ASYMPTOTIC EXPANSION IN TIME OF THE SCHRÖDINGER GROUP ON CONICAL MANIFOLDS

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Abstract. For Schrödinger operator P on Riemannian manifolds with conical end, we study the contribution of zero energy resonant states to the singularity of the resolvent of P near zero. Long-time expansion of the Schrödinger group $U(t) = e^{-itP}$ is obtained under a non-trapping condition at high energies.

Contents

1. Introduction

In this work, we study long-time asymptotic behavior of the dynamics generated by Schrödingers operator on Riemannian manifolds with conical end. This problem is closely related to low-energy spectral analysis, which has been studied for Schrödinger operators on $\mathbb{R}^n$ since a long time ([1, 5, 16, 18, 23, 25, 26, 28]). These works are concerned with perturbation of a constant elliptic differential operator by a term decaying like $O(|x|^{-2-\epsilon})$ as $x \to \infty$. See also [35] for spectral analysis of N-body Schrödinger operator near its first threshold, the bottom of the essential spectrum. The relevant issue in these works is the asymptotic expansion of resolvent near the threshold. The main difficulty arises from possible existence of zero energy resonances, which is invisible in the L^2-setting. See also [13, 24, 39] for threshold spectral analysis for slowly decreasing potentials on $\mathbb{R}^n$ where no zero energy resonant state is present. Thresholds are exceptional points where new physical phenomena appear ([2, 3, 36]). Low-energy spectral analysis of Schrödinger operators has many interesting applications in spectral and scattering theories ([1, 5, 18, 25, 26]). In particular, a mathematical proof of the phenomenon claimed in [2] should pass by the analysis of threshold energy resonances in their setting. For the role of zero energy resonance in

1991 *Mathematics Subject Classification.* 35P25, 47A40, 81U10.

Key words and phrases. resolvent expansion, zero energy resonance, Schrödinger operator with metric.

The author is grateful to G. Carron for his stimulating work [12] and for many helpful discussions. Research supported in part by MSRI in Berkeley and by a research program of the Chinese Academy of Sciences.

extended index theory in Riemannian geometry, we refer to [4, 7, 10, 11, 20, 22].

The present work concerns the perturbation of a non-trivial model operator P_0 on manifold with conical end, and is motivated by a recent paper of G. Carron [12] on the jump of the spectral shift function at zero. Let $\mathbf{M}$ be a connected Riemannian manifold which, outside some compact, is isometric to a conical space $\mathbb{R}_+ \times \mathbf{\Sigma}$, $\mathbf{\Sigma}$ being a compact $(n-1)$-dimensional manifold with or without boundary. If $\mathbf{\Sigma}$ is of boundary, the Dirichlet condition is used. We want to study Schrödinger operator $P = -\Delta_g + W(x)$ which is perturbation of a model operator P_0 of the form

$$P_0 = -\Delta_{g_0} + \frac{q(\theta)}{r^2}. \tag{1.1}$$

Here $(r, \theta) \in \mathbb{R}_+ \times \mathbf{\Sigma}$ is some polar coordinates, $q(\theta)$ is a real continuous function and g_0 is a metric on $\mathbb{R}_+ \times \mathbf{\Sigma}$ of the form

$$g_0 = dr^2 + r^2 h(\theta, d\theta)$$

with h an arbitrary Riemannian metric on $\mathbf{\Sigma}$ independent of r. The term $\frac{q(\theta)}{r^2}$ can not be treated by method of perturbation usually used in low-energy spectral analysis. New phenomenon occurs due to the non trivial metric h. Assume $-\Delta_h + q(\theta) \geq -\frac{(n-2)^2}{4}$. Set

$$\sigma_\infty = \left\{\nu;\ \nu = \sqrt{\lambda + \frac{(n-2)^2}{4}}, \lambda \in \sigma(-\Delta_h + q)\right\}, \quad \sigma_k = \sigma_\infty \cap [0, k], \qquad k \in \mathbb{N}. \tag{1.2}$$

If $h = (d\theta)^2$ is the Euclidean metric on the sphere $\mathbb{S}^{n-1}$, $n \geq 2$ and $q = 0$, then, $P_0 = -\Delta$ and

$$\sigma_\infty = \left\{\frac{n-2}{2} + k;\ k \in \mathbb{N}\right\}. \tag{1.3}$$

In this case, σ_∞ consists of either only half-integers (n odd) or only integers (n even). In particular, for Laplace operator $-\Delta$, one has

$$\sigma_1 = \begin{cases} \{0, 1\}, & n = 2, \\ \{\frac{1}{2}\}, & n = 3, \\ \{1\}, & n = 4, \\ \emptyset, & n \geq 5. \end{cases} \tag{1.4}$$

The main part of this work is devoted to analyzing the contribution of the zero energy resonant states to the singularities of resolvent $R(z) = (P - z)^{-1}$ at $z = 0$ for general second order perturbations of P_0. The essential difference from the previous works [1, 5, 16, 18, 23, 25, 26, 28] on perturbation of constant elliptic operators on $\mathbb{R}^n$ is the presence of a non-trivial metric h. For perturbation of the Laplacian $-\Delta$ on $\mathbb{R}^n$, it is well-known that there are at most one s-wave resonant state and two p-wave resonant states for $n = 2$, and one s-wave resonant state for $n = 3, 4$ and no zero energy resonant state for $n \geq 5$. It is well known that the case $n = 2$ is the most difficult, due to the interaction between the three zero resonant states. New phenomenon occurs in geometric setting. In fact, even if we simply replace the standard metric $(d\theta)^2$ on $\mathbb{S}^{n-1}$ by a rescaled one $h = \hbar^2 (d\theta)^2$, where $\hbar > 0$ is a small constant, the multiplicity of zero resonance may be arbitrarily large (see an example for a family of Hodge-De Rham Laplacian given in [37]). The interaction between the zero resonant states is the main difficulty to determine the singularity of the resolvent at zero in this work.

The final result of this work can roughly be described as follows. Assume that $W(x) = \frac{q(\theta)}{r^2} + V_0(x)$ with

$$|\partial_x^\alpha V_0(x)| \le C\langle x\rangle^{-\rho_0-|\alpha|}, \quad \forall x \tag{1.5}$$

and that the classical Hamiltonian $p(x,\xi) = |\xi|_g^2 + \frac{q(\theta)}{r^2}$ is non-trapping at high energies. We obtain the asymptotic expansion of the Schrödinger group $U(t) = e^{-itP}$ in the form

$$U_c(t) = \sum_{j=1}^{\kappa_0} \Phi_{-\varsigma_j,-\delta_{\varsigma_j}}(t) t^{\varsigma_j - 1}\Pi_{r,j} + \cdots + O(|t|^{-N-1-\epsilon}) \tag{1.6}$$

in $\mathcal{L}(L^{2,s}; L^{2,-s})$, $s > 2N+1$, where N depends on ρ_0 and the spectral nature of 0 w. r. t. P,

$$U_c(t) = U(t) - \sum_{\lambda_j\in\sigma_p(P)} e^{-it\lambda_j}\Pi_{\lambda_j},$$

Π_{λ_j} is the orthogonal projection onto the eigenspace associated to the eigenvalue λ_j of P,

$$\Pi_{r,j} = e^{i\pi\varsigma_j}\sum_{l=1}^{m_j} < \cdot, u_j^{(l)} > u_j^{(l)}, \quad j = 1,\cdots,\kappa_0$$

with $u_j^{(l)}$ the zero energy resonant states of P having the asymptotic behavior

$$\frac{\varphi(\theta)}{r^{\frac{n-2}{2}+\nu}}(1+o(1)), \qquad r\to\infty, \tag{1.7}$$

for some $\varphi \ne 0$ and $\nu = \varsigma_j \in \sigma_1$ (see (4.28)), and $\Phi_{\lambda,m}(t)$ is defined by (6.9). If there is no resonant state with energy zero, some corresponding terms disappear and our analysis shows that

$$U_c(t) \simeq C_0|t|^{-\nu_0-1}, \tag{1.8}$$

where if 0 is not an eigenvalue of P, $\nu_0 = \nu_m$ with

$$\nu_m = \sqrt{\frac{(n-2)^2}{4} + \lambda_0},$$

λ_0 being the smallest eigenvalue of $-\Delta_h + q(\theta)$; and if 0 is an eigenvalue of P,

$$\nu_0 = \min\{\nu_m, \min\{\nu - 1; \nu \in \sigma_\infty \cap]\ [\nu_m]+1, [\nu_m]+2]\}\},$$

$[\nu]$ being the largest integer less than or equal to ν. For example, in the case $P = P_0 = -\Delta_{g_0}$ on $\mathbb{R}^2 \setminus ([0,\infty[\times\{0\})$ with the Dirichlet condition on $[0,\infty[\times\{0\}$, where $g_0 = dr^2 + r^2 h$, $h = a^2(d\theta)^2$ with $a > 0$ chosen such that the unit circle is of length one in the metric h. Then, (1.8) gives

$$e^{it\Delta_{g_0}} \simeq C|t|^{-\pi-1}, \quad t\to\infty, \tag{1.9}$$

in appropriate spaces. This is to compare with the well-known asymptotics of the two-dimensional free Schrödinger group

$$e^{it\Delta} \simeq C'|t|^{-1}, \quad t\to\infty.$$

The relevance of the eigenvalues of $-\Delta_h + q(\theta)$ in the long-time asymptotics of $U(t)$ is as follows. Small eigenvalues of $-\Delta_h + q(\theta)$ determines leading terms in long-time expansion of the Schrödinger group $U(t)$. Our study on zero energy resonant states shows that the eigenfunctions of $-\Delta_h + q(\theta)$ associated to an eigenvalue $\lambda_\nu = \nu^2 - \frac{(n-2)^2}{4}$ with $\nu \in [0,1]$

may produce resonant states of P at 0 with space behavior (1.7) which in turn create singularity of the form z_ν^{-1} of $R(z)$ at 0, where

$$z_\nu^{-1} = \begin{cases} \ln z, & \text{if } \nu = 0 \\ z^{-\nu}, & \text{if } 0 < \nu < 1, \\ (z \ln z)^{-1}, & \text{if } \nu = 1. \end{cases}$$

These singularities of $R(z)$ give the leading terms in presence of zero energy resonant states in long-time expansion of the Schrödinger group of the order

$$\Phi_{-\nu}(t) = \begin{cases} t^{\nu-1}, & \text{if } 0 \le \nu < 1, \\ (\ln t)^{-1}, & \text{if } \nu = 1. \end{cases}$$

To obtain long-time expansion for $U_c(t)$, we need information about the spectral measure over the whole real axis $\mathbb{R}$, in particular, the structure of the singularity of the resolvent $R(z) = (P - z)^{-1}$ at zero, where the zero energy resonant states play an important role. Their large multiplicity causes the main difficulty to the proof of existence and the calculation of asymptotic expansion of $R(z)$ near zero. The absence of positive eigenvalues and the local smoothness away from 0 of spectral measure are proved by Mourre's method. The high energy part of spectral measure is studied under a non-trapping condition which is necessary to obtain high energy resolvent estimates (see [33, 34] for the semi-classical case where the Planck constant $\hbar$ can be taken as $\hbar = \frac{1}{\sqrt{\lambda}} \to 0$).

We expect the low-energy resolvent asymptotics obtained in this work to be useful in the study of a Levinson-type theorem and of some complex phenomenon similar to the Efimov effect in geometric scattering.

The plan of this work is as follows. In Sections 2 and 3, we recall some results announced in [37] on the model operator P_0 and on the characterization for resonant states for P. In particular, we show that the space-behavior of resonant states with zero energy are governed by small eigenvalues of $-\Delta_h + q(\theta)$. In Section 4, we give a representation formula for the resolvent $R(z)$ and obtain its asymptotic expansions in the case $0 \notin \sigma_\infty$. The case $0 \in \sigma_\infty$ has specific features and is studied in Section 5. After the proof of absence of positive eigenvalue of P and of local smoothness of the boundary value of resolvent, long-time expansion for wave functions are obtained in Section 6 under a non-trapping condition on the classical Hamiltonian $p = |\xi|_g^2 + \frac{q(\theta)}{r^2}$ at high energies.

Notation. The scalar product on $L^2(\mathbb{R}_+; r^{n-1}dr)$ and $L^2(\mathbf{M}; dv)$ is denoted by $< \cdot, \cdot >$ and that on $L^2(\mathbf{\Sigma})$ by $(\cdot, \cdot)$. $H^{r,s}(\mathbf{M})$, $r \in \mathbb{Z}$, $s \in \mathbb{R}$, denotes the weighted Sobolev space of order r with volume element $\langle x \rangle^{2s} dv$. The duality between $H^{1,s}$ and $H^{-1,-s}$ is identified with L^2-product. Denote $H^{0,s} = L^{2,s}$. $\mathcal{L}(H^{r,s}; H^{r',s'})$ stands for the space of continuous linear operators from $H^{r,s}$ to $H^{r',s'}$. In this work, P and P_0 are considered as self-adjoint operators in L^2 with form domain H_0^1. The complex plane $\mathbb{C}$ is slit along positive real axis so that $z^\nu = e^{\nu \ln z}$ and $\ln z = \log |z| + i \arg z$ with $0 < \arg z < 2\pi$ are holomorphic for z near 0 in the slit complex plane.

2. The model operator

In this Section, we state some results on the resolvent and the Schrödinger group for the unperturbed operator

$$P_0 = -\Delta_{g_0} + \frac{q(\theta)}{r^2} \tag{2.1}$$

on $\mathbf{M}_0 = \mathbb{R}_+ \times \mathbf{\Sigma}$, where $\mathbf{\Sigma}$ is an $(n-1)$-dimensional compact manifold, $n \geq 2$. Here (r, θ) is the polar coordinates on $\mathbf{M}_0$, $q(\theta)$ is a real continuous function and g_0 is a metric of the form

$$g_0 = dr^2 + r^2 h$$

with h a Riemannian metric on $\mathbf{\Sigma}$ independent of r. If $\mathbf{\Sigma}$ is of boundary, the Dirichlet condition is used for P_0 on $\partial\mathbf{M}_0$. Let Δ_h denote Laplace-Beltrami operator on $(\mathbf{\Sigma}, h)$. Assume

$$-\Delta_h + q(\theta) \geq -\frac{(n-2)^2}{4}, \qquad \text{on } L^2(\mathbf{\Sigma}). \tag{2.2}$$

Then, $P_0 \geq 0$ in $L^2(\mathbf{M}; dv)$ (see [12]). One has

$$P_0 = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(-\Delta_h + q(\theta)).$$

Put

$$\sigma_\infty = \left\{\nu;\ \nu = \sqrt{\lambda + \frac{(n-2)^2}{4}}, \lambda \in \sigma(-\Delta_h + q)\right\}. \tag{2.3}$$

Denote

$$\sigma_k = \sigma_\infty \cap [0, k], \qquad k \in \mathbb{N}.$$

For $\nu \in \sigma_\infty$, let n_ν denote the multiplicity of $\lambda_\nu = \nu^2 - \frac{(n-2)^2}{4}$ as eigenvalue of $-\Delta_h + q(\theta)$. Let $\{\varphi_\nu^{(j)}; \nu \in \sigma_{h,q}, 1 \leq j \leq n_\nu\}$ denote an orthonormal basis of $L^2(\mathbf{\Sigma})$ consisting of eigenfunctions of $-\Delta_h + q(\theta)$:

$$(-\Delta_h + q(\theta))\varphi_\nu^{(j)} = \lambda_\nu \varphi_\nu^{(j)}, \quad (\varphi_\nu^{(i)}, \varphi_\nu^{(j)}) = \delta_{ij}.$$

Let π_ν denote the orthogonal projection in $L^2(\mathbf{M})$ onto the subspace spanned by the eigenfunctions of $-\Delta_h + q$ associated with the eigenvalue λ_ν

$$\pi_\nu f = \sum_{j=1}^{n_\nu} (f, \varphi_\nu^{(j)}) \otimes \varphi_\nu^{(j)}, \quad f \in L^2(\mathbf{M}).$$

Set

$$Q_\nu = -\frac{d^2}{dr^2} - \frac{n-1}{r}\frac{d}{dr} + \frac{\nu^2 - \frac{(n-2)^2}{4}}{r^2}, \qquad \text{in } L^2(\mathbb{R}_+; r^{n-1}dr). \tag{2.4}$$

Then, we have the orthogonal decomposition for the resolvent $R_0(z) = (P_0 - z)^{-1}$

$$R_0(z) = \sum_{\nu \in \sigma_\infty} (Q_\nu - z)^{-1}\pi_\nu, \quad z \notin \mathbb{R}. \tag{2.5}$$

To establish the asymptotic expansion of the resolvent $R_0(z)$ for z near 0, we look for the asymptotic expansion as $z \to 0$ for each $(Q_\nu - z)^{-1}$ and estimate the remainders w.r.t.

ν. Remark that Q_ν can be diagonalized by the Hankel transform of order ν ([38]). The Schwartz kernel of $(Q_\nu - z)^{-1}$ is given by:

$$\begin{aligned} K_\nu(r,\tau;z) &= -(r\tau)^{-\frac{n-2}{2}} \int_0^\infty e^{-\frac{r^2+\tau^2}{4it}+izt-i\frac{\pi\nu}{2}} J_\nu(\frac{r\tau}{2t})\,\frac{dt}{2t} \\ &= -(r\tau)^{-\frac{n-2}{2}} \int_0^\infty e^{i\frac{\rho}{t}+izr\tau t-i\frac{\pi\nu}{2}} J_\nu(\frac{1}{2t})\,\frac{dt}{2t} \end{aligned} \tag{2.6}$$

for $(r,\tau)\in\mathbb{R}_+^2$ and z with $\Im z>0$ (see [12]). Here J_ν is the Bessel function of the first kind of order ν and

$$\rho=\rho(r,\tau)\equiv\frac{r^2+\tau^2}{4r\tau}.$$

J_ν can be represented as

$$J_\nu(\lambda)=\frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)}(\frac{\lambda}{2})^\nu\int_{-1}^1 e^{i\lambda t}(1-t^2)^{\nu-1/2}dt,\quad \Re\,\nu>-1/2, \tag{2.7}$$

and has the asymptotic behavior: $J_\nu(\lambda)=O(\lambda^\nu)$ as $\lambda\to0$, $J_\nu(\lambda)=O(\lambda^{-1/2})$ as $\lambda\to\infty$. See [27, 38].

