x^c|\geq |x_a\cdot\omega||\pi^c\omega| - |y| - |x^a|$, where y is the orthogonal projection of x_a to Π^ω. Since $\pi^c\omega\neq 0$ for all $c\not\subseteq a$, one has $|x^c|\geq c_0|x_a|$, $c_0>0$, for $\varepsilon>0$ sufficiently small. So in this region $I_a = O(\langle x_a\rangle^{-\rho})$. An easy interpolation of the results of Lemma 3.5 gives that for any $\varepsilon>0$ and $0\leq t\leq 1$,

$$\|\langle x_a\cdot\omega\rangle^{-1/2-\varepsilon-t} r_\gamma\| \leq C/\lambda^{(1+t)/2}.$$

This proves

$$< \chi_-(\frac{|x^a|}{\varepsilon\langle x\rangle})\chi_+(\frac{|x_a\cdot\omega|}{\langle x_a\rangle}) I_a r_\beta, h_\alpha >= O(\lambda^{-1-(\rho-1-\varepsilon)/2})$$

for any $\varepsilon>0$. Similar arguments to the proof of (3.10) show that

$$< \chi_+(\frac{|x^a|}{\varepsilon\langle x\rangle})\chi_+(\frac{|x_a\cdot\omega|}{\langle x_a\rangle}) I_a r_\beta, h_\alpha >= O(\lambda^{-1-(\varepsilon_\alpha+\varepsilon_\beta-1-\varepsilon)/2}).$$

This shows $< I_a r_\beta, h_\alpha >= O(\lambda^{-1-\delta_0})$. The same estimates hold also for $< I_a h_\beta, r_\alpha >$ and $< I_a r_\beta, r_\alpha >$. Theorem 3.1 is proved. □

4 Proof of Theorem 1.1

The proof of Theorem 1.1 uses an induction on $\#a - \#b$. Let us begin with the case $\#a = 2$. In this case, the only cluster decomposition $b\in\mathcal{A}$ with $a\subset b$ is $b = a_{\max}$, and $\Sigma_a = \mathbf{S}_a$. There is then only one effective potential $I_{\alpha\beta}$, which is of short-range by the assumptions (1.4) and (1.9):

$$I_{\alpha\beta} = O(\langle x_a\rangle^{-\rho_1})$$

for some $\rho_1 > 1$. Theorem 3.1 says then that

$$\lim_{\lambda\to\infty} \lambda(T_{\alpha\beta}(\lambda) f_{2,\beta}, f_{1,\alpha})_a = \frac{1}{16\pi^2} < I^\omega_{\alpha\beta} f_2^\omega, f_1^\omega >_{L^2(\Pi^\omega)}$$

for all $f_j \in \mathcal{S}(\mathbf{X}_a)$ with compactly supported Fourier transform. This means that the high energy asymptotics of scattering matrices determine $I^\omega_{\alpha\beta}$ for all $\omega \in \mathbf{S}_a$. Since $\dim X_a \geq 2$, $\cup_{\omega\in\mathbf{S}_a}\Pi^\omega = \mathbf{X}_a$, $I_{\alpha\beta}$ can be recontructed by the standard argument using Radon transform (see, for example, [5]). Here we present a self-contained argument based on the following elementary lemma.

Lemma 4.1 *Let $k \geq 2$ and*

$$\omega = (\cos\theta, \sin\theta, 0, \cdots, 0) \in \mathbf{S}^{k-1}, \nu = (\sin\theta, -\cos\theta, 0, \cdots, 0), \theta \in [0, \pi].$$