We first give a formal expansion for resolvent $(Q_\nu-z)^{-1}$, $\nu\in\sigma_\infty$. By an abuse of notation, we use the same letter K to denote the operator defined by the Schwartz kernel $K(r,\tau)$ in $L^2(\mathbb{R}_+, r^{n-1}dr)$. Let $\nu\in\sigma_\infty$ and $l\in\mathbb{N}$ with $l\le\nu<l+1$. If $l\ge1$, $K_\nu(z)$ can be first expanded in the form

$$K_\nu(r,\tau;z) = \sum_{k=0}^{l-1} z^k F_{\nu,k} + R_{\nu,l-1}(z) \qquad \text{with} \tag{2.8}$$

$$F_{\nu,k} = -(r\tau)^{-\frac{n-2}{2}+k}\frac{i^k}{k!}\int_0^\infty e^{i\frac{\rho}{t}-i\frac{\pi\nu}{2}} t^k J_\nu(\frac{1}{2t})\,\frac{dt}{2t} \tag{2.9}$$

$$R_{\nu,l-1}(z) = -(r\tau)^{-\frac{n-2}{2}}\int_0^\infty e^{i\frac{\rho}{t}-i\frac{\pi\nu}{2}}\mathcal{O}_{l-1}(e^{ir\tau zt})J_\nu(\frac{1}{2t})\,\frac{dt}{2t}. \tag{2.10}$$

Here and in the following, $\mathcal{O}_N(g(s))$ denotes the remainder in Taylor expansion of g up to the N-th order:

$$\mathcal{O}_N(g(s))=g(s)-\sum_{j=0}^N\frac{g^{(k)}(0)}{k!}s^k=\frac{1}{N!}\int_0^1(1-\theta)^N s^{N+1}g^{(N+1)}(s\theta)\,d\theta.$$

If $l=0$ and $0\le\nu<1$, set $R_{\nu,l-1}(z)=K_\nu(z)$. Split $R_{\nu,l-1}(z)$ into two pieces

$$R_{\nu,l-1}(z)=R_{\nu,l-1,1}(z)+R_{\nu,l-1,2}(z)$$

with $R_{\nu,l-1,1}(z)$ defined by the integral for $t\in]0,1]$ in (2.10). It is clear that $R_{\nu,l-1,1}(z)$ has the following asymptotic expansion for any N

$$R_{\nu,l-1,1}(r,\tau;z)=\sum_{j=l}^N z^j K_{\nu,1,j}(r,\tau)+R_{\nu,N,1}(r,\tau;z), \tag{2.11}$$

where

$$K_{\nu,1,j}(r,\tau)=-(r\tau)^{-\frac{n-2}{2}+j}\int_0^1 e^{i\frac{\rho}{t}-i\frac{\pi\nu}{2}}\frac{(it)^j}{j!}J_\nu(\frac{1}{2t})\,\frac{dt}{2t}$$

and

$$R_{\nu,N,1}(r,\tau;z)=-(r\tau)^{-\frac{n-2}{2}}\int_0^1 e^{i\frac{\rho}{t}-i\frac{\pi\nu}{2}}J_\nu(\frac{1}{2t})\,\mathcal{O}_N(e^{izr\tau t})\,\frac{dt}{2t}. \tag{2.12}$$

The asymptotic expansion of $R_{\nu,l-1,2}(z)$ can be deduced from Lemma A.1 in [37]. Let us begin with the case $l=0$ and $\nu \in [0,1[$. Then, $K_{\nu,2}(z) \equiv R_{\nu,l-1,2}(z)$ can be written in the form

$$K_{\nu,2}(r,\tau;z) = \int_1^\infty e^{izr\tau t} t^{-1-\nu} f(\frac{1}{t};r,\tau,\nu)\, dt \tag{2.13}$$

where

$$f(s;r,\tau,\nu) = D_\nu(r,\tau)\int_{-1}^1 e^{i(\rho+\theta/2)s}(1-\theta^2)^{\nu-1/2}d\theta, \quad \nu \geq 0, \tag{2.14}$$

with

$$D_\nu(r,\tau) \;=\; a_\nu (r\tau)^{-\frac{n-2}{2}}, \quad a_\nu = -\frac{e^{-i\pi\nu/2}}{2^{2\nu+1}\pi^{1/2}\Gamma(\nu+1/2)}.$$

Write $f(s;r,\tau,\nu) = \sum_{j=0}^\infty s^j f_j(r,\tau,\nu)$, $s\in\mathbb{R}$, where

$$f_j(r,\tau,\nu) = (r\tau)^{-\frac{n-2}{2}} P_{j,\nu}(\rho), \tag{2.15}$$

with $P_{j,\nu}(\rho)$ a polynomial in ρ of degree j:

$$P_{j,\nu}(\rho) = \frac{i^j a_\nu}{j!}\int_{-1}^1 (\rho+\theta/2)^j (1-\theta^2)^{\nu-1/2}\, d\theta. \tag{2.16}$$

In particular,

$$f_0(r,\tau,\nu) \;=\; d_\nu (r\tau)^{-\frac{n-2}{2}}, \quad d_\nu = -\frac{e^{-i\pi\nu/2}}{2^{2\nu+1}\Gamma(\nu+1)} \tag{2.17}$$

$$f_1(r,\tau,\nu) \;=\; i d_\nu (r\tau)^{-\frac{n-2}{2}}\rho. \tag{2.18}$$

If $\nu \in [0,1[$, Lemma A.1 in [37] can be applied to obtain an asymptotic expansion for $K_{\nu,2}(z)$.

In the case $l \geq 1$, inserting the integral remainder formula of $\mathcal{O}_{l-1}(e^{ir\tau z t})$ into (2.10), we can still apply Lemma A.1 in [37] to obtain the formal expansion of $R_{\nu,l-1,2}(z)$. Summing up, we have the following

Proposition 2.1. *Let $\nu \in \sigma_\infty$ and $l \in \mathbb{N}$ with $l \leq \nu < l+1$. Set $\nu' = \nu - l \in [0,1[$.*

(a). If $l<\nu<l+1$, one has

$$(Q_\nu - z)^{-1} = \sum_{j=0}^{N} z^j F_{\nu,j} + z^{\nu'}\sum_{j=l}^{N-1} z^j G_{\nu,j} + R_{\nu,N}(z) \tag{2.19}$$

with $F_{\nu,j}$ given by (2.9) for $0\leq j\leq l$ and for $l+1\leq j\leq N$,

$$F_{\nu,j} \;=\; \frac{i^l (j-l)!}{j!}(r\tau)^j C_{\nu',j-l} + \frac{(ir\tau)^j}{j!}\int_0^1 t^{j-\nu-1} f(\frac{1}{t};r,\tau,\nu)\, dt, \tag{2.20}$$

$$G_{\nu,j} \;=\; (r\tau)^{j+\nu'} b_{\nu',j} f_{j-l}, \quad l\leq j\leq N, \tag{2.21}$$

$$\begin{aligned} R_{\nu,N}(z) \;=\;& z^{\nu'+N}G_{\nu,N} + \frac{(ir\tau z)^l}{(l-1)!}\int_0^1 (1-\theta)^{l-1}(\tilde{R}_{\nu',N-l,2}(\theta z r\tau) \\ &+ \int_0^1 \mathcal{O}_{N-l}(e^{it\theta z r\tau}) t^{l-1-\nu} f(\frac{1}{t};r,\tau,\nu)dt)d\theta \end{aligned} \tag{2.22}$$

When $l=0$, the integral in θ is absent.

(b). If $\nu = l \in \mathbb{N}$, then,

$$(Q_\nu - z)^{-1} = \sum_{j=0}^{N} z^j F_{\nu,j} + \ln z \sum_{j=l}^{N} z^j G_{\nu,j} + R_{\nu,N}(z) \tag{2.23}$$

with $F_{\nu,j}$ given by (2.9) for $0 \le j \le l-1$ and

$$\begin{aligned}
F_{\nu,j} &= (r\tau)^j \{ \frac{i^l (j-l)!}{j!} C_{0,j-l} - \ln(r\tau) \frac{i^j f_{j-l}}{j!} - c_{l,j} f_{j-l} \} && (2.24)\\
&\quad + \frac{(ir\tau)^j}{j!} \int_0^1 t^{j-l-1} f(\frac{1}{t}; r, \tau, \nu)\, dt, \quad l \le j \le N, && (2.25)\\
G_{\nu,j} &= -(ir\tau)^j \frac{f_{j-l}}{j!}, \quad l \le j \le N, && (2.26)\\
R_{0,N}(z) &= (izr\tau)^{N+1} f_{N+1} b_{N+1} + \tilde{R}_{0,N,2}(zr\tau) \\
&\quad + \int_0^1 \mathcal{O}_N(e^{itzr\tau}) t^{-1} f(\frac{1}{t}; r, \tau, \nu) dt \qquad \textit{and} && (2.27)\\
R_{l,N}(z) &= \frac{(ir\tau z)^{N+1} (N-l+1)!}{(N+1)!} f_{N-l+1} b_{N-l+1} + \int_0^1 (1-\theta)^{l-1} (\tilde{R}_{0,N-l,2}(\theta z r\tau) \\
&\quad + \int_0^1 \mathcal{O}_{N-l}(e^{it\theta z r\tau}) t^{-1} f(\frac{1}{t}; r, \tau, \nu) dt) d\theta && (2.28)
\end{aligned}$$

for $\nu = l \ge 1$. Here $c_{0,j} = 0$ for all j and

$$c_{l,j} = -\frac{i^j}{(l-1)!(j-l)!} \int_0^1 (1-\theta)^{l-1} \theta^{j-l} \ln\theta \, d\theta, \quad l \ge 1, j \ge l.$$

Here, $C_{\nu',j}$ and $b_{\nu',j}$ are defined in Lemma A.1, [37], *for $0 \le \nu' < 1$.*

Estimating the remainders of Proposition 2.1 uniformly with respect to ν, we can derive the asymptotic expansion of the resolvent of P_0. Define for $\nu \in \sigma_\infty$

$$z_\nu = \begin{cases} z^{\nu'}, & \text{if } \nu \notin \mathbb{N} \\ z \ln z, & \text{if } \nu \in \mathbb{N} \end{cases}$$

Let $\sigma_N = \sigma_\infty \cap [0, N]$. For $\nu > 0$, let $[\nu]_-$ be the largest integer strictly less than ν. When $\nu = 0$, set $[\nu]_- = 0$. Define δ_ν by $\delta_\nu = 1$, if $\nu \in \sigma_\infty \cap \mathbb{N}$; 0, otherwise. One has $[\nu] = [\nu]_- + \delta_\nu$.

Theorem 2.2. *The following asymptotic expansion holds for z near 0 with $\Im z > 0$.*

$$R_0(z) = \delta_0 \ln z \, G_{0,0} + \sum_{j=0}^{N} z^j F_j + \sum_{\nu \in \sigma_N} z_\nu \sum_{j=[\nu]_-}^{N-1} z^j G_{\nu, j+\delta_\nu} \pi_\nu + R_0^{(N)}(z), \tag{2.29}$$

in $\mathcal{L}(-1, s; 1, -s)$, $s > 2N+1$. Here

$$G_{\nu,j}(r,\tau) = \begin{cases} b_{\nu',j} (r\tau)^{j+\nu'} f_{j-[\nu]}(r, \tau; \nu'), & \nu \notin \mathbb{N}, \\ -\frac{(ir\tau)^j}{j!} f_{j-[\nu]}(r, \tau; 0), & \nu \in \mathbb{N}. \end{cases} \tag{2.30}$$

$$F_j = \sum_{\nu \in \sigma_\infty} F_{\nu,j} \pi_\nu \in \mathcal{L}(-1, s; 1, -s), \quad s > 2j+1 \tag{2.31}$$

$$R_0^{(N)}(z) = O(|z|^{N+\epsilon}) \in \mathcal{L}(-1, s; 1, -s), \quad s > 2N+1, \quad \epsilon > 0. \tag{2.32}$$

$f_{j-[\nu]}(r, \tau; \nu')$ is given by (2.15) and $b_{\nu',j}$ by Lemma A.1 in [37].

For the proof of Theorem 2.2, see [37]. The long-time expansion of solutions to the Schrödinger equation associated with P_0 can be easily deduced from Theorem 2.2 by using Mourre's multiple commutator method ([17]).

Theorem 2.3. *Let $U_0(t) = e^{-itP_0}$. One has in $\mathcal{L}(0,s;0,-s)$ with $s > 2N+1$*

$$U_0(t) = \sum_{\nu\in\sigma_N} \sum_{j=[\nu]}^{N-1+\delta_\nu} t^{-j-1-\nu'} a_{\nu,j} G_{\nu,j}\pi_\nu + O(|t|^{-N-1-\epsilon}). \tag{2.33}$$

Here

$$a_{\nu,j} = \begin{cases} \frac{i}{\pi}(-i)^j \sin(\nu\pi) e^{i\pi\nu/2}\Gamma(\nu+j+1), & \nu \notin \mathbb{N}, \\ i(-i)^{j+\nu}(\nu+j)!, & \nu \in \mathbb{N}. \end{cases} \tag{2.34}$$

Remark 2.4. *(a). In [12], it is proved that if $0 \notin \sigma_\infty$, $R_0(z)$ is continuous up to 0, and if $[0,1]\cap\sigma_\infty = \emptyset$, $R_0(z)$ is C^1 up to 0.*

(b). In [8], global Strichartz estimates are proved for Schrödinger and wave equations associated to $-\Delta+\frac{a}{r^2}$, where $-\Delta$ is the Laplacian on $\mathbb{R}^n$ and a is a constant with $a > -(n-2)^2/4$. We believe that similar estimates should remain true for the model operator $P_0 = -\Delta_{g_0}+\frac{q(\theta)}{r^2}$ under the condition $0 \notin \sigma_1$, or more generally for the perturbation P of P_0 studied here under the condition that zero is neither a resonance or an eigenvalue of P.

The remaining part of this work is devoted to proving similar results for the resolvent and the Schrödinger group of the perturbed operator P.

3. Characterization of resonant states at zero

Let $\mathbf{M}$ be a Riemannian manifold which outside some compact, K, is isometric to $\mathbb{R}_+ \times \mathbf{\Sigma}$. Consider the perturbation of P_0 in the form

$$P = -\Delta_g + \frac{q(\theta)}{r^2} + V_0(x) \tag{3.1}$$

on $\mathbf{M}$, where $x = r\theta$ is polar coordinates around some point $x_0 \in \mathbf{M}$, g is a Riemannian metric on $\mathbf{M}$ and V_0 is real function such that

$$g - g_0 = O(|x|^{-\rho_0}), \qquad V_0(x) = O(|x|^{-\rho_0}) \tag{3.2}$$

for some $\rho_0 > 2$ as $|x| = d(x,x_0) \to \infty$. g is assumed to be C^2 and V_0 bounded. We assume that

$$\frac{q(\theta)}{r^2} \geq -a(-\Delta_g) - b \tag{3.3}$$

in the sense of self-adjoint operators for some $0 \leq a < 1$ and $b \in \mathbb{R}_+$. This condition ensures that P is self-adjoint as operator induced by the corresponding form on $H_0^1(\mathbf{M})$. Let K be a compact of $\mathbf{M}$ such that $\mathbf{M}_0 = \mathbf{M}\setminus K$ is isometric to $\mathbb{R}_+ \times \mathbf{\Sigma}$. On $\mathbf{M}_0$, P can be written as

$$P = P_0 + V, \quad V = -\Delta_g + \Delta_{g_0} + V_0(x).$$

Remark that the extension by 0 of $v \in H_0^1(\mathbf{M}_0)$ is in $H_0^1(\mathbf{M})$, and the restriction of $w \in H^{-1}(\mathbf{M})$ to $\mathbf{M}_0$ belongs to $H^{-1}(\mathbf{M}_0)$.

Definition *Set $\mathcal{N} = \{u; Pu = 0, u \in H_0^{1,-s}, \forall s > 1\}$. A function $u \in \mathcal{N}\setminus L^2$ is called a resonant state of P at zero.*

The asymptotic expansion for solutions to second order elliptic differential equation on manifolds with corners has been studied [19, 20]. To study the singularity of $z \to R(z) = (P-z)^{-1}$ near $z=0$, it is important for us to have a characterization for resonant states and to distinguish them from eigenfunctions.

Theorem 3.1. *Assume $\rho_0 > 3$ and that $0 \notin \sigma_1$. Let $u \in \mathcal{N}$. Then,*

$$u(r\theta) = \sum_{0<\nu\leq 1} \sum_{j=1}^{n_\nu} -\frac{1}{2\nu} < Vu, |y|^{-\frac{n-2}{2}+\nu}\varphi_\nu^{(j)} > \frac{\varphi_\nu^{(j)}(\theta)}{r^{\frac{n-2}{2}+\nu}} + v \tag{3.4}$$

where $v \in L^2$, and $< \cdot, \cdot >$ is the scalar product in $L^2(\mathbf{M}_0; dv_{g_0})$. In particular,

$$u \in L^2 \Longleftrightarrow < Vu, |y|^{-\frac{n-2}{2}+\nu}\varphi_\nu^{(j)} >= 0, \quad \forall\, \nu \in \sigma_1,\ 1 \leq j \leq n_\nu. \tag{3.5}$$

Let $\mathcal{C}$ denote the linear span of all vectors of the form

$$c(u) = (\frac{1}{2\nu} < Vu, -|y|^{-\frac{n-2}{2}+\nu}\varphi_\nu^{(j)} >; \nu \in \sigma_1,\ 1 \leq j \leq n_\nu) \in \mathbb{C}^\kappa,$$

with $u \in \mathcal{N}$, $\kappa = \sum_{\nu\in\sigma_1} n_\nu$. Then,

$$\dim(\mathcal{N}/(\ker_{L^2} P)) = \dim \mathcal{C}. \tag{3.6}$$

Note that Theorem 3.1 implies the finiteness of the multiplicity of the zero resonance. A function $u \in \mathcal{N}/\ker_{L^2} P$ will be called a ν-resonant state of P if

$$u(r\theta) = \frac{\phi(\theta)}{r^{\frac{n-2}{2}+\nu}} + O(\frac{1}{r^{\frac{n-2}{2}+\nu+\epsilon}}), \quad r \to \infty, \quad \epsilon > 0,$$

for some $\phi \neq 0$. A family of ν-resonant states is said linearly independent if their leading parts are linearly independent.

Let u be a ν-resonant state. For $\rho_0 > 2$, one has $Vu \in H^{-1,\rho_0-1+\nu-\epsilon}$ for any $\epsilon > 0$. The proof of Theorem 3.1 gives for $u_{\nu,j} = (u, \varphi_\nu^{(j)})$

$$u_{\nu,j}(r) = -\frac{1}{2\nu} < Vu, \varphi_\nu^{(j)}|y|^{-\frac{n-2}{2}+\nu} > r^{-\frac{n-2}{2}-\nu}(1+O(r^{-\epsilon})), \quad r \to \infty, \tag{3.7}$$

for any $0 < \epsilon < \rho_0 - 2$.

Assume now that $0 \in \sigma_\infty$. By Theorem 2.2, $R_0(z) = \ln z\ G_{0,0}\pi_0 + F_0 + O(|z|^\epsilon)$ in $\mathcal{L}(-1, s; 1, -s)$ for any $s > 1$. Let $\phi_0(x) = \varphi_0^{(1)}(\theta) r^{-\frac{n-2}{2}}$, where $\varphi_0^{(1)}$ is a normalized eigenfunction of $-\Delta_h + q(\theta)$ with eigenvalue $-(n-2)^2/4$. Denote still by ϕ_0 its extension by 0 outside $\mathbf{M}_0$. Then

$$G_{0,0}\pi_0 = \frac{1}{2} < \cdot, \phi_0 > \phi_0$$

is defined on $\mathbf{M}$.

Theorem 3.2. *Assume $\rho_0 > 3$ and $0 \in \sigma_1$. Let $u \in \mathcal{N}$.*

(a). One has

$$\begin{aligned} u \quad = \quad & < Vu, -\ln r\, \phi_0 > \phi_0 \\ & + \sum_{0<\nu\leq 1} \sum_{j=1}^{n_\nu} \frac{1}{2\nu} < Vu,\ -|y|^{-\frac{n-2}{2}+\nu}\varphi_\nu^{(j)} > \frac{\varphi_\nu^{(j)}(\theta)}{r^{\frac{n-2}{2}+\nu}} + v \end{aligned} \tag{3.8}$$

where $v \in L^2$.

(b). Assume $\mathbf{M} = \mathbf{M}_0$. Let $u \in \mathcal{N}$. Then, one has $< Vu, \phi_0 >= 0$ and $(u + F_0 Vu) = \beta\phi_0$ with $\beta = \frac{1}{2} < Vu, -\ln r\, \phi_0 >$.

Part (b) of Theorem 3.2 shows that $(1+F_0V)\phi = 0$ if $\phi \in \mathcal{N}$ is not a 0-resonant state. If $(1+F_0V)u = \beta\phi_0$ and if

$$\gamma_0 \equiv < \phi_0, -V\phi_0 > \neq 0, \tag{3.9}$$

then, $\beta = -\gamma_0^{-1} < F_0Vu, V\phi_0 >$. See [37] for the proof of Theorems 3.1 and 3.2.

4. Asymptotic expansion of $R(z)$

Let $K \subset \mathbf{M}$ be a compact such that $\mathbf{M}_0 = \mathbf{M} \setminus K$ is isometric to $\mathbb{R}_+ \times \mathbf{\Sigma}$. Let $x = r\theta$ be polar coordinates around some point $x_0 \in \mathbf{M}$. Let $1 < R_1 < R_2$ be large enough so that $K \subset B_{R_1}$, where $B_{R_j} = \{x \in \mathbf{M}; r < R_j\}$, $j = 1, 2$. Let $0 \le \chi_j(x) \le 1$ be smooth functions on $\mathbf{M}$ such that supp $\chi_1 \subset B_{R_2}$, $\chi_1(x) = 1$ near B_{R_1} and

$$\chi_1(x)^2 + \chi_2(x)^2 = 1.$$

Let $-\Delta_g^D$ denote the Dirichlet realization of $-\Delta_g$ on B_{R_2} Set $R_D(z) = (-\Delta_g^D - z)^{-1}$, which is holomorphic near 0. For $z \in \mathbb{C} \setminus \mathbb{R}_+$, z near 0, one has

$$(\chi_1 R_D(z)\chi_1 + \chi_2 R_0(z)\chi_2)(P - z) = 1 + \tilde{F}(z), \tag{4.1}$$

where

$$\tilde{F}(z) = \chi_1 R_D(z)(W\chi_1 + [\chi_1, -\Delta_g]) + \chi_2 R_0(z)(\chi_2 V + [\chi_2, -\Delta_{g_0}])$$

with $W = \frac{q(\theta)}{r^2} + V_0(x)$ and $V = P - P_0$ on $\mathbf{M}_0$.

Assume that $0 \notin \sigma_\infty$. By Theorem 2.2, $R_0(z) = F_0 + o(1)$ in $\mathcal{L}(-1, s; 1, -s)$, $s > 1$, as $z \to 0$. $\tilde{F}(0)$ exists in $\mathcal{L}(1, -s; 1, -s)$, $s > 1$ and close to 1.

Lemma 4.1. $\ker(1 + \tilde{F}(0))$ *coincides with the kernel,* $\mathcal{N}$*, of* P *in* $H^{1,-s}$*.* $1 + \tilde{F}(0)$ *is a Fredholm operator in* $\mathcal{L}(1, -s; 1, -s)$*.*

Proof. It is clear that $\mathcal{N} \subseteq \ker(1 + \tilde{F}(0))$. If $u \in H^{1,-s}$ is in $\ker(1 + \tilde{F}(0))$, then $Pu = -P\tilde{F}(0)u \in H^{-1,s}$ for some $s > 1$, because $P = P_0 + O(r^{-\rho_0})$ for some $\rho_0 > 2$. This means that $v = Pu$ satisfies

$$(\chi_1 R_D(0)\chi_1 + \chi_2 F_0 \chi_2)v = 0$$

in $H^{1,-s}$. Since $R_D(0)$ and F_0 are positive, taking the dual product of the above equation with v, we deduce that $\chi_j v = 0$ for $j = 1, 2$. Therefore $v = 0$ and $u \in \mathcal{N}$.

Note that due to the local singularity and the second order perturbation, $\tilde{F}(0)$ is not a compact operator. Since $\ker(1 + \tilde{F}(0)) = \mathcal{N}$, one has

$$\ker(1 + \tilde{F}(0)) = \ker(1 + i\tilde{F}(0)(P + i)^{-1}).$$

One can check that $\tilde{F}(0)(P + i)^{-1}$ is a compact operator on $H^{1,-s}$. So $\ker(1 + \tilde{F}(0))$ is of finite dimension. Similarly, one can show that the cokernel of $1 + \tilde{F}(0)$ is also of finite dimension. This proves that $1 + \tilde{F}(0)$ is a Fredholm operator on $H^{1,-s}$. □

Denote

$$\vec{\nu} = (\nu_1, \cdots, \nu_k) \in (\sigma_N)^k, \quad z_{\vec{\nu}} = z_{\nu_1} \cdots z_{\nu_k},$$

$$\{\vec{\nu}\} = \sum_{j=1}^k \nu'_j, \quad [\vec{\nu}]_- = \sum_{j=1}^k [\nu_j]_-, \quad [\vec{\nu}] = \sum_{j=1}^k [\nu_j].$$

Here $\nu'_j = \nu_j - [\nu_j]_-$ for $\nu_j > 0$.

Corollary 4.2. *Assume $\mathcal{N} = \{0\}$. Let $N \in \mathbb{N}$ and $\rho_0 > 4N+2$, $R(z)$ has the following expansion in $\mathcal{L}(-1,s;1,-s)$ with $s > 2N+1$*

$$R(z) = \sum_{j=0}^{N} z^j R_j + \sum_{k=1}^{N_0} \sum_{\vec{\nu}\in(\sigma_N)^k} z_{\vec{\nu}} \sum_{j=[\vec{\nu}]_-}^{N-1} z^j R_{\vec{\nu},j} + O(|z|^{N+\epsilon}) \tag{4.2}$$

Here N_0 is some integer large enough depending on σ_∞ and N,

$$R_0 = (\chi_1 R_D(0)\chi_1 + \chi_2 F_0 \chi_2)(1+\tilde{F}(0))^{-1}.$$

and R_j (resp., $R_{\vec{\nu};j}$) are in $\mathcal{L}(1,-s;1,-s)$ for $s > 2j+1$ (resp., for $s > 2j + \{\vec{\nu}\} + 1$).

Proof. By Lemma 4.1, $1+\tilde{F}(0)$ has a bounded inverse on $H^{1,-s}$ if $\mathcal{N} = \{0\}$. It follows that $(1+\tilde{F}(z))^{-1}$ exists for z small. Since $R_D(z)$ is holomorphic near 0, we can use Theorem 2.2 to calculate the its asymptotic expansion. The conclusion for $R(z)$ then follows from the equation

$$R(z) = (\chi_1 R_D(z)\chi_1 + \chi_2 R_0(z)\chi_2)(1+\tilde{F}(z))^{-1}.$$

□

When $\mathcal{N} \neq \{0\}$, we use the Grushin's method to reduce the calculation of the $(1+\tilde{F}(z))^{-1}$ to a finite dimensional space. The main difficulty arises from the interaction between different resonant states. To avoid further complication due to cut-offs, we assume from now on that $\mathbf{M} = \mathbf{M}_0$ is isometric to $\mathbb{R}_+ \times \mathbf{\Sigma}$. In this case, we can choose $\chi_1 = 0$ and $\chi_2 = 1$ on $\mathbf{M}$ and $\tilde{F}(z)$ becomes $\tilde{F}(z) = R_0(z)V$ with $V = P - P_0 = \Delta_{g_0} - \Delta_g + V_0(x)$. Set

$$\mathcal{N} = \text{ Ker } (1+F_0V) \subset H^{1,-s}, \mathcal{N}^* = \text{ Ker } (1+F_0V)^* \subset H^{-1,s}, \quad 1 < s < \rho_0 - 1.$$

Since $V(1+F_0V) = (1+VF_0)V = (1+F_0V)^*V$, one can check that V is injective from $\mathcal{N}$ into $\mathcal{N}^*$ and $V^* = V$ is injective from $\mathcal{N}^*$ into $\mathcal{N}$. Consequently, V is bijective from $\mathcal{N}$ onto $\mathcal{N}^*$. This shows that $\mathcal{N}$ is independent of s with $1 < s < \rho_0 - 1$, $\dim\mathcal{N} = \dim\mathcal{N}^*$, and the quadratic form

$$\phi \to < \phi, -V\phi >$$

is non-degenerate on $\mathcal{N}$. Since $P_0 \geq 0$, this quadratic form is positive definite. Let

$$\mu = \dim\mathcal{N}, \quad \mu_r = \dim\mathcal{N}/(\ker_{L^2} P).$$

We can choose a basis $\{\phi_1, \cdots, \phi_\mu\}$ of $\mathcal{N}$ such that

$$< \phi_i, -V\phi_j > = \delta_{ij}. \tag{4.3}$$

Without loss, assume that ϕ_j, $1 \leq j \leq \mu_r$, are resonant states. Note that

$$R(z) = (1 + R_0(z)V)^{-1} R_0(z).$$

The main task is to establish the asymptotic expansion of $W(z) = (1+R_0(z)V)^{-1}$ for z near zero and $\Im z > 0$. To do this, we study as in [35] a Grushin problem associated to the operator

$$A(z) = \begin{pmatrix} W(z) & T \\ S & 0 \end{pmatrix} : H^{1,-s} \times \mathbb{C}^\mu \to H^{1,-s} \times \mathbb{C}^\mu,$$

where $s > 1$, T and S are given by

$$\begin{aligned} Tc &= \sum_{j=1}^{\mu} c_j \phi_j, \quad c = (c_1, \cdots, c_\mu) \in \mathbb{C}^\mu, \\ Sf &= (< f, -V\phi_1 >, \cdots, < f, -V\phi_\mu >) \in \mathbb{C}^\mu, \quad f \in H^{1,-s}. \end{aligned}$$

Define $Q : H^{1,-s} \to H^{1,-s}$ by

$$Qf = \sum_{j=1}^{\mu} < f, -V\phi_j > \phi_j. \tag{4.4}$$

Then,

$$TS = Q \text{ on } H^{1,-s} \text{ and } ST = I_\mu \text{ on } \mathbb{C}^\mu.$$

Decompose Q as $Q = Q_r + Q_e$ where $Q_r = \sum_{j=1}^{\mu_r} < \cdot, \; -V\phi_j > \phi_j$. Then,

$$Q_r^2 = Q_r, \quad Q_e^2 = Q_e, \quad Q_rQ_e = Q_eQ_r = 0. \tag{4.5}$$

Lemma 4.3. *(a). One has the decomposition*

$$H^{1,-s} = \mathcal{N} \oplus \text{ Range } (1 + F_0V). \tag{4.6}$$

Q is the projection from $H^{1,-s}, s > 1$, onto $\mathcal{N}$ with Ker $Q =$ Range $(1 + F_0V)$.

(b). Let $Q' = 1 - Q$. Then, $Q'(1 + F_0V)Q'$ is invertible on the range of Q' and $(Q'(1 + F_0V)Q')^{-1}Q' \in \mathcal{L}(1, -s; 1, -s), s > 1$.

Proof. (a). It is easy to check that $\mathcal{N} \cap$ Range $(1+F_0V) = \{0\}$. Since $1+F_0V$ is continuous on $H^{1,-s}$ and is an operator of index, Range $(1 + F_0V)$ is closed (Theorem 2.3 in [6]) and is therefore equal to (Ker $(1 + F_0V)^*)^\perp$. For any $u \in H^{1,-s}$, one has $u = Qu + (u - Qu)$ with

$$u - Qu \in (\text{ Ker } (1 + F_0V)^*)^\perp = \text{ Range } (1 + F_0V).$$

This proves $H^{1,-s} = \mathcal{N} \oplus$ Range $(1 + F_0V)$. It is easy to verify that Q is the projection onto $\mathcal{N}$ w.r.t. this decomposition of $H^{1,-s}$.