Let u be a measurable function on $\mathbf{R}^k$ such that

$$t \to \sup_{\theta\in[0,\pi]} |u(x + tw)| \in L^1(\mathbf{R}; dt),$$

for a.e. $x \in \Pi^\omega = \{y \in \mathbf{R}^k, y \cdot \omega = 0\}$. Let $u^\omega = \int_{\mathbf{R}} u(x + t\omega)dt$. Suppose that $u^\omega \in L^1_{\mathrm{loc}}(\Pi^\omega)$ and $\lim_{x\in\Pi^\omega, x\to\infty} u^\omega(x) = 0$ uniformly in $\theta \in [0, \pi]$. Then for $v \in \mathcal{S}(\mathbf{X}^k)$, one has

$$< \hat{u}, \overline{\hat{v}} >_{\mathcal{S}',\mathcal{S}} = (2\pi)^{k-1} \int_0^\pi (\int_{\Pi^\omega} u^\omega(x') \overline{|D_\nu| v^\omega(x')} dx') d\theta. \tag{4.1}$$

Here D_ν is the derivation in the direction ν.

Proof. Assume first that $u \in \mathcal{S}$. To prove (4.1) in this case, we introduce the new variables

$$\xi_1 = \eta_2 \sin\theta, \xi_2 = -\eta_2\cos\theta, \xi_j = \eta_j, j = 3 \cdots k,$$

with $\eta_j \in \mathbf{R}$ and $\theta \in [0, \pi]$. We have the formula for this change of variables:

$$\int_{\mathbf{R}^k} f(\xi) d\xi = \int_0^\pi (\int_{\mathbf{R}^{k-1}} f(\eta_2\sin\theta, -\eta_2\cos\theta, \cdots, \eta_k)|\eta_2| d\eta') d\theta. \tag{4.2}$$

Write $x' = (x_2, \cdots, x_k)$, $\eta' = (\eta_2, \cdots, \eta_k) \in \mathbf{R}^{k-1}$ and $\omega = \omega(\theta) = (\cos\theta, \sin\theta, 0, \cdots, 0) \in \mathbf{S}^{k-1}$. An easy computation shows that the inverse Fourier transform in η' of

$$\hat{f}(\eta_2\sin\theta, -\eta_2\cos\theta, \eta_3, \cdots, \eta_k)$$

is

$$\frac{1}{(2\pi)^{k-1}} \int_{\mathbf{R}^{k-1}} e^{i\eta'\cdot x'} \hat{f}(\eta_2\sin\theta, -\eta_2\cos\theta, \cdots, \eta_k) d\eta'$$

$$= \int_{\mathbf{R}} f(y_1\cos\theta + x_2\sin\theta, y_1\sin\theta - x_2\cos\theta, x_3, \cdots, x_k) dy_1 = f^\omega(x').$$

Here we have identified $x' \in \mathbf{R}^{k-1}$ with the orthogonal projection of $x \in \mathbf{R}^k$ onto Π^ω (which is $(x_2\sin\theta, -x_2\cos\theta, x_3, \cdots, x_k)$). This gives

$$\int \hat{u}(\xi)\overline{\hat{v}(\xi)} d\xi = \int_0^\pi (\int_{\mathbf{R}^{k-1}} \hat{u}(\eta_2\sin\theta, -\eta_2\cos\theta, \cdots, \eta_k) \overline{\hat{v}(\eta_2\sin\theta, -\eta_2\cos\theta, \cdots, \eta_k)} |\eta_2| d\eta') d\theta$$

$$= (2\pi)^{k-1} \int_0^{\pi} (\int_{\mathbf{R}^{k-1}} u^{\omega}(x') \overline{|D_{x_2}| v^{\omega}(x')} dx') d\theta,$$

which proves (4.1) in the case $u \in \mathcal{S}$.

In the general case, notice that the assumptions on u ensure that $|D_\nu| u^\omega \in \mathcal{S}'(\Pi^\omega)$ and that one can construct by cut-off and regularization a sequence $\{u_n\}$ with $u_n \in \mathcal{S}$, $u_n \to u$ in $\mathcal{S}'$ and $|D_\nu| u_n^\omega \to |D_\nu| u^\omega$ in $\mathcal{S}'(\Pi^\omega)$ uniformly for $\theta \in [0, \pi]$. The lemma follows by taking the limit $n \to \infty$ in (4.1) applied to u_n. $\square$