(b). $Q' = 1 - Q$ is a projection from $H^{1,-s}$ onto Range $(1 + F_0V)$. For $u \in F$ such that $Q'(1+F_0V)Q'u = 0$, we have $Q'u = u$ and $0 = Q'(1+F_0V)Q'u = (1+F_0V)u - Q(1+F_0V)u = (1 + F_0V)u$. This means $u \in \mathcal{N}$. By (a), $u = 0$. This proves that $Q'(1 + F_0V)Q'$ is injective on Range $(1 + F_0V)$. Since Range $Q' =$ Range $(1 + F_0V)$, we can show also that $Q'(1+F_0V)Q'$ is surjective on Range $(1+F_0V)$. Therefore, $Q'(1+F_0V)Q'$ is bijective on Range $(1 + F_0V)$. Since Range $(1 + F_0V)$ is closed, $Q'(1 + F_0V)Q'$ is invertible on Range $(1 + F_0V)$ and

$$D_0 = (Q'(1 + F_0V)Q')^{-1}Q' \in \mathcal{L}(1, -s; 1, -s), \quad s > 1. \tag{4.7}$$

□

From the asymptotic expansion of $R_0(z)$ and Lemma 4.3, it follows that $Q'W(z)Q'$ is invertible on Range $(1 + F_0V)$ with bounded inverse. Let

$$D(z) = (Q'(1 + R_0(z)V)Q')^{-1}Q', \quad D_0 = (Q'(1 + F_0V)Q')^{-1}Q'.$$

Using the operator $D(z)$, we can construct an approximate inverse of $A(z)$ as in [35] to prove that for $z \in U_\delta$, the operator $A(z)$ is invertible from $H^{1,-s} \times \mathbb{C}^\mu$ to $H^{1,-s} \times \mathbb{C}^\mu$. The inverse $A(z)^{-1}$ can be written in the form

$$A(z)^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{+-}(z) \end{pmatrix},$$

where

$$E(z) = D(z), \qquad E_+(z) = T - D(z)Y, \tag{4.8}$$

$$E_-(z) = S - SX, \qquad E_{+-}(z) = -SW(z)T + SXY, \tag{4.9}$$

with

$$X = QW(z)Q'D(z), Y = Q'W(z)T.$$

It follows that the inverse of $W(z)$ is given by

$$W(z)^{-1} = E(z) - E_+(z)E_{+-}(z)^{-1}E_-(z). \tag{4.10}$$

If $\rho_0 > 2$, one has

$$D(z) = D_0 + O(|z|^\epsilon), \quad \epsilon > 0, \tag{4.11}$$

in $\mathcal{L}(1,-s;1,-s)$, $s > 1$. More generally, if $k \geq 1$ and $\rho_0 > 4N+2$, one has in $\mathcal{L}(-1,s;1,-s)$, $s > 2N+1$,

$$D(z) = \sum_{j=0}^{N} z^j D_j + \sum_{k=1}^{N_0} \sum_{\vec{\nu}\in(\sigma_N)^k} z_{\vec{\nu}} \sum_{j=[\vec{\nu}]_-}^{N-1} z^j D_{\vec{\nu},j} + O(|z|^{N+\epsilon}) \tag{4.12}$$

D_j (resp., $D_{\vec{\nu};j}$) are in $\mathcal{L}(1,-s;1,-s)$ for $s > 2j+1$ (resp., for $s > 2j + \{\vec{\nu}\} + 1$) and $D_{\vec{\nu},j}$ are operators of finite rank. Here and in the following, N_0 is again some integer large enough depending on σ_∞ and N. In the present case, N_0 can be taken as the largest integer such that $N_0\nu_0 \leq N$, where $\nu_0 = \min\{\nu \in \sigma_\infty\} > 0$. Since the terms with $\{\vec{\nu}\} + j > N$ can be put into the remainder, (4.12) can be rewritten as

$$D(z) = \sum_{j=0}^{N} z^j D_j + \sum_{\{\vec{\nu}\}+j\leq N}^{(1)} z_{\vec{\nu}} z^j D_{\vec{\nu},j} + O(|z|^{N+\epsilon}) \tag{4.13}$$

where $D_{\vec{\nu},j} = 0$ if $j < [\vec{\nu}]_-$, and for $\ell \geq 1$, $\sum_{\{\vec{\nu}\}+j\leq N}^{(\ell)}$ stands for the sum over all $\vec{\nu} \in (\sigma_N)^k$, $\ell \leq k \leq N_0$ and $[\vec{\nu}]_- \leq j \leq N$ with $\{\vec{\nu}\} + j \leq N$. One has in particular,

$$D_1 = -D_0F_1VD_0, \quad D_{\nu_0,0} = -D_0G_{\nu_0,\delta_{\nu_0}}\pi_{\nu_0}VD_0. \tag{4.14}$$

Similarly, $E_+(z)$ (resp., $E_-(z)$) has a similar asymptotic expansion in z up to $O(|z|^{k+\epsilon})$ in $\mathcal{L}(\mathbb{C}^\kappa; H^{1,-s})$ (resp., in $\mathcal{L}(H^{-1,s};\mathbb{C}^\kappa)$) for $s > 2k+1$ provided that $\rho_0 > 2k+2$. The following Proposition is important to prove the existence of asymptotic expansion of resolvent.

Proposition 4.4. *Let $\rho_0 > 3$ if $\mu_r = 0$, and $\rho_0 > 4$ if $\mu_r \neq 0$. $E_{+-}(z)$ is invertible for z small enough and its inverse is given by*

$$\begin{aligned} E_{+-}(z)^{-1} &= \begin{pmatrix} (\mathcal{T}^*\mathcal{D}_1(z)\mathcal{T})^{-1} & -(\mathcal{T}^*\mathcal{D}_1(z)\mathcal{T})^{-1}C\Phi_e^{-1} \\ -\Phi_e^{-1}C^*(\mathcal{T}^*\mathcal{D}_1(z)\mathcal{T})^{-1} & z^{-1}\Phi_e^{-1} \end{pmatrix} \\ &\quad \times \left(I_\nu + \begin{pmatrix} O(|z|/|z_{\varsigma_{\kappa_0}}|) + O(|z|^\epsilon) & O(|z|^\epsilon) \\ O(|z|^\epsilon) & O(|z|^\epsilon) + O(|z|/|z_{\varsigma_{\kappa_0}}|) \end{pmatrix} \right) \end{aligned}$$

Here

$$\Phi_e = (<\phi_i,\ \phi_j>)_{\mu_r<i,j\leq\mu}, \quad C = (<F_1V\phi_j,\ V\phi_i>)_{1\leq i\leq\mu_r,\ \mu_r<j\leq\mu},$$

$\mathcal{T}$ is an invertible matrix,

$$\mathcal{D}_1(z) = \begin{pmatrix} (c'_{\varsigma_1}z_{\varsigma_1})I_{m_1} & & 0 \\ & \ddots & \\ 0 & & (c'_{\varsigma_1}z_{\varsigma_{\kappa_0}})I_{m_{\kappa_0}} \end{pmatrix}$$

with $c'_\nu = 4\nu^2c_\nu \neq 0$, and $0 < \varsigma_1 < \cdots < \varsigma_{\kappa_0} \leq 1$ are those of $\nu \in \sigma_1$ for which there exist m_j linearly independent ς_j-resonant states with $\sum_{j=1}^{\kappa_0} m_j = \mu_r$.

Proof. We calculate the asymptotics of the $\mu \times \mu$ matrix

$$E_{+-}(z) = \left(< (W(z) - W(z)Q'D(z)Q'W(z))\phi_j, V\phi_i >\right)_{1\le i,j\le \mu}.$$

Set

$$L_1(z) = \sum_{\nu\in\sigma_1} z_\nu G_{\nu,\delta_\nu}\pi_\nu. \tag{4.15}$$

For $\rho_0 > 4$, $3 < s < \rho_0 - 1$, we have in $H^{1,-s}$

$$W(z)\phi_i = (L_1(z) + zF_1)V\phi_i + O(|z|^{1+\epsilon}), \quad \epsilon > 0.$$

For $\mu_r < j \le \mu$, $\phi_i \in L^2$ and the above expansion remains true for $\rho_0 > 3$. By Theorem 3.1, $L_1(z)\phi_i = 0$ if ϕ_i is an eigenfunction. Therefore, $W(z)\phi_i$ is of the order $O(|z|)$ if ϕ_i is an eigenfunction and of the order $O((|z|^\epsilon)$ if ϕ_i is a resonant state. Since $D(z)$ is uniformly bounded in $\mathcal{L}(1,-s;1,-s)$ for $s > 1$, the (i,j)-th entry of $E_{+-}(z)$ has the asymptotics

$$(E_{+-}(z))_{ij} = z < F_1V\phi_j, V\phi_i > +O(|z|^{1+\epsilon}) \tag{4.16}$$

if at least one of ϕ_i and ϕ_j is an eigenfunction and $\rho_0 > 4$. In the case both ϕ_i and ϕ_j are eigenfunctions, since $V\phi_i$, $V\phi_j$ are in H^{-1,ρ_0}, we can prove as in [35] that $< F_1V\phi_i, V\phi_j >=< \phi_i, \phi_j >$ for $\rho_0 > 3$. We obtain under the assumption $\rho_0 > 3$

$$(E_{+-}(z))_{ij} = z < \phi_j, \phi_i > +O(|z|^{1+\epsilon}), \quad \mu_r < i, j \le \mu. \tag{4.17}$$

For $1 \le i, j \le \mu_r$, $W(z)\phi_i = L_1(z)V\phi_i + O(|z|)$ in $H^{1,-s}$, for $3 < s < \rho_0 - 1$ and

$$\begin{aligned}
&(E_{+-}(z))_{ij}\\
&= \; < L_1(z)V\phi_j, V\phi_i > - < L_1(z)VD(z)W(z)\phi_j, V\phi_i > +O(|z|) \qquad (4.18)\\
&= \; \sum_\nu c'_\nu z_\nu \sum_{l=1}^{n_\nu}\left(u^{(l)}_{\nu,j}\overline{u^{(l)}_{\nu,i}} + \sum_{\nu'}\sum_{l'=1}^{n_{\nu'}} v^{(l)}_{\nu,j}(z)\overline{u^{(l')}_{\nu',i}}\right) + O(|z|)
\end{aligned}$$

where

$$\begin{aligned}
u^{(l)}_{\nu,i} &= < V\phi_i, -|x|^{-\frac{n-2}{2}+\nu}\varphi^{(l)}_\nu > /(2\nu)\\
v^{(l)}_{\nu,j}(z) &= < VD(z)W(z)\phi_j, -|x|^{-\frac{n-2}{2}+\nu}\varphi^{(l)}_\nu > /(2\nu)\\
c'_\nu &= 4\nu^2 c_\nu
\end{aligned}$$

with c_ν given by

$$c_\nu = -\frac{e^{-i\pi\nu}\Gamma(1-\nu)}{\nu 2^{2\nu+1}\Gamma(\nu+1)}, \quad \text{for } 0<\nu<1, \text{ and } \quad c_1 = \frac{1}{8}. \tag{4.19}$$

Let $\kappa = \sum_\nu n_\nu$ and let $\mathcal{U}$, $\mathcal{V}(z)$ denote the $\kappa \times \mu_r$ matrices with entries $u^{(l)}_{\nu,j}$ and $v^{(l)}_{\nu,j}(z)$, $1 \le j \le \mu_r$, respectively. Let $\mathcal{D}(z)$ denote the diagonal $\kappa\times\kappa$ matrix: $\mathcal{D}(z) = \text{ Diag }(c'_\nu z_\nu I_{n_\nu})$. Then,

$$(E_{+-}(z))_{1\le i,j\le\mu_r} = \mathcal{U}^*\mathcal{D}(z)(\mathcal{U} + \mathcal{V}(z)) + O(|z|). \tag{4.20}$$

Remark that the j-th column of $\mathcal{U}$ is just $c(\phi_j)$ defined in Theorem 3.1. Since $\phi_1, \cdots, \phi_{\mu_r}$ are linearly independents as resonant states, by Theorem 3.1, $\mathcal{U}$ is of maximum rank μ_r. $\mathcal{U}^*\mathcal{D}(z)(\mathcal{U}+\mathcal{V}(z))$ is the matrix of the Hermitian form

$$\Phi(\cdot,\cdot) =< L_1(z)V(1 - D(z)W(z))\cdot,\; V\cdot >$$

in the basis $\{\phi_1, \cdots, \phi_{\mu_r}\}$ of $\mathcal{N}/(\ker_{L^2} P)$.

It is not clear from (4.20) whether the inverse of $\mathcal{U}^*\mathcal{D}(z)\mathcal{U}$ gives the leading term of the inverse of $E_{+-}(z)$, because due to different values of ν, not all of the entries in $\mathcal{U}^*\mathcal{D}(z)\mathcal{V}(z)$ are of higher order in z than those in $\mathcal{U}^*\mathcal{D}(z)\mathcal{U}$. To prove that $\mathcal{U}^*\mathcal{D}(z)(\mathcal{U}+\mathcal{V}(z))$ is invertible for $z \in U_\delta$ with an explicit leading term, we compute the matrix of the Hermitian form $\Phi(\cdot,\cdot)$ in another basis $\{\psi_j; 1 \le j \le \mu_r\}$ constructed in the following way. Let $0 < \varsigma_1 < \varsigma_2 < \cdots < \varsigma_{\kappa_0} \le 1$ be those of $\nu \in \sigma_\infty$ for which there are m_{ς_j} linearly independent ς_j-resonant states with $m_{\varsigma_j} \ge 1$ and $\sum_{j=1}^{\kappa_0} m_{\varsigma_j} = \mu_r$. Let $\{\varphi_{\varsigma_j}^{(l)}(\theta); l = 1, \cdots, n_{\varsigma_j}\}$ $(n_{\varsigma_j} \ge m_{\varsigma_j})$ be an orthonormal basis of the eigenspace of $-\Delta_h + q(\theta)$ associated with the eigenvalue $\varsigma_j^2 - (n-2)^2/4$. Modifying the orthonormal basis $\varphi_\nu^{(l)}$ used before if necessary, we can assume that there are m_{ς_j} linearly independent ς_j-resonant states in the form

$$\frac{\varphi_{\varsigma_j}^{(l)}(\theta)}{r^{\frac{n-2}{2}+\varsigma_j}} + O(r^{-\frac{n-2}{2}-\varsigma_j-\epsilon}), \quad 1 \le l \le m_{\varsigma_j}.$$

By an induction on j, we can construct from these resonant states m_{ς_j} linearly independent ς_j-resonant states such that

$$\psi_{\varsigma_j}^{(l)}(r\theta) = \frac{\varphi_{\varsigma_j}^{(l)}(\theta)}{r^{\frac{n-2}{2}+\varsigma_j}} + \sum_{\nu > \varsigma_j, 1 \le l' \le n_\nu} c_{\nu,l';j,l} \frac{\varphi_\nu^{(l')}(\theta)}{r^{\frac{n-2}{2}+\nu}} + O_{L^2}(1). \tag{4.21}$$

Here

$$c_{\nu,l';j,l} = < V\psi_{\varsigma_j}^{(l)}, -\frac{1}{2\varsigma_j}\frac{\varphi_\nu^{(l')}}{r^{\frac{n-2}{2}-\nu}} > .$$

Subtracting if necessary a suitable multiple of $\psi_{\varsigma_i}^{(l')}$ from $\psi_{\varsigma_j}^{(l)}$ which leaves unchanged the leading term of $\psi_{\varsigma_j}^{(l)}$, one can assume without loss that

$$c_{\nu,l';j,l} = 0, \text{ for } \nu = \varsigma_i, i > j, 1 \le l' \le m_{\varsigma_i}. \tag{4.22}$$

Let $\{\varphi_m;\ 1 \le m \le \kappa\}$ be a rearrangement of the basis $\{\varphi_\nu^{(l)}, \nu \in \sigma_1, 1 \le l \le n_\nu\}$ such for $1 \le m \le \mu_r$

$$\varphi_m = \varphi_{\varsigma_j}^{(l)}, \quad \text{if } m = \sum_{s=1}^{l-1} m_{\varsigma_s} + j.$$

Correspondingly, set $\psi_m = \psi_{\varsigma_j}^{(l)}, 1 \le m \le \mu_r$. The matrix of $\Phi(\cdot,\cdot)$ in this new basis $\{\psi_m\}$ is given by $\mathcal{M}(z) = \mathcal{U}'^*\mathcal{D}'(z)(\mathcal{U}' + \mathcal{V}'(z))$, where $\mathcal{V}'(z) = O(|z|^\epsilon)$, $\mathcal{D}'(z) = \text{ Diag } (c'_{\nu_{\sigma(j)}} z_{\nu_{\sigma(j)}})$, with σ an appropriate permutation of $\{1, \cdots, \kappa\}$, c'_ν being defined in Proposition 4.4 and

$$\mathcal{U}' = (u'_{i,j})_{1 \le i \le \kappa, 1 \le j \le \mu_r},$$

with $u'_{ij} = \delta_{ij}$ for $1 \le i, j \le \mu_r$ and for $i > \mu_r$, $u'_{ij} = 0$ if $\nu_{\sigma(i)} \le \nu_{\sigma(j)}$. In fact, u'_{ij} is given by

$$u'_{ij} = -\frac{1}{2\nu_{\sigma(i)}} < V\psi_j, |x|^{-\frac{n-2}{2}+\nu_{\sigma(i)}}\varphi_i >$$

and these properties follow from (4.21) and (4.22). Write the matrices in blocs

$$\mathcal{U}' = \begin{pmatrix} I_{\mu_r} \\ \mathcal{U}_2 \end{pmatrix}, \quad \mathcal{V}'(z) = \begin{pmatrix} \mathcal{V}_1(z) \\ \mathcal{V}_2(z) \end{pmatrix}, \quad \mathcal{D}'(z) = \begin{pmatrix} \mathcal{D}_1(z) & 0 \\ 0 & \mathcal{D}_2(z) \end{pmatrix}.$$

One has:

$$\mathcal{D}_1(z) = \text{ Diag } (c'_{\varsigma_1} z_{\varsigma_1} I_{m_{\varsigma_1}}, \cdots, c'_{\varsigma_{\kappa_0}} z_{\varsigma_{\kappa_0}} I_{m_{\varsigma_{\kappa_0}}}) \tag{4.23}$$

and

$$\begin{aligned}\mathcal{M}(z) &= \mathcal{D}_1(z) + \mathcal{U}_2^*\mathcal{D}_2(z)\mathcal{U}_2 + \mathcal{D}_1(z)\mathcal{V}_1(z) + \mathcal{U}_2^*\mathcal{D}_2(z)\mathcal{V}_2(z) \\ &= \mathcal{D}_1(z)\left(1 + \mathcal{V}_1(z) + \mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)(\mathcal{U}_2 + \mathcal{V}_2(z))\right).\end{aligned}$$

$\mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)$ is a $\mu_r \times (\kappa - \mu_r)$ matrix whose entries are

$$\left[\mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)\right]_{ij} = \overline{u'_{\mu_r+j,i}}\frac{c'_{\nu_{\sigma(\mu_r+j)}} z_{\nu_{\sigma(\mu_r+j)}}}{c'_{\nu_{\sigma(i)}} z_{\nu_{\sigma(i)}}},$$

for $1 \le i \le \mu_r, 1 \le j \le \kappa - \mu_r$. Since $u'_{ij} = 0$ if $\nu_{\sigma(i)} \le \nu_{\sigma(j)}$, $\left[\mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)\right]_{ij} \ne 0$ only when $\nu_{\sigma(\mu_r+j)} > \nu_{\sigma(i)}$. This proves that

$$\mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z) = O(|z|^\epsilon).$$

Consequently, $\mathcal{M}(z)$ is invertible and its inverse is given by

$$\begin{aligned}\mathcal{M}(z)^{-1} &= \left(1 + \mathcal{V}_1(z) + \mathcal{D}_1(z)^{-1}\mathcal{U}_2^*\mathcal{D}_2(z)(\mathcal{U}_2 + \mathcal{V}_2(z))\right)^{-1}\mathcal{D}_1(z)^{-1} \\ &= (1 + O(|z|^\epsilon))\mathcal{D}_1(z)^{-1}. \end{aligned}\tag{4.24}$$

Since $\mathcal{U}^*\mathcal{D}(z)(\mathcal{U} + \mathcal{V}(z))$ is related to $\mathcal{M}(z)$ by

$$\mathcal{U}^*\mathcal{D}(z)(\mathcal{U} + \mathcal{V}(z)) = \mathcal{T}^*\mathcal{M}(z)\mathcal{T}$$

where $\mathcal{T}$ is the transfer matrix from $\{\psi_1, \cdots, \psi_{\mu_r}\}$ to $\{\phi_1, \cdots, \phi_{\mu_r}\}$, it is also invertible. The leading term of its inverse is $(\mathcal{T}^*\mathcal{D}_1(z)\mathcal{T})^{-1}$ which is of the order $O(|z_{\varsigma_{\kappa_0}}|^{-1})$. This proves Proposition 4.4 when zero is not an eigenvalue of P under the assumption $\rho_0 > 3$.

When zero is an eigenvalue of P, we obtain with $\rho_0 > 4$

$$E_{+-}(z) = \begin{pmatrix} T^*\mathcal{M}(z)T + O(|z|) & zC + O(|z|^{1+\epsilon}) \\ zC^* + O(|z|^{1+\epsilon}) & z\Phi_e + O(|z|^{1+\epsilon}) \end{pmatrix}, \tag{4.25}$$

where Φ_e and C are given in Proposition 4.4. Let $S(z) = (T^*\mathcal{M}(z)T)^{-1}$. One has

$$\begin{aligned}&E_{+-}(z)\begin{pmatrix} S(z) & -S(z)C\Phi_e^{-1} \\ -\Phi_e^{-1}C^*S(z) & z^{-1}\Phi_e^{-1}) \end{pmatrix} \\ &= I_\nu + \begin{pmatrix} O(|z/z_{\varsigma_{\kappa_0}}| + |z|^\epsilon) & O(|z|^\epsilon) \\ O(|z|^\epsilon) & O(|z/z_{\varsigma_{\kappa_0}}| + |z|^\epsilon) \end{pmatrix}.\end{aligned}$$

This proves that $E_{+-}(z)$ is invertible for $z \in U_\delta$ with δ small enough. Proposition 4.4 is proved. □

Remark 4.5. *(a). The proof of Proposition 4.4 is easier if V is radial (depending only on r and ∂_r) outside a compact, because the resonant states $\{\phi_j; 1 \le j \le \mu_r\}$ can then be chosen such that in the expansion given by Theorem 3.1, there is only one non zero term. In this case, the matrix $(E_{+-}(z))_{1\le i,j\le\mu_r}$ has a diagonal leading part which is invertible.*

(b). It is clear from the proof of Proposition 4.4 that the statement in the presence of both resonance and eigenvalue at zero remains true when $\rho_0 > 4 - \varsigma_1$, because $\phi_j \in H^{1,-1+\varsigma_1-\epsilon}$ for $1 \le j \le \mu$. In particular, assume that σ_1 consists only of one point $\nu_0 \in]0,1]$ and that 0 is not an eigenvalue of P. In this case, $\dim \mathcal{N} \le 1$. *Assume* $\dim \mathcal{N} = 1$. *From Remark 2.4 of* [37], *we have*

$$W(z) = 1 + F_0V + z_{\nu_0}G_{\nu_0,\delta_{\nu_0}}V + O(|z|^\delta), \quad \textit{in } \mathcal{L}(1,-s',1,-s) \tag{4.26}$$

for any $\nu_0 \le \delta \le 1$ *and* s, $\rho_0 - s' > 1 + 2\delta - \nu_0$. *Since* $\phi_1 \in H^{1,-1+\nu_0-\epsilon}$, *taking* δ *sufficiently close to* ν_0, *we see that the leading term of* $E_{+-}(z)$ *can be calculated under the assumption* $\rho_0 > 2$:

$$E_{+-}(z) = c'_{\nu_0} z_{\nu_0} |u_{\nu_0,1}|^2 + O(|z|^\delta) \tag{4.27}$$

Therefore, under the assumption $\rho_0 > 2$, *the result of Proposition 4.4 holds:*

$$E_{+-}(z)^{-1} = \frac{1}{c'_{\nu_0} z_{\nu_0}} |u_{\nu_0,1}|^{-2}(1 + O(|z|^\delta/|z_{\nu_0}|)).$$

If $\nu_0 = 1$, $O(|z|^\delta/|z_{\nu_0}|) = O(|\ln z|^{-1})$.

Proposition 4.4 ensures that the asymptotic expansion for $E_{+-}(z)$ is invertible. From the representation of the resolvent $R(z)$, it follows that the asymptotic expansion of $R(z)$ exists. A technical issue in the following theorem is to give the form of this expansion and calculate a few leading terms. Let $0 < \varsigma_1 < \cdots < \varsigma_{\kappa_0} \le 1$ be the points in σ_1 such that P has m_j linearly independent ς_j-resonant states with $\sum_{j=1}^{\kappa_0} m_j = \mu_r$. Then there exists a basis of ς_j-resonant states, $u_j^{(i)}$, $i = 1, \cdots, m_j$ verifying

$$\frac{|c_{\varsigma_j}|^{1/2}}{4\varsigma_j^2} < V u_j^{(l)}, -|x|^{-\frac{n-2}{2}+\varsigma_j} \varphi_j^{(l')} >= \delta_{ll'}, \quad 1 \le l, l' \le m_j, \quad 1 \le j \le \kappa_0, \tag{4.28}$$

where c_{ς_j} is given by (4.19) and $\delta_{ll'} = 1$ if $l = l'$; 0 otherwise.

Theorem 4.6. *Assume* $0 \notin \sigma_\infty$. *Let* $\mu = \dim \mathcal{N} \neq 0$. *Assume* $\rho_0 > \max\{4N-6, 2N+1\}$ *if* $\mu_r = 0$ *and* $\rho_0 > \max\{4N-6, 2N+2\}$ *if* $\mu_r \neq 0$. *One has the following asymptotic expansion for* $R(z)$ *in* $\mathcal{L}(-1, s; 1, -s)$, $s > \max\{2N-3, 2\}$:

$$R(z) = \sum_{j=0}^{N-2} z^j T_j + \sum_{\{\vec{\nu}\}+j \le N-2}^{(1)} z_{\vec{\nu}} z^j T_{\vec{\nu};j} + T_e(z) + T_r(z) + T_{er}(z) + O(|z|^{N-2+\epsilon}) \tag{4.29}$$

Here T_j *(resp.,* $T_{\vec{\nu},j}$*) is in* $\mathcal{L}(1, -s; -1, s)$ *for* $s > 2j+1$ *(resp., for* $s > 2j + 1 + \{\vec{\nu}\}$*),*

$$T_0 = (1 + F_0 V)^{-1} F_0, T_1 = -T_0 F_1 V T_0 F_0.$$

The sum $\sum_{\{\vec{\nu}\}+j \le N}^{(1)}$ *has the same meaning as in (4.13) and the first singular term in this sum is* z_{ν_0} *with coefficient* $T_{\nu_0,0}$ *given by*

$$T_{\nu_0,0} = T_0 G_{\nu_0,\delta_{\nu_0}} \pi_{\nu_0} (1 - V T_0 F_0),$$

where ν_0 *is the smallest value of* $\nu \in \sigma_\infty$. $T_e(z)$, $T_r(z)$ *describe the contributions up to the order* $O(|z|^{N-2+\epsilon})$ *from eigenfunctions and resonant states, respectively, and* $T_{er}(z)$ *the* *interaction between eigenfunctions and resonant states. One has*

$$\begin{aligned}
T_e(z) &= -z^{-1}\Pi_0 + \sum_{j\geq -1,\, \{\vec{\nu}\}+j\leq N-2}^{(1)} z_{\vec{\nu}} z^j T_{e;\vec{\nu};j} \\
T_r(z) &= \sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1}(\Pi_{r,j} + \sum_{\alpha,\beta,\vec{\nu},l}^{+,N-1} z_{\vec{\nu}} z^{|\beta|}(z_{\vec{\varsigma}})^{-\alpha-\beta} z^l T_{r;\vec{\nu},\alpha,\beta,l,j}), \qquad \textit{with} \\
\Pi_{r,j} &= e^{i\pi\varsigma_j}\sum_{l=1}^{m_j} < \cdot, u_j^{(l)} > u_j^{(l)}, \quad j=1,\cdots,\kappa_0 \\
T_{er}(z) &= \sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1}(\Pi_0 V Q_e F_1 V \Pi_{r,j} + \Pi_{r,j} V Q_r F_1 V \Pi_0 + \sum_{\alpha,\beta,\vec{\nu},l}^{+,N-1} z_{\vec{\nu}} z^{|\beta|}(z_{\vec{\varsigma}})^{-\alpha-\beta} z^l T_{er;\vec{\nu},\alpha,\beta,l,j}).
\end{aligned}$$

Π_0 *is the spectral projection of* P *at* 0*, and* $T_e(z)$ *is of rank not exceeding Rank* Π_0 *with leading singular parts given by* $\nu_j \in \sigma_2$*:*

$$T_{e;\vec{\nu};-1} = (-1)^{k'+1}(\Pi_0 V G_{\nu_1,1+\delta_{\nu_1}}\pi_{\nu_1}V)\cdots(\Pi_0 V G_{\nu_{k'},1+\delta_{\nu_{k'}}}\pi_{\nu_{k'}}V)\Pi_0, \tag{4.30}$$

for $\vec{\nu} = (\nu_1,\cdots,\nu_{k'}) \in \sigma_2^{k'}$ *with* $\{\vec{\nu}\} \leq 1$*,* $(z_{\vec{\varsigma}})^{-\alpha} = (z_{\varsigma_1})^{-\alpha_1}\cdots(z_{\varsigma_{\kappa_0}})^{-\alpha_{\kappa_0}}$*. The summation* $\sum_{\alpha,\beta,\vec{\nu},l}^{+,N-1}$ *is taken over all possible* α*,* $\beta \in \mathbb{N}^{\kappa_0}$ *with* $1 \leq |\alpha| \leq N_0$*,* $|\beta| \geq 1$*,* $\vec{\nu} = (\nu_1,\cdots,\nu_{k'}) \in \sigma_N^{k'}$*,* $k' \geq 2|\alpha|$*, for which there are at least* α_k *values of* ν_j*'s belonging to* σ_1 *with* $\nu_j \geq \varsigma_k$*, for* $1 \leq k \leq \kappa_0$*,* $l \in \mathbb{N}$*, satisfying*

$$|\beta| + \{\vec{\nu}\} + l - \sum_{k=1}^{\kappa_0}(\alpha_k+\beta_k)\varsigma_k \leq N-1.$$

Proof. We only give the proof of (4.29) based on the representation formula $R(z) = (E(z) - E_+(z)E_{+-}(z)^{-1}E_-(z))R_0(z)$ in the case $N=2$ and $\rho_0 > 6$. The proof for general case is the same. It is clear that the asymptotic expansion of $E(z)R_0(z)$ gives arise to the first two sums in (4.29). Let us study the leading singularities and the form of asymptotic expansion related to the term $E_+(z)E_{+-}(z)^{-1}E_-(z)R_0(z)$ which is of rank $\leq \mu$. One has

$$\begin{aligned}
&-E_+(z)E_{+-}(z)^{-1}E_-(z) \\
&\quad= -(T - D(z)Y)(-SW(z)T + SXY)^{-1}(S - SX) \\
&\quad= -(1 - D(z)W(z))Q\left(Q(-W(z) + W(z)D(z)W(z))Q\right)^{-1}Q(1 - W(z)D(z))
\end{aligned}$$

and

$$\begin{aligned}
QW(z) &= Q((L_1(z) + zF_1 + L_2(z) + z^2F_2)V + O(|z|^{2+\epsilon})), \\
D(z) &= D_0 + D_1(z) + D_2(z) + O(|z|^{2+\epsilon}),
\end{aligned}$$

in $\mathcal{L}(1,-s;1,-s), 5 < s < \rho_0 - 1$, where

$$\begin{aligned}
L_1(z) &= \sum_{\nu\in\sigma_1} z_\nu G_{\nu,\delta_\nu}\pi_\nu, &\qquad L_2(z) &= \sum_{\nu\in\sigma_2} z_\nu z G_{\nu,1+\delta_\nu}\pi_\nu \\
D_1(z) &= zD_1 + \sum_{\nu\in\sigma_1} z_\nu D_{\nu,0}, &\qquad D_2(z) &= z^2 D_2 + \sum_{\nu\in\sigma_2} z_\nu z D_{\nu,1}
\end{aligned}$$

with $\sigma_j = \sigma_\infty \cap [0, j]$. It follows that

$$\begin{aligned} & Q(-W(z) + W(z)D(z)W(z))Q \\ = \; & Q\{-[L_1(z) + zF_1 + L_2(z)]V + [L_1(z) + zF_1 + L_2(z)]V(D_0 + D_1(z))L_1(z)V \\ & +L_1(z)V[D_2(z)L_1(z) + (D_0 + D_1(z))(zF_1 + L_2(z))]V + O(|z|^2)\}Q \end{aligned}$$

Let $S(z) = Q(-W(z) + W(z)D(z)W(z))Q$. Set

$$\nu_0 = \min\{\min\{\nu \in \sigma_1\}, \min\{\nu - 1; \nu \in \sigma_2 \setminus \sigma_1\}\}.$$

Assume first that 0 is not a resonance of P. Then $QL_1(z) = L_1(z)VQ = 0$ and $QW(z) = Q(zF_1 + L_2(z))V + O(|z|^2))$. We have for $\rho_0 > 5$

$$\begin{aligned} S(z) &= Q(-(zF_1 + L_2(z))V + O(|z|^2))Q \\ &= -Q(QF_1VQ)[z + Q(QF_1VQ)^{-1}L_2(z)V + O(|z|^2)]Q \end{aligned}$$

As in [35], it can be shown that $(QF_1VQ)^{-1}Q = \Pi_0 V$, where Π_0 is the orthogonal projection onto the zero eigenspace of P. Since $z^{-1}L_2(z) = O(z_{\nu_0})$ is small, we obtain

$$S(z)^{-1} = -z^{-1}\left(1 + \sum_{j\geq 1}^{N_0}(-1)^j(z^{-1}\Pi_0 V L_2(z)VQ)^j\right)\Pi_0 V_0 + O(1),$$

with N_0 large enough so that $N_0\nu_0 \geq 1$.

$$\begin{aligned} & -E_+(z)E_{+-}(z)^{-1}E_-(z) \\ = \; & z^{-1}\left(1 + \sum_{j\geq 1}^{N_0}(-1)^j(z^{-1}\Pi_0 V L_2(z)V)^j\right)\Pi_0 V + O(1). \end{aligned}$$

Since $R_0(z) = F_0 + L_1(z) + O(z)$ and $\Pi_0 V L_1(z) = 0$, we obtain that

$$\begin{aligned} & -E_+(z)E_{+-}(z)^{-1}E_-(z)R_0(z) \\ = \; & -z^{-1}\left(1 + \sum_{j\geq 1}^{N_0}(-1)^j(z^{-1}\Pi_0 V L_2(z)V)^j\right)\Pi_0 + O(1). \end{aligned}$$

This gives the formula for $T_{e,\vec{\nu},-1}$ in the case when there is no resonant state. Since $Q'W(z)Q = zQ'F_1VQ + O(|z|^{1+\nu_0})$, it follows that

$$-(1-\Pi_0)E_+(z)E_{+-}(z)^{-1}E_-(z)R_0(z) = -(1-\Pi_0)D_0F_1V\Pi_0(z^{-1}L_2(z))V\Pi_0 + O(|z|) \tag{4.31}$$

This shows that $(1-\Pi_0)T_e(z) = O(|z|^{\nu_0})$.