By pushing further the arguments used in the above lemma, we can also give a reconstruction formula of u from u^ω, but it is equivalent to a two-dimensional inverse Radon transform([5]). Notice that all the assumptions on u are satisfied if $|u(x)| \le C\langle x \rangle^{-1-\varepsilon}, \forall x$. Since $\dim \mathbf{X}_a \ge 2$, Lemma 4.1 shows that $I_{\alpha\beta}$ is uniquely determined by $\{I_{\alpha\beta}^\omega, \omega \in \mathbf{S}_a\}$. Remark that in the case $\#a = 2$, Theorem 2.1 is not needed.

Assume now that $\#a = k \ge 3$. For each $b \in \mathcal{A}$ with $b \supset a$, there is an effective potential $V_{\alpha\beta}^b$. When $\#b = \#a - 1$, P^a can be regarded as a two-cluster Subhamiltonian of P^b. The results on relative scattering (Theorems 2.1 and 3.1) applied to this b allow us to reconstruct $V_{\alpha\beta}^b$ from $T_{\alpha\beta}(\lambda)$ by the arguments used above. In general case, we use an induction on $\#a - \#b$. Assume that $\#a - \#b = k$ and that all effective potentials $V_{\alpha\beta}^c$ with $c \supset a, \#a - \#c < k$ are reconstructed. To see that the scattering matrices $T_{\alpha\beta}(\lambda)$ determine the effective potential $V_{\alpha\beta}^b$, we take a family of cluster decompositions $b_j \in \mathcal{A}$ such that

$$a = b_1 \subset b_2 \subset b_2 \cdots \subset b_k = b$$

and $\#b_j = \#b_{j+1} + 1$. Applying Theorems 2.1 and 3.1 to b, we see that $T_{\alpha\beta}(\lambda)$ determines

$$I_{\alpha\beta}^{b,\omega} = \int_{\mathbf{R}} I_{\alpha\beta}^b(x_a^b + t\omega) dt, \quad \forall \omega \in \mathbf{S}_a^b \setminus (\cup_{c \not\subseteq a} \mathbf{X}_c).$$

Remark that $\{c \in \mathcal{A}; c \not\subseteq a, c \subset b\} = \cup_{j=2}^k \mathcal{A}_{a,b_j}$ and

$$I_{\alpha\beta}^{b,\omega} = \sum_{j=2}^{k-1} V_{\alpha\beta}^{b_j,\omega} + V_{\alpha\beta}^{b,\omega}.$$

By the induction hypothesis, all $V_{\alpha\beta}^{b_j,\omega}, 2 \le j \le k-1$, are known. So $V_{\alpha\beta}^{b,\omega} = I_{\alpha\beta}^{b,\omega} - \sum_{j=2}^{k-1} V_{\alpha\beta}^{b_j,\omega}$ is recovered from the high energy asymptotics of $T_{\alpha\beta}^b(\lambda)$ for all $\omega \in \mathbf{S}_a^b \setminus (\cup_{c \not\subseteq a} \mathbf{X}_c)$. However, $V_{a,b} = \sum_{c \cup a = b} V_c(x^c)$ and in this case, $|x^a| + |x^c| \ge c|x^b|$ for all $c \in \mathcal{A}_{a,b}$. The assumptions (1.4) and (1.9) give

$$V_{\alpha\beta}^b = O(\langle x_a^b \rangle^{-\rho_1})$$

for some $\rho_1 > 1$. This means that $V_{\alpha\beta}^{b,\omega}$ is well defined for all $\omega \in \mathbf{S}_a^b$ (actually continuous in ω under our assumptions). So $V_{\alpha\beta}^{b,\omega}$ is uniquely determined by $T_{\alpha\beta}^b(\lambda)$ for all $\omega \in \mathbf{S}_a^b$. Since $\dim \mathbf{X}_a^b \ge 2$, the arguments used before allow to reconstruct the effective potential $V_{\alpha\beta}^b$, $b \supset a$. Theorem 1.1 is proved. $\square$

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