Assume now that zero is a resonance of P. One has in $H^{1,-s}$ with $s > 1$ sufficiently close to 1:

$$\begin{aligned} S(z) &= S_r(z) + S_e(z) + S_{re}(z) + S_{er}(z) + O(|z|^2) \qquad \text{with} \\ S_e(z) &= -Q_e(zF_1 + L_2(z))VQ_e \\ S_r(z) &= Q_r\{-[L_1(z) + zF_1 + L_2(z)] + [L_1(z) + zF_1 + L_2(z)]V(D_0 + D_1(z))L_1(z) \\ &\quad +L_1(z)V[D_2(z)L_1(z) + (D_0 + D_1(z))(zF_1 + L_2(z))\,]\}VQ_r \\ S_{re}(z) &= Q_r\{-(zF_1 + L_2(z)) + L_1(z)V(D_2(z)L_1(z) + (D_0 + D_1(z))[zF_1 + L_2(z)]\,)\}VQ_e \\ S_{er}(z) &= Q_e\{-(zF_1 + L_2(z)) + [zF_1 + L_2(z)]V[D_0 + D_1(z)]L_1(z)\}VQ_r. \end{aligned}$$

From the proof of Proposition 4.4,

$$I_r(z) \equiv S_r(z)^{-1}Q_r, \quad I_e(z) \equiv S_e(z)^{-1}Q_e$$

exist. We have

$$S(z)(I_r(z)+I_e(z)) \;=\; Q + S_{er}(z)I_r(z) + S_{re}(z)I_e(z) + O(|z|)$$

Note that $I_e(z) = O(|z|^{-1})$, $I_r(z) = O(|z_{\varsigma_{\kappa_0}}|^{-1})$. It follows that $S_{er}(z)I_r(z) \to 0$, $S_{re}(z)I_e(z)) = O(1)$ as $z \to 0$, which implies

$$(S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^2 = S_{er}(z)I_r(z)S_{re}(z)I_e(z) + S_{re}(z)I_e(z)S_{er}(z)I_r(z) \to 0$$

as $z \to 0$. Therefore, $Q + S_{er}(z)I_r(z) + S_{re}(z)I_e(z)$ is invertible on the range of Q and we have the convergent expansion:

$$(Q + S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^{-1}Q = Q + \sum_{j=1}^{\infty}(-1)^j(S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^j$$

in $\mathcal{L}(1,-s;1,-s)$ for $s>1$. $S(z)^{-1}$ is then given by

$$(I_e(z)+I_r(z))(Q + \sum_{j=1}^{\infty}(-1)^j(S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^j) + O(1).$$

It follows that

$$\begin{aligned}
&-E_+(z)E_{+-}(z)^{-1}E_-(z)\\
&= ((1-(D_0+D_1(z))L_1(z)V)I_r(z) + I_e(z))\\
&= (1-(D_0+D_1(z))L_1(z)V)I_r(z)Q_r(1-L_1(z)V(D_0+D_1(z))) + I_e(z) + I_{er}(z) + O(1).
\end{aligned}$$

Here, $I_{er}(z)$ is defined by

$$\begin{aligned}
I_{er}(z) &= ((1-(D_0+D_1(z))L_1(z)V)I_r(z) + I_e(z))\\
&\quad\times\left(\sum_{j=1}^{\infty}(-1)^j(S_{er}(z)I_r(z) + S_{re}(z)I_e(z))^j\right)(Q_r(1-L_1(z)V(D_0+D_1(z))) + Q_e),
\end{aligned}$$

$I_{er}(z)$ is the contribution from the interaction between resonant states and eigenfunctions. $I_e(z)$ has the same asymptotic expansion as in the case $\mu_r = 0$. The contribution from resonant states is given by

$$(1-(D_0+D_1(z))L_1(z)V)I_r(z)Q_r(1-L_1(z)V(D_0+D_1(z))).$$

By the analysis made in Proposition 4.4, $Q_r(-L_1(z)V + L_1(z)V(D_0+D_1(z))L_1(z)V)Q_r$ is invertible on the range of Q_r. Let $I_{r,0}(z)$ denote its inverse. By (4.24),

$$I_{r,0}(z) = T(\mathcal{T}^*\mathcal{D}_1(z)\mathcal{T})^{-1}S(1+O(|z|^\epsilon)),$$

where $\mathcal{T}$ is the transfer matrix from $\{\psi_1,\cdots,\psi_{\mu_r}\}$ to $\{\phi_1,\cdots,\phi_{\mu_r}\}$ and $\mathcal{D}_1(z)$ is given in Proposition 4.4. Note that $S = -T^*V$, where $T^* : H^{1,-s} \to \mathbb{C}^\mu$ is the formal adjoint of T. Let

$$\Pi_r(z) = T(\mathcal{T}^{-1}\mathcal{D}_1(z)^{-1}(\mathcal{T}^{-1})^*)T^*$$

One can verify that

$$\Pi_r(z) \;=\; \sum_{j=1}^{\kappa_0}(z_{\varsigma_j})^{-1}\sum_{l=1}^{m_j}\frac{4\varsigma_j^2}{c_{\varsigma_j}} < \cdot,\ \psi_j^{(l)} > \psi_j^{(l)}. \tag{4.32}$$

See [35] for detailed calculation in a similar situation. Since $I_{r,0}(z) = -\Pi_r(z)V(1+O(|z|^\epsilon))$, we obtain

$$I_{r,0}(z)R_0(z) = \Pi_r(z)(1+O(|z|^\epsilon)).$$

By Theorem 3.1 and (4.21), $\psi_j^{(l)}$ satisfies

$$< V\psi_j^{(l)}, -\frac{1}{2\varsigma_j}|y|^{-\frac{n-2}{2}+\varsigma_j}\varphi_{\varsigma_j}^{(l')} >= \delta_{ll'}.$$

It suffices to take

$$u_j^{(l)} = \frac{2\varsigma_j}{|c_{\varsigma_j}|^{\frac{1}{2}}}\psi_j^{(l)} \tag{4.33}$$

in order to obtain the leading part of the singularity from resonant states as stated in Theorem 4.6. For z small enough, $I_{r,0}(z)$ has a convergent expansion

$$I_{r,0}(z) = -\left(1+\sum_{j=1}^{\infty}(\Pi_r(z)V(L_1(z)V(D_0+D_1(z))L_1(z)V)Q_r)^j\right)\Pi_r(z)V. \tag{4.34}$$

We need only to sum up to $j = N_0$ for some N_0 large enough such that the remainder is $O(|z|^{N-2+\epsilon})$. By Theorem 3.1 and (4.21), $\Pi_{r,j}G_{\nu,\delta_\nu}\pi_\nu = 0$ if $\nu < \varsigma_j$. Therefore, $\Pi_r(z)V(L_1(z)V(D_0+D_1(z))L_1(z)V)Q_r$ can be written as

$$\sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1}\Pi_{r,j}\left(\sum_{\vec{\nu}\in(\sigma_1)^s, s=2,3,k=0,1}^{+} z_{\vec{\nu}}z^k J_{r;\vec{\nu},k,j}\right)$$

where the notation $\sum^+_{\vec{\nu}\in(\sigma_1)^l}$, $l = 2,3$, means that the summation is taken over those $\vec{\nu} = (\nu_1, \cdots, \nu_l)$, which has at least one component, say ν_1, verifying $\nu_1 \geq \varsigma_j$ and $J_{r;\vec{\nu},k,j} = 0$ for $\vec{\nu} \in (\sigma_1)^3$ and $k = 1$. It follows that $I_{r,0}(z)$ can be expanded as

$$\begin{aligned} I_{r,0}(z) &= -\left(1+\sum_{l=1}^{N_0}(\sum_{j=1}^{\kappa_0} z_{\varsigma_j}^{-1}\Pi_{r,j}\sum_{\vec{\nu}\in(\sigma_1)^s, s=2,3,k=0,1,\nu_1\geq\varsigma_j} z_{\vec{\nu}}z^k J_{r;\vec{\nu},k,j})^l\right)\Pi_r(z)V + O(|z|^{N-2+\epsilon}) \\ &= -\Pi_r(z)V + \sum_{\alpha\in\mathbb{N}^{\kappa_0}, 1\leq|\alpha|\leq N_0}\ \sum^{+}_{\vec{\nu}\in\sigma_1^s, 2|\alpha|\leq s\leq 3|\alpha|; k\leq|\alpha|} z_{\vec{\nu}}z^k(z_{\vec{\varsigma}})^{-\alpha}I_{r;\vec{\nu},\alpha,k,j}\Pi_r(z)V + O(|z|^{N-2+\epsilon}). \end{aligned}$$

Here $(z_{\vec{\varsigma}})^{-\alpha} = (z_{\varsigma_1})^{-\alpha_1}\cdots(z_{\varsigma_{\kappa_0}})^{-\alpha_{\kappa_0}}$ and the summation $\sum^+$ is taken over all possible $\vec{\nu} = (\nu_1, \cdots, \nu_s) \in \sigma_1^s$ for which there are at least α_l of the ν_j's belonging to σ_1 with $\nu_j \geq \varsigma_l$ for all $1 \leq l \leq \kappa_0$.

Since $I_{r,0}(z)S_{r,1}(z) = O(\frac{z}{z_{\varsigma_{\kappa_0}}}) = O(\frac{1}{\ln z})$, one has the following convergent series in $\mathcal{L}(1,-s;1,-s)$, $s > 1$, for $z \in U_\delta$ with $\delta > 0$ small enough,

$$I_r(z) = S_r(z)^{-1}Q_r = I_{r,0}(z) + \sum_{j=1}^{\infty}(-1)^j(I_{r,0}(z)S_{r,1}(z))^jI_{r,0}(z)$$

where

$$\begin{aligned} S_{r,1}(z) &= Q_r(-[zF_1+L_2(z)] + [zF_1+L_2(z)]V[D_0+D_1(z)]L_1(z) \\ &\quad +L_1(z)V[D_2(z)L_1(z) + (D_0+D_1(z))(zF_1+L_2(z))]\ )VQ_r \\ &= zQ_r(-F_1 + \sum_{\vec{\nu}\in(\sigma_2)^s, 1\leq s\leq 3, j=0,1} z_{\vec{\nu}}z^jS_{r;\vec{\nu},j})VQ_r. \end{aligned}$$

Inserting the expansions of $I_{r,0}(z)$ and $S_{r,1}(z)$ into $I_r(z)$ and rearranging the terms, we obtain

$$I_r(z) = -\Pi_r(z)V + \sum_{j,\alpha,\beta,\vec{\nu}\in\sigma_2^k,k\geq 2|\alpha|}^{+,N-1} z_{\vec{\nu}} z^{|\beta|+j}(z_{\vec{\varsigma}})^{-\alpha-\beta} I_{r;\vec{\nu},\alpha,\beta,j}\Pi_r(z)V + O(|z|^{N-2+\epsilon}).$$

Note that here $N=2$ and only $\nu\sigma_2$ is needed. In the case $\varsigma_{\kappa_0} < 1$, a finite sum on β is sufficient in order to obtain an asymptotic expansion of $R(z)$ up to $O(|z|^{N-2+\epsilon})$. In the case $\varsigma_{\kappa_0} = 1$, $z_{\varsigma_{\kappa_0}} = z\ln z$. It is then necessary first to sum over all β in order to expand $R(z)$ up to $O(|z|^{N-2+\epsilon})$. It is now clear that

$$(1-(D_0+D_1(z))L_1(z)V)I_r(z)Q_r(1-L_1(z)V(D_0+D_1(z)))R_0(z)$$

has the asymptotic expansion of $T_r(z)$.

For the interaction between resonant states and eigenfunctions, note that $S_{er}(z) = -zQ_e(F_1V+O(|z|^\epsilon))Q_r$ and $S_{re}(z) = -zQ_r(F_1V+O(|z|^\epsilon))Q_e$. It follows that

$$\begin{aligned} I_e(z)S_{er}(z)I_r(z) &= -\Pi_0 VQ_eF_1V\Pi_r(z)V + O(|z|^\epsilon/|z_{\varsigma_{\kappa_0}}|) + O(|z|/(|z_{\varsigma_{\kappa_0}}|)^2) \\ I_r(z)S_{re}(z)I_e(z) &= -\Pi_r(z)VQ_rF_1V\Pi_0V + O(|z|^\epsilon/|z_{\varsigma_{\kappa_0}}|) + O(|z|/(|z_{\varsigma_{\kappa_0}}|)^2) \end{aligned}$$

which gives

$$I_{er}(z) \;=\; -(\Pi_0VQ_eF_1V\Pi_r(z)V + \Pi_r(z)VQ_eF_1V\Pi_0V) + O(|z|^\epsilon/|z_{\varsigma_{\kappa_0}}|) + O(|z|/(|z_{\varsigma_{\kappa_0}}|)^2)$$

The remainder terms have asymptotic expansions of the form of $T_{er}(z)$. Theorem 4.6 is proved. □

Theorem 4.6 shows that the asymptotic expansion of $R(z)$ may contain any terms of the form $z_{\vec{\nu}}z^l$, $(\frac{z_\varsigma}{z^{\varsigma_j}})^k$, $\varsigma\in\sigma_1$ with $\varsigma_j < \nu \leq 1$, and $(\frac{1}{\ln z})^m$. If P has only 1-resonant states (*i.e.*, $\kappa_0 = 1$ and $\varsigma_1 = 1$) which may, however, still have an arbitrarily large multiplicity, α is absent in the summation $\sum_{\alpha,\beta,\vec{\nu},l}^{+,N-1}$ and the sum on β is infinite and gives rise to convergent series in $\frac{1}{\ln z}$. In this case, $T_r(z)$ in Theorem 4.6 can be written in the form

$$T_r(z) = \frac{1}{z\ln z}\{\Pi_{r,1} + \sum_{\vec{\nu}\in(\sigma_N)^s, s\leq N_0, l\geq 0, \{\vec{\nu}\}+l\leq N-1} z_{\vec{\nu}}z^l\Psi_{\vec{\nu},l}(z)\}$$

where $\Psi_{\vec{\nu},l}(z)$ is a convergent series of the form

$$\Psi_{\vec{\nu},l}(z) = \sum_{k=1}^{\infty}\frac{1}{\ln^k z}T_{r;\vec{\nu},k,l}.$$

5. The case $0\in\sigma_\infty$

Assume now $0\in\sigma_\infty$. By Theorem 2.2, $R_0(z) = \ln z\, G_{0,0}\pi_0 + F_0 + O(|z|^\epsilon)$ in $\mathcal{L}(-1,s;1,-s)$ for any $s>1$. Let $\phi_0(x) = \varphi_0^{(1)}(\theta)r^{-\frac{n-2}{2}}$. Since $d_0 = -\frac{1}{2}$,

$$G_{0,0}\pi_0 = \frac{1}{2} < \cdot, \phi_0 > \phi_0.$$

Let V be the perturbation as before. Set

$$\mathcal{N} = \{u\in H^{1,-s};\ < Vu, \phi_0 >= 0,\ \exists\beta\in\mathbb{C},\ (1+F_0V)u = \beta\phi_0\} \tag{5.1}$$

for $s>1$ close to 1. If $u\in\mathcal{N}$, $(1+R_0(z)V)u = \beta\phi_0 + O(|z|^\epsilon)$ in $H^{1,-s}$, and since $P_0\phi_0 = 0$,

$$(P-z)u = (P_0-z)(1+R_0(z)V)u = O(|z|^\epsilon)$$

in $H^{-1,-s}$. It follows that $Pu=0$. $\mathcal{N}$ is of finite dimension. The eigenfunctions of P at 0 belong to $\mathcal{N}$. u is called a resonant state of P at 0 if $u \in \mathcal{N}\setminus L^2$.

Part (b) of Theorem 3.2 shows that $(1+F_0V)\phi = 0$ if $\phi\in\mathcal{N}$ is not a 0-resonant state. Assume from now on that

$$\gamma_0 \equiv < \phi_0, -V\phi_0 > \neq 0. \tag{5.2}$$

In the case $n=2$, $P_0=-\Delta$ and $P=-\Delta+V(x)$, the condition (5.2) is reduced to $\int V(x)\,dx \neq 0$, which is used in [5]. Under the condition (5.2), $\beta = -\gamma_0^{-1} < F_0Vu, V\phi_0 >$.

To establish the asymptotics of $R(z)$ as $z\to 0$, we modify the Grushin problem for $(1+R_0(z)V$ studied in Section 4 by setting

$$A(z)=\begin{pmatrix} W(z) & T \\ S & 0\end{pmatrix} \;:\; H^{1,-s}\times\mathbb{C}^{\mu+1}\to H^{1,-s}\times\mathbb{C}^{\mu+1},$$

where $s>1$ and

$$T\,(c_0,c_1,\cdots,c_\mu)=\sum_{j=0}^{\mu} c_j\phi_j, \quad (c_0,c_1,\cdots,c_\mu)\in\mathbb{C}^{\mu+1},$$

$$Sf=\left(\gamma_0^{-1}< f,-V\phi_0 >, < f,-V\phi_1 >,\cdots, < f,-V\phi_\mu >\right)\in\mathbb{C}^{\mu+1}, \quad f\in H^{1,-s}.$$

Here $\phi_j\in\mathcal{N}$, $j=1\cdots,\mu$ are chosen as before. Assume that ϕ_1 is a 0-resonant state. It follows from Theorem 3.2 that

$$(1+F_0V)\phi_j=\beta_j\phi_0, \quad \beta_1\neq 0, \quad \beta_j=0, \quad j\geq 2.$$

Define

$$Q_0f=\gamma_0^{-1} < f,-V\phi_0 > \phi_0, \quad Q_1f=\sum_{j=1}^{\mu} < f,-V\phi_j > \phi_j, \quad Q=Q_0+Q_1. \tag{5.3}$$

Then Q is a projection. Let $Q'=1-Q$.

$$Q'(1+R_0(z)V)Q'=Q'(1+F_0V)Q'+O(|z|^\epsilon).$$

As in Section 4, we can deduce that $Q'(1+R_0(z)V)Q'$ is invertible on the range of Q' with the inverse

$$D(z)=(1+Q'R_0(z)VQ')^{-1}Q'\in\mathcal{L}(1,-s;1,-s)$$

bounded for $s>1$ and z near zero. $D(z)$ has an asymptotic expansion of the form (4.13) with the convention that $\nu'=1$ for $\nu=0$. By means of the formula for the inverse of $A(z)$ we obtain a representation for $W(z)^{-1}=(1+R_0(z)V)^{-1}$

$$W(z)^{-1}=E(z)-E_+(z)E_{+-}(z)^{-1}E_-(z). \tag{5.4}$$

Here operators are defined as in Section 4 with obvious modifications. Let $\delta_{0j}=1$ if $j=0$; 0 otherwise. The entries

$$[E_{+-}(z)]_{ij} =< (W(z)-W(z)Q'D(z)Q'W(z))\phi_j, -\gamma_0^{-\delta_{0i}}V\phi_i >, \quad 0\leq i,j\leq\mu,$$

can be calculated as in Proposition 4.4. Let $\rho_0>4$. Set

$$W(z)=1+(\ln z\, G_{0,0}\pi_0+F_0+L_1(z)+zF_1)V+O(|z|^{1+\epsilon})$$

with $L_1(z)=\sum_{0\leq\nu\leq 1} z_\nu G_{\nu,\delta_\nu}\pi_\nu$, in $\mathcal{L}(1,-s;1,-s')$ with $1<s<\rho_0-3$ and $s'>3$. $Q'W(z)\phi_j$ is bounded if $j=0$, $Q'W(z)\phi_j=Q'(L_1(z)+zF_1)V\phi_j+O(|z|^{1+\epsilon})$ for $1\leq j\leq\mu_r$

and $Q'W(z)\phi_j = zQ'F_1V\phi_j + O(|z|^{1+\epsilon})$ for $\mu_r < j \le \mu$.

$$[E_{+-}(z)]_{0,j} = \begin{cases} \frac{1}{2}\gamma_0 \ln z + e_0 + O(|z|^\epsilon), & j=0 \\ \beta_j + b_{0,j}(z) + O(|z|^{1+\epsilon}), & 1 \le j \le \mu \end{cases}$$

where

$$\begin{aligned} e_0 &= <(1+F_0V)(1-D_0(1+F_0V))\phi_0, -\gamma_0^{-1}V\phi_0>, \\ b_{0,j}(z) &= <(1-(1+(F_0+L_1(z))V)D(z))(L_1(z)+zF_1)V\phi_j, -\gamma_0^{-1}V\phi_0> . \end{aligned}$$

From Theorem 3.2, $b_{0,j}(z) = O(|z|^\epsilon)$ for $1 \le j \le \mu_r$ and $b_{0,j}(z) = O(|z \ln z|)$ for $\mu_r < i \le \mu$. Remark that

$$[E_{+-}(z)]_{j,0} = \gamma_0 [E_{+-}(\overline{z})]_{0,j}, \quad j \ge 1.$$

Since $G_{0,0}\pi_0V\phi_i = 0$ and $< G_{0,1}\pi_0V\phi_i, V\phi_j >= 0$ for $i,j \ge 1$, the entries $[E_{+-}(z)]_{ij}$, $i,j \ge 1$, are the same as in Proposition 4.4. Let $b_{i,1}(z)$ be defined as $b_{i,0}(z)$ with ϕ_0 replaced by ϕ_1. Decompose $E_{+-}(z)$ into blocs

$$E_{+-}(z) = \begin{pmatrix} E_{+-}^{11}(z) & E_{+-}^{12}(z) \\ E_{+-}^{21}(z) & E_{+-}^{22}(z) \end{pmatrix}$$

with

$$E_{+-}^{11}(z) = ((E_{+-}(z))_{i,j})_{0 \le i,j \le 1}, \quad E_{+-}^{22}(z) = ((E_{+-}(z))_{i,j})_{2 \le i,j \le \mu} .$$

The inverse of $E_{+-}^{22}(z)$ can be calculated in a special basis in the same way as in the proof of Proposition 4.4. By an explicit calculation similar to that made in the proof of Proposition 4.4, we can prove that $E_{+-}^{12}(z)E_{+-}^{22}(z)^{-1} = O(|\ln z|)$ and $E_{+-}^{12}(z)E_{+-}^{22}(z)^{-1}E_{+-}^{21}(z) = O(|z|^\epsilon)$. Since

$$E_{+-}^{11}(z) = \begin{pmatrix} \frac{1}{2}\gamma_0 \ln z + O(1) & \overline{\beta_1} + O(|z|^\epsilon) \\ \beta_1\gamma_0 + O(|z|^\epsilon) & O(|z|^\epsilon) \end{pmatrix}$$

with $\beta_1 \ne 0$, we obtain

$$T(z) = E_{+-}^{11}(z) - E_{+-}^{12}(z)E_{+-}^{22}(z)^{-1}E_{+-}^{21}(z)$$

is invertible with $T(z)^{-1} = E_{+-}^{11}(z)^{-1}(1 + O(|z|^\epsilon))$. It follows that

$$E_{+-}(z)^{-1} =$$

$$\begin{pmatrix} E_{+-}^{11}(z)^{-1} & -E_{+-}^{11}(z)^{-1}E_{+-}^{12}(z)E_{+-}^{22}(z)^{-1} \\ -E_{+-}^{22}(z)^{-1}E_{+-}^{21}(z)E_{+-}^{11}(z)^{-1} & E_{+-}^{22}(z)^{-1} \end{pmatrix} \times (1 + O(|z|^\epsilon)) \quad (5.5)$$

It is clear that equation (5.5) gives the leading singularities of $R(z)$ in the most general case and can be simplified if some part is absent. In particular, if there is no 0-resonant state, $E'_{+-}(z)$ is a scalar of the form $\frac{1}{2}\gamma_0 \ln z + O(1)$.

Theorem 5.1. *(a). Suppose that $\mu = 0$. Let $\rho_0 > 4N+2$. Then, in $\mathcal{L}(-1, s; 1-s)$ with $s > 2N+1$, one has*

$$R(z) = \sum_{j=0}^{N} z^j T_j + \sum_{\{\vec{\nu}\}+j \le N}^{(1)} z_{\vec{\nu}} z^j T_{\vec{\nu},j} + T_s(z) + O(|z|^{N+\epsilon}) \quad (5.6)$$

Here $T_0 = (1+Q'F_0VQ')^{-1}Q'F_0$*, and* $T_s(z)$ *is an operator of rank* 1 *having an asymptotic expansion of the form*

$$T_s(z) \quad = \quad T_{s,0} + \sum_{\ell=0}^{N_0} \sum_{\{\vec{\nu}\}+j\le N}^{(\ell)} z_{\vec{\nu}} z^j \sum_{m=\ell}^{\infty} (\ln z)^{-m-1} T^{(\ell)}_{s;\vec{\nu},j;m} + O(|z|^{N+\epsilon}) \tag{5.7}$$

Here $\phi_0^* = (1-D_0F_0V)\phi_0$*, the infinite sum in* m *is convergent, and the first two coefficients are given by*

$$T_{s,0} \quad = \quad -\frac{1}{\gamma_0} < \cdot, \phi_0 > \phi_0^* \tag{5.8}$$

$$T^{(0)}_{s;\vec{\nu},j;0} \quad = \quad \frac{2}{\gamma_0^2}(-e_0 < \cdot, \phi_0 > + < (1-F_0VD_0)F_0\cdot, V\phi_0 >)\phi_0^* \tag{5.9}$$

for $\{\vec{\nu}\}+j=0$*.*
(b). Assume that there is a 0*-resonant state and* $\mu=1$*. Assume* $\rho_0 > 4N+2$*. Then, (5.6) holds in* $\mathcal{L}(-1,s;1,-s)$ *with* $s > 2N+1$ *with now* $T_s(z)$ *an operator of rank two describing the contributions from* 0*-resonant states of* P_0 *and* P*:* $T_s(z) = T_0(z) + T_1(z) + T_{01}(z)$ *with*

$$T_0(z) \quad = \quad \sum_{\ell=1}^{N_0} \sum_{\{\vec{\nu}\}+j\le N}^{(\ell)} \sum_{m=0}^{\ell} (\ln z)^m z_{\vec{\nu}} z^j T^{(j)}_{0;\vec{\nu},j,m} + O(|z|^{N+\epsilon}) \tag{5.10}$$

$$T_{01}(z) \quad = \quad -\frac{1}{|\beta_1|^2\gamma_0}(\overline{\beta_1} < \cdot, -V\phi_1 > \phi_0 + \beta_1 < \cdot, -V\phi_0 > \phi_1 + T^{(0)}_{01}(z) \tag{5.11}$$

$$T_1(z) \quad = \quad -\ln z < \cdot, u_0 > u_0 + T_{1,1} + T_1^{(0)}(z) \tag{5.12}$$

Here $T^{(0)}_{01}(z)$ *and* $T^{(0)}_1(z)$ *have asymptotic expansions of the form (5.10), and* u_0 *is a* 0*-resonant state verifying*

$$\frac{1}{\sqrt{2}} < Vu_0, -\ln r\phi_0 >= 1. \tag{5.13}$$

(c). Assume that $\mu \ge 1$ *and there exists* 0*-resonant state. Assume* $\rho_0 > \max\{4N-6, 2N+2\}$*. Then, the asymptotic expansion of the form (5.6) with* N *replaced by* $N-2$ *holds in* $\mathcal{L}(-1,s;,-s)$ *with* $s > \max\{2N-3, 2\}$ *with* $T_s(z) = T_0(z)+T_1(z)+T_{01}(z)$*, where* $T_0(z)$ *is of rank one having the expansion (5.7) in (a) with the same leading terms,* $T_{01}(z)$ *is also of the form (5.7), and* $T_1(z)$ *arises from resonant states and eigenfunctions of* P *with energy* 0 *whose leading terms are given by those of* $T_e(z)+T_r(z)+T_{er}(z)$ *in (b) of Theorem 4.4.*

Proof. We only compute the leading singularities of the resolvent. It is again based on the representation formula

$$R(z) = (E(z) - E_+(z)E_{+-}(z)^{-1}E_-(z))R_0(z).$$

Since $E(z) = D(z) = (1+Q'R_0(z)VQ')^{-1}Q'$ and $Q'G_{0,0} = 0$, $E(z)R_0(z)$ contains no singular $\ln z$-term. The asymptotic expansions of $D(z)R_0(z)$ and $W(z)D(z)R_0(z)$ are of the form (4.13).

(a). Let $\mu = 0$. The calculation made in the proof of Theorem 3.2 ([37]) shows that

$$E_{+-}(z) = \frac{1}{2}\gamma_0 \ln z + e_0 + r(z)$$

where $r(z) = \sum_{\{\vec{\nu}\}+j\leq N}^{(1)} z_{\vec{\nu}} z^j e_{\vec{\nu},j} + O(|z|^{N+\epsilon})$. Let $e(z) = (1 + 2e_0/(\gamma_0 \ln z))^{-1}$ which has a convergent expansion in $(\ln z)^{-1}$. Let $s(z) = E_{+-}(z)^{-1}$. Then

$$\begin{aligned} s(z) &= \frac{2e(z)}{\gamma_0 \ln z} \sum_{j=0}^{\infty} (-1)^j \left(\frac{2e(z)}{\gamma_0 \ln z} r(z) \right)^j \\ &= \frac{2e(z)}{\gamma_0 \ln z} \sum_{l=0}^{N_0} \left(\frac{2e(z)}{\gamma_0 \ln z} \right)^l \sum_{\{\vec{\nu}\}+j\leq N}^{(l)} z_{\vec{\nu}} z^l s_{\vec{\nu},l} + O(|z|^{N+\epsilon}). \end{aligned} \tag{5.14}$$

The $\ln z$-singularity in $R_0(z)$ is divided off. Let $\phi_0^*(z) = (1 - D(z)W(z))\phi_0$. Then,

$$\begin{aligned} &-E_+(z)E_{+-}(z)^{-1}E_-(z)R_0(z) \\ &= \frac{s(z)}{\gamma_0} < (1 - W(z)D(z))R_0(z)\cdot, \, V\phi_0 > \phi_0^*(z) \\ &= -\frac{e(z)}{\gamma_0} < \cdot, \, \phi_0 > \phi_0^*(z) + \frac{s(z)}{\gamma_0} < ((1 - F_0VD_0)F_0 + O(|z|^\epsilon))\cdot, \, V\phi_0 > \phi_0^*(z) \end{aligned}$$

This allows to give the first two terms in $T_s(z) = -E_+(z)E_{+-}(z)^{-1}E_-(z)R_0(z)$ as stated in (a).

(b). Let $\nu = 1$ and ϕ_1 be a 0-resonant state chosen as before. Let $Q = Q_0 + Q_1$ and $d(z) = \det E_{+-}(z)$. Then for $\rho_0 > 2N + 1$, one has

$$d(z) = -|\beta_1|^2\gamma_0 + P_1'(\ln z) \sum_{\{\vec{\nu}\}+j\leq N}^{(1)} z_{\vec{\nu}} z^j d_{\vec{\nu},j} + O(|z|^{N+\epsilon}) \tag{5.15}$$

with $P_1'(\ln z)$ a polynomial of degree 1 in $\ln z$. It follows that

$$d(z)^{-1} = -\frac{1}{|\beta_1|^2\gamma_0} + \sum_{l=1}^{N_0} P_l(\ln z) \sum_{\{\vec{\nu}\}+j\leq N}^{(l)} z_{\vec{\nu}} z^j c_{l;\vec{\nu},j} + O(|z|^{N+\epsilon}) \tag{5.16}$$

where $P_l(\ln z)$ is a polynomial of degree l in $\ln z$. Set

$$E_{+-}(z)^{-1} = \begin{pmatrix} e_{11}(z) & e_{12}(z) \\ e_{21}(z) & e_{22}(z) \end{pmatrix} = \frac{1}{d(z)} \begin{pmatrix} O(|z|^\epsilon) & -\overline{\beta_1} + O(|z|^\epsilon) \\ -\beta_1\gamma_0 + O(|z|^\epsilon) & \frac{1}{2}\gamma_0 \ln z + O(1) \end{pmatrix}$$

Decompose $-(1-D(z)W(z))T(E_{+-}(z)^{-1})S(1-W(z)D(z))R_0(z)$ as $T_0(z)+T_1(z)+T_{01}(z)$ with

$$\begin{aligned} T_0(z) &= -e_{11}(z)(1 - D(z)W(z))Q_0(1 - W(z)D(z))R_0(z) \\ T_1(z) &= -(1 - D(z)W(z)) \left(e_{22}(z)Q_1 + \frac{e_{21}(z)}{\gamma_0} < \cdot, \, -V\phi_0 > \phi_1 \right) (1 - W(z)D(z))R_0(z) \\ T_{01}(z) &= -e_{12}(z) < (1 - W(z)D(z))R_0(z)\cdot, \, -V\phi_1 > (1 - D(z)W(z))\phi_0 \end{aligned}$$

$T_0(z)$ and $T_{01}(z)$ admit asymptotic expansions of the form stated in (b). The details of calculation are omitted. For $T_1(z)$ which gives the leading term in (b), one sees that

$$\begin{aligned} T_1(z) &= -\frac{\ln z}{2|\beta_1|^2} < \cdot, \beta_1\phi_0 - F_0V\phi_1 > \phi_1 + O(1) \\ &= -\frac{\ln z}{2|\beta_1|^2} < \cdot, \phi_1 > \phi_1 + T_{1,1} + O(|z|^\epsilon) \end{aligned}$$

because $\phi_1 + F_0 V\phi_1 = \beta_1\phi_0$. Here $T_{1,1}$ is a rank one operator and the $O(|z|^\epsilon)$ term has an expansion of the form (5.10). According to (b) of Theorem 3.2, $\beta_1 = \frac{1}{2} < V\phi_1, -\ln r\,\phi_0 >$. It suffices to take

$$u_0 = \frac{\sqrt{2}\phi_1}{< V\phi_1, -\ln r\,\phi_0 >}.$$

(c). It follows from a similar calculation as in (b). Set as before $Q = Q_0 + Q_1$. Let $S = (S_0, S_1)$ and $T = (T_0, T_1)$ be the corresponding decomposition with T_0 of rank one and T_1 of rank μ. Set

$$E_{+-}(z)^{-1} = \left(\begin{array}{cc} E_{11}(z) & E_{12}(z) \\ E_{21}(z) & E_{22}(z) \end{array} \right).$$

Comparing it with (5.5), one sees that $E_{11}(z) = s(z)(1 + O(|z|^\epsilon))$ has an asymptotic expansion of the form (5.14):

$$E_{11}(z) \;=\; \sum_{l=0}^{N_0} \sum_{\{\vec{\nu}\}+j\leq N}^{(l)} z_{\vec{\nu}} z^j \sum_{m=l}^{\infty} (\ln z)^{-m-1} e_{\vec{\nu},j;l,m} + O(|z|^{N+\epsilon}) \tag{5.17}$$

with the same leading coefficients as $s(z)$. $E_{12}(z)$ and $E_{21}(z)$ have the similar asymptotic expansions of the form (5.17) with $(\ln z)^{-m-1}$ replaced by $(\ln z)^{-m}$. Finally, $E_{22}(z) = E_{+-}^{22}(z)^{-1}(1 + O(|z|^\epsilon))$ and the asymptotic expansion of $E_{+-}^{22}(z)^{-1}$ is studied in the proof of Theorem 4.4. Set

$$-E_+(z)E_{+-}(z)^{-1}E_-(z)R_0(z) = T_0(z) + T_1(z) + T_{01}(z)$$

with

$$\begin{aligned} T_0(z) &= -E_{11}(z)(1 - D(z)W(z))Q_0(1 - W(z)D(z))R_0(z) \\ T_1(z) &= -(1 - D(z)W(z))T_1E_{22}(z)Q_1S_1(1 - W(z)D(z))R_0(z) \\ T_{01}(z) &= -(1 - D(z)W(z))[T_0E_{12}(z)S_1 + T_1E_{21}(z)S_0](1 - W(z)D(z))R_0(z). \end{aligned}$$

$T_0(z)$, $T_1(z)$ and $T_{0,1}(z)$ admit asymptotic expansions stated in (c). □

If there is a 0-resonant state and if $\nu \geq 2$, making use of (5.5), one can show that the asymptotic expansion of $R(z)$ exists and is a mixture of (b) and (c). The result is more complicated and will not be stated in this work.

As a consequence of the existence of asymptotic expansion of $R(z)$ for z near 0 obtained in Sections 4 and 5 (Corollary 4.2, Theorem 4.6, Theorem 5.1), we conclude that for suitable $\rho_0 > 2$, the point spectrum, $\sigma_p(P)$, of $P = -\Delta_g + \frac{q(\theta)}{r^2} + V_0(x)$ is finite in $]-\infty, 0[$. In fact, a bound of the form $R(z) = O(\frac{1}{|z|})$ in suitable space for $|z| > 0$ small and $\Im z > 0$ implies that 0 is not an accumulating point of eigenvalues of P. As an example, $P = -\Delta + \frac{a}{r^2} + V_0(x)$ on $\mathbb{R}^n$, $n \geq 3$ and $-\frac{(n-2)^2}{4} \leq a < 0$, has only a finite number of negative eigenvalues for suitable $\rho_0 > 2$. In this example,

$$\int_{\mathbb{R}^n} (\frac{a}{|x|^2} + V_0(x))_-^{\frac{n}{2}}\, dx = +\infty,$$

so that the well-known Cwick-Lieb-Rosenbloum bound can not directly be applied to.

6. Long-time expansion of the Schrödinger group

Let $U(t) = e^{-itP},\ t \in \mathbb{R}$. Then,

$$U(t) = \frac{1}{2i\pi} \int_{\mathbb{R}} e^{-it\lambda} R(\lambda + i0)\, d\lambda, \quad t > 0. \tag{6.1}$$

The long-time expansion of $U(t)$ is determined by the low-energy behavior of the resolvent. To estimate the remainder, we need the smoothness of $R(\lambda + i0)$ for all $\lambda > 0$ and some high-energy estimates on the resolvent. Assume that g and V_0 are smooth and satisfy

$$|\partial_x^\alpha V_0(x)| \le C_\alpha \langle x \rangle^{-\rho_0 - |\alpha|}, \quad |\partial_x^\alpha (g - g_0)| \le C_\alpha \langle x \rangle^{-\rho_0 - |\alpha|} \tag{6.2}$$

for some $\rho_0 > 0$ and for all $\alpha \in \mathbb{N}^n$. Let $A = -i(r\partial_r + \partial_r r)/2$. Since $i[P_0, A] = 2P_0$ as forms defined on a core of $D(P_0)$, the Mourre's estimate is true for P at every positive energy: for any $E > 0$, there exists $c_0,\ \delta > 0$ and a compact operator K such that

$$i\chi_{E,\delta}(P)[P, A]\chi_{E,\delta}(P) \ge c_0 \chi_{E,\delta}(P) + K, \tag{6.3}$$

where $\chi_{E,\delta}(P)$ is the spectral projection of P onto the interval $[E - \delta, E + \delta]$.

Proposition 6.1. *Under the assumption (6.2), the eigenvalue of P is absent in $]0, +\infty[$. The boundary values of the resolvent $R(\lambda \pm i0)$ exist in $\mathcal{L}(L^{2,s}, L^{2,-s})$, $s > 1/2$, for all $\lambda \in \mathbb{R}_+^*$ and is C^k in $\lambda \in \mathbb{R}_+^*$ in $\mathcal{L}(L^{2,s}, L^{2,-s})$ if $s > k + 1/2$.*

Proof. The absence of positive eigenvalue is well-known for g a flat metric on $\mathbb{R}^n$ and $q = 0$ and is proved in [31] (Theorem 17.6) in geometric scattering for $q = 0$ by using ideas from [14, 21]. When $q \neq 0$, we can use the same ideas to prove the absence of positive eigenvalue of P. In fact, suppose $E_0 > 0$ and $u \in D(P) = H^1$ such that $Pu = E_0 u$. Let $\chi(x)$ be a smooth cut-off on $\mathbf{M}$ which equals 0 for $|x| < R_0/2$, and 1 for $|x| > R_0$, R_0 being fixed and large enough. Let $u_1 = \chi(x)u(x)$. Then

$$(\tilde{P} - E_0)u_1 = v$$

where $\tilde{P} = -\Delta_g + \frac{q(\theta)}{r^2}\chi'(x) + V_0(x)$, with $\chi'\chi = \chi$ and $\chi'(x) = 0$ for $|x| < R_0/4$, and $v \in L^2$ with support contained in $\{|x| \le R_0\}$. Then $\tilde{P}$ is of smooth coefficients and the ellipticity of P implies $u_1 \in H^2$. Since (6.3) holds for any $E \ge E_0$, we can use the Mourre's estimate (6.3) as exploited in [14, 31] to show that $e^{E^{\frac{1}{2}}|x|}u_1 \in L^2$ for any $E > 0$ and

$$\|e^{E^{\frac{1}{2}}\rho(|x|)}u_1\| \le C\|\langle x \rangle e^{E^{\frac{1}{2}}\rho(|x|)}v\|$$

uniformly in $E > 0$. Here $\rho(r)$ is some smooth function which equals r for r large enough ([14]). Since v is of compact support, the above estimate shows that $u(x) = u_1(x) = 0$ for $|x| \ge R_1$, $R_1 > R_0$ large enough. Since $\mathbf{M}$ is connected, the unique continuation theorem for P implies that $u = 0$ on $\mathbf{M}$. Therefore, $\sigma_p(P) \cap]0, +\infty[= \emptyset$.

Now, the existence of the boundary values $R(\lambda \pm i0)$, $\lambda > 0$, follows from the standard Mourre's method, and their smoothness is a consequence of multiple commutator method ([17]). □

The estimates of $R(\lambda \pm i0)$ for large λ require a non-trapping condition on the metric g. This kind of conditions is necessary. See [33, 34] for semi-classical Schrödinger operators on $\mathbb{R}^n$ where $\hbar = \lambda^{-1/2} \to 0$, and [9, 32] for Schrödinger operators on manifolds. In present case, due to the singularity of $\frac{q(\theta)}{|x|^2}$ at the origin is too strong to be treated as perturbation, we assume that there exists $c_0 > 0$, $M > 1$ such that

$$\{p(x, \xi), x \cdot \xi\} \ge c_0 p(x, \xi), \qquad \text{for all } (x, \xi) \in p^{-1}([M, \infty[). \tag{6.4}$$

Here $\{\cdot,\cdot\}$ is the Poisson bracket and $p = |\xi|_g^2 + \frac{q(\theta)}{r^2}$. Condition (6.4) is satisfied if one of the following conditions is verified: (a). g satisfies a virial condition: $\{g(x,\xi), x\cdot\xi\} - 2g(x,\xi) \geq 0$ for all $(x,\xi) \in p^{-1}(1)$ and q is arbitrary, where $g(x,\xi) = |\xi|_g^2$; (b). $\{g(x,\xi), x\cdot\xi\} \geq c_0 > 0$ for all $(x,\xi) \in p^{-1}(1)$ and $q(\theta) \geq 0$. Let us indicate that if $g(x,\xi)$ is of the form $g(x,\xi) = (1+a(x))|\xi|^2$ and $q(\theta) < 0$ where a is a smooth function with

$$a(x) = O(|x|^{-\rho_0}), \quad a(x) > -1, \quad 2(1+a(x)) - x\cdot\nabla a(x) \geq c_0 > 0$$

for all x, then g is a non-trapping metric. If $\partial_r a(0) = 0$, (6.4) is satisfied; while if $\partial_r a(0) > 0$, (6.4) is not satisfied.

Proposition 6.2. *Assume (6.4). Then there exists $c > 0$ and $\lambda_0 > 0$ such that*

$$i[P,A] \geq c\lambda, \qquad \text{on the range of } E_{](1-\epsilon)\lambda,(1+\epsilon)\lambda[}(P) \tag{6.5}$$

for $\lambda \geq \lambda_0$.

Proof. Let $\chi \in C_0^\infty(1-2\epsilon, 1+2\epsilon)$, $0 \leq \chi \leq 1$ with $\chi(s) = 1$ for $s \in [1-\epsilon, 1+\epsilon]$. Under the assumption (6.4), one can show that there exists a first order differential operator, B, with smooth coefficients of the order $O(|x|^{-\rho_0})$ such that

$$i\chi(\frac{P}{\lambda})[P,A]\chi(\frac{P}{\lambda}) \geq \chi(\frac{P}{\lambda})(c\lambda + B)\chi(\frac{P}{\lambda}) \tag{6.6}$$

for some $c > 0$, and all λ large enough. In sense of quadratic forms, one has from (3.3)

$$0 \leq \chi(\frac{P}{\lambda})(-\Delta_g)\chi(\frac{P}{\lambda}) \quad \leq \quad (1+\epsilon)\lambda + b + a\chi(\frac{P}{\lambda})(-\Delta_g)\chi(\frac{P}{\lambda}).$$

for some $a < 1$ and $b \in \mathbb{R}$. It follows that

$$\|(-\Delta_g)^{1/2}\chi(\frac{P}{\lambda})\| \leq \sqrt{\frac{(1+\epsilon)\lambda + b}{1-a}},$$

for $\lambda > 1$ large enough. Consequently, by the ellipticity of $-\Delta_g$, the term $\chi(\frac{P}{\lambda})B\chi(\frac{P}{\lambda})$ can bounded by $C\sqrt{\lambda}$. Proposition 6.2 is proved for $\lambda > 1$ large enough. □

Note that multiple commutators of P with A are well defined on the form domain of P. By Mourre's method with multiple commutators, Proposition 6.2 implies the following high energy resolvent estimates

$$\|\langle x\rangle^{-s}\frac{d^k}{d\lambda^k}R(\lambda+i0)\langle x\rangle^{-s}\| \leq C_{k,s}\lambda^{-(k+1)/2}, \qquad \lambda \geq \epsilon > 0, \quad s > k+1/2. \tag{6.7}$$

For $\lambda \in \mathbb{R}$ and $m \in \mathbb{Z}$, define $\Phi_{\lambda,m}(t)$ by:

$$\Phi_{\lambda,m}(t) = \frac{t^{\lambda+1}}{2\pi i}\int (s+i0)^\lambda \ln^m(s+i0)e^{-its}ds, \quad t > 0, \tag{6.8}$$

where the Fourier transform is taken in the sense of distributions. Then,

$$\Phi_{\lambda,m}(t) = \begin{cases} \frac{i}{\pi}\sum_{l=0}^m C_m^l \left(\frac{d^{m-l}}{d\lambda^{m-l}}\left(\sin(\pi\lambda)\, e^{i\pi\lambda/2}\Gamma(\lambda+1)\right)\right)\ln^l t, & \text{if } m \in \mathbb{N}, \\ \frac{i(-1)^m}{\pi}\sum_{l=m}^M C_{l+m-1}^{m-1}\frac{d^{l-m}}{d\lambda^{l-m}}(\sin(\pi\lambda)\, e^{i\pi\lambda/2}\Gamma(\lambda+1))\ln^{-l} t + O(\ln^{-M-1} t), & \text{if } m < 0, \end{cases} \tag{6.9}$$

for any $M > 1$. See Lemma A.2 in [37]. Set

$$U_c(t) = U(t) - \sum_{\lambda_j \in \sigma_p(P)} e^{-it\lambda_j} \Pi_{\lambda_j},$$

where Π_{λ_j} is the spectral projection onto the eigenspace associated to the eigenvalue λ_j of P. Remark that as a by-product of the existence of asymptotic expansion of $R(z)$ near $z = 0$, P has only a finite number of eigenvalues.

Theorem 6.3. *Assume (6.4) and that $0 \notin \sigma_\infty$.*

(a). Let $\mu = 0$ and $\rho_0 > 4N + 2$. One has the following asymptotic expansion in $\mathcal{L}(0, s; 0, -s)$, $s > 2N + 1$ for $t > 0$ large enough.

$$U_c(t) \quad = \quad \sum_{\{\vec{\nu}\}+j \le N}^{(1)} \Phi_{\{\vec{\nu}\}+j, m_{\vec{\nu}}}(t) t^{-\{\vec{\nu}\}-j-1} T_{\vec{\nu};j} + O(|t|^{-N-1-\epsilon}). \tag{6.10}$$

Here $T_{\vec{\nu};j}$ is the same as in Theorem 4.6 and $m_{\vec{\nu}}$ is the number of integer components in $\vec{\nu}$.

(b). Assume from now on $\mathbf{M} = \mathbf{M}_0$. Let $\mu \neq 0$. Assume $\mu_r = 0$ and $\rho_0 > \max\{4N - 6, 2N + 1\}$. One has the following asymptotic expansion in $\mathcal{L}(0, s; 0, -s)$ for $s > 2N + 1$.

$$\begin{aligned} U_c(t) = & \sum_{\{\vec{\nu}\}+j \le N-2}^{(1)} \Phi_{\{\vec{\nu}\}+j, m_{\vec{\nu}}}(t) t^{-\{\vec{\nu}\}-j-1} T_{e;\vec{\nu};j} \\ & + \sum_{j \ge -1; \{\vec{\nu}\}+j \le N-2}^{(1)} \Phi_{\{\vec{\nu}\}+j, m_{\vec{\nu}}}(t) t^{-\{\vec{\nu}\}-j-1} T_{\vec{\nu};j} + O(|t|^{-N+1-\epsilon}), \end{aligned} \tag{6.11}$$

for $t \to \infty$. Here $T_{e;\vec{\nu};j} = 0$ if $\{\vec{\nu}\} + j < 0$.

(c). Assume $\mu_r \neq 0$ and $\rho_0 > \max\{4N - 6, 2N + 2\}$. One has the following asymptotic expansion in $\mathcal{L}(0, s; 0, -s)$, $s > \max\{2N - 3, 2\}$.

$$\begin{aligned} U_c(t) = & \sum_{j=1}^{\kappa_0} \Phi_{-\varsigma_j, -\delta_{\varsigma_j}}(t) t^{\varsigma_j - 1} (\Pi_{r,j} + \Pi_0 V Q_e F_1 V \Pi_{r,j} + \Pi_{r,j} V Q_r F_1 V \Pi_0) \\ & + \sum_{\{\vec{\nu}\}+j \le N-2}^{(1)} \Phi_{\{\vec{\nu}\}+j, m_{\vec{\nu}}}(t) t^{-\{\vec{\nu}\}-j-1} T_{\vec{\nu};j} + \sum_{j \ge -1; \{\vec{\nu}\}+j \le N-2}^{(1)} \Phi_{\{\vec{\nu}\}+j, m_{\vec{\nu}}}(t) t^{-\{\vec{\nu}\}-j-1} T_{e;\vec{\nu};j} \\ & + \sum_{j=1}^{\kappa_0} \sum_{\vec{\nu}, \alpha, \beta, l, j}^{+, N-1} \Phi_{\lambda, m}(t) t^{-\lambda-1} T_{e,r;\vec{\nu},\alpha,\beta,l,j} + O(|t|^{-N+1-\epsilon}) \end{aligned} \tag{6.12}$$

with

$$T_{e,r;\vec{\nu},\alpha,\beta,l,j} = T_{r;\vec{\nu},\alpha,\beta,l,j} + T_{er;\vec{\nu},\alpha,\beta,l,j}$$

determined by the asymptotic expansion of $R(z)$ near 0, $\lambda = \{\vec{\nu}\} + |\beta| + l - \varsigma_j - \vec{\varsigma} \cdot (\alpha + \beta)$, and $m \in \mathbb{Z}$ the degree of the power of $\ln z$ in $z_{\vec{\nu}} z_{\varsigma_j}^{-1} (z_{\vec{\varsigma}})^{-\alpha-\beta}$

When $0 \in \sigma_\infty$, we only give the asymptotic expansions of the Schrödinger group in the cases (a) and (b) of Theorem 5.1.

Theorem 6.4. *Assume (6.4) and that $0 \in \sigma_\infty$. Let $\rho_0 > 4N + 2$ and $s > 2N + 1$.*

(a). Let $\mu = 0$. One has the following asymptotic expansion in $\mathcal{L}(0,s;0,-s)$,

$$U_c(t) = \sum_{\{\vec{\nu}\}+j\le N}^{(1)} \Phi_{\{\vec{\nu}\}+j,m_{\vec{\nu}}}(t) t^{-\{\vec{\nu}\}-j-1} T_{\vec{\nu};j} + \sum_{\ell=0}^{N_0} \sum_{\{\vec{\nu}\}+j\le N}^{(\ell)} \sum_{m=\ell}^{\infty} \Phi_{\{\vec{\nu}\}+j,m_{\vec{\nu}}-m-1}(t) t^{-\{\vec{\nu}\}-j-1} T_{s;\vec{\nu};j,m} + O(|t|^{-N-1-\epsilon}). \quad (6.13)$$

(b). Assume $\mu = 1$ and that there is 0-resonant state. One has the following asymptotic expansion in $\mathcal{L}(0,s;0,-s)$,

$$U_c(t) = \sum_{\{\vec{\nu}\}+j\le N}^{(1)} \Phi_{\{\vec{\nu}\}+j,m_{\vec{\nu}}}(t) t^{-\{\vec{\nu}\}-j-1} T_{\vec{\nu};j} - it^{-1} < \cdot, u_0 > u_0 + \sum_{\ell=0}^{N_0} \sum_{\{\vec{\nu}\}+j\le N}^{(\ell)} \sum_{m=0}^{\ell} \Phi_{\{\vec{\nu}\}+j,m_{\vec{\nu}}+m}(t) t^{-\{\vec{\nu}\}-j-1} T^{(\ell)}_{s;\vec{\nu};j,m} + O(|t|^{-N-1-\epsilon}). \quad (6.14)$$

Here $T_{\vec{\nu};j}$ is the same as in (b) of Theorem 5.1 and

$$T^{(\ell)}_{s;\vec{\nu};j,m} = T^{(\ell)}_{0;\vec{\nu};j,m} + T^{(\ell)}_{1;\vec{\nu};j,m} + T^{(\ell)}_{01;\vec{\nu};j,m}$$

with $T^{(\ell)}_{1;\vec{\nu};j,m}$ and $T^{(\ell)}_{01;\vec{\nu};j,m}$ the coefficients of $T^{(0)}_1(z)$ and $T^{(0)}_{01}(z)$ expanded as in (5.10).

Theorems 6.3 and 6.4 follow from (6.7), Theorems 4.6 and 5.1, and Lemma A.2 of [37].

Finally, we mention that microlocal propagation properties in finite time of the Schrödinegr group on manifold are studied in a recent work [15].

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