EMBEDDED EIGENVALUES AND RESONANCES OF SCHRÖDINGER OPERATORS WITH TWO CHANNELS

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Abstract. In this article, we give a necessary and sufficient condition in the perturbation regime on the existence of eigenvalues embedded between two thresholds. For an eigenvalue of the unperturbed operator embedded at a threshold, we prove that it can produce both discrete eigenvalues and resonances. The locations of the eigenvalues and resonances are given.

1. Introduction

In this work, we study the spectral properties of two-channel type Schrödinger operators of the form

$$P = \begin{pmatrix} -\Delta + E_1 & 0 \\ 0 & -\Delta + E_2 \end{pmatrix} + V(x), \qquad \text{in } L^2(\mathbb{R}^d; \mathbb{C}^2), \tag{1.1}$$

where $E_1 < E_2$, $x \in \mathbb{R}^d$, $V(x)$ is a 2×2 Hermitian matrix-valued function:

$$V(x) = \begin{pmatrix} V_1(x) & V_{12}(x) \\ V_{21}(x) & V_2(x) \end{pmatrix}, \quad V(x)^* = V(x).$$

Assume $V(x)$ is $-\Delta$-compact. The unperturbed operator

$$P_0 = \begin{pmatrix} -\Delta + E_1 + V_1(x) & 0 \\ 0 & -\Delta + E_2 + V_2(x) \end{pmatrix} \tag{1.2}$$

may have eigenvalues embedded in the interval $[E_1, E_2]$ which is contained in the continuous spectrum of P_0. We want to study what happens to these eigenvalues under the off-diagonal perturbation. The perturbation of embedded eigenvalues has its origin in quantum mechanics and quantum field theory. See [5, 15]. A well-known example in quantum mechanics is the operator of the helium atom given by

$$H_\beta = H_0 + \beta W \quad \text{in} \quad L^2(\mathbb{R}^6), \tag{1.3}$$

where $W = \frac{1}{|x_1 - x_2|}$, $x_1, x_2 \in \mathbb{R}^3$, and

$$H_0 = -\Delta_{x_1} - \frac{2}{|x_1|} - \Delta_{x_2} - \frac{2}{|x_2|}. \tag{1.4}$$

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The spectrum of H_0 is composed of the eigenvalues $\{-\frac{1}{n^2} - \frac{1}{m^2}; n, m \in \mathbb{N}^*\}$ and the continuous spectrum

$$\sigma_{ess}(H_0) = [-1, \infty[.$$

The eigenvalues $-\frac{1}{n^2} - \frac{1}{m^2}$ with $n, m \geq 2$ are embedded in the continuum. It is believed that these embedded eigenvalues produce resonances of the helium, which are relevant to the allure of scattering cross-section near the thresholds $\{-\frac{1}{n^2}\}$. See [15]. In [17], it is proved that H_β has no eigenvalues in $]-\frac{1}{2}-\epsilon_0, -\frac{1}{2}+\epsilon_0[$ in some symmetry reduced subspace, if the integral

$$I = \int_{\mathbb{R}^6} \frac{1}{|x_1 - x_2|} \overline{\psi(x_1)}\, \overline{\psi(x_2)} \phi(x_1)\eta(x_2) dx_1 dx_2 \neq 0 \tag{1.5}$$

where ϕ and ψ are eigenfunctions of $-\Delta_y - \frac{2}{|y|}$ associated with eigenvalues -1 and $-\frac{1}{4}$, respectively, η a confluent hypergeometric function. By method of complex dilation, one sees that the eigenvalue of H_0 at $-\frac{1}{2}$ dissolves into a resonance of H_β for $\beta > 0$ small. The resonances of the helium atom near thresholds are not yet fully understood. Note that the spectral analysis of H_β between the first two thresholds -1 and $-\frac{1}{4}$ can be reduced to a non-linear spectral problem with leading term given by

$$\begin{pmatrix} -\Delta - 1 - \frac{2}{|x|} & \beta C(x) \\ \beta C(x)^* & (-\Delta - \frac{1}{4} - \frac{2}{|x|}) I_4 \end{pmatrix}, \quad x \in \mathbb{R}^3,$$

where $C(x) = O(\frac{1}{|x|^2})$ is a 1×4 matrix arising from interaction between scattering channels associated with energies -1 and $-\frac{1}{4}$. Therefore, two-channel type Schrödinger operators may be considered as a simplified model of N-body Schrödinger operators.

To simplify notation, we take $E_1 = 0$ and $E_2 = E_0 > 0$. Consider the operator

$$P = P_0 + \beta \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix}, \quad \text{in } L^2(\mathbb{R}^d; \mathbb{C}^2),$$

where the unperturbed operator is

$$P_0 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

with $P_1 = -\Delta + V_1(x)$ and $P_2 = -\Delta + E_0 + V_2(x)$ and the off-diagonal part is considered as perturbation. β is a real small parameter in Sections 3 and 4, and is fixed to be 1 in Sections 2 and 5. We assume that V is a Hermitian matrix valued function satisfying

$$(x \cdot \nabla)^k V_j \textit{ and } (x \cdot \nabla)^k V_{12} \textit{ are } -\Delta\textit{-compact on } \mathbb{R}^d \textit{ for } k = 0, 1 \textit{ and } j = 1, 2 \tag{1.6}$$

and for x large enough,

$$|V_j(x)| \leq C\langle x\rangle^{-\rho_j}, \quad |V_{12}(x)| \leq C\langle x\rangle^{-\rho_0} \tag{1.7}$$

for some $\rho_j > 0$, $j = 0, 1, 2$, where $\langle x\rangle = (1 + |x|^2)^{1/2}$. The spectral property of the unperturbed operator P_0 is similar to that of H_0, while the off-diagonal part in V can be regarded as interaction between channels. In particular, the discrete spectrum of $-\Delta + E_0 + V_2$ in $[0, E_0[$ becomes eigenvalues of P_0 embedded in its continuous spectrum. We shall show that under some conditions, the eigenvalues of P_0 embedded in the interval $]0, E_0[$ dissolve into resonances under the off-diagonal perturbation, while the eigenvalue zero of P_0 can partly be shifted to the left to become discrete spectrum and partly to the right to induce resonances of $P(\beta)$.

Let us describe briefly the results of this work on weak perturbation. Denote $R_j(z) = (P_j - z)^{-1}$, $j = 0, 1, 2$. For $a \in \mathbb{C}$, $r > 0$, set $D(a, r) = \{z \in \mathbb{C}; |z - a| < r\}$. The result for dilation-analytic potential is easy to state (and to prove). If $e \in]0, E_0[$ is an eigenvalue of P_2 with multiplicity m and φ_k, $1 \le k \le m$, the orthonormal eigenfunctions of P_2 associated with e. Set

$$A_1(z) = (< R_1(z)V_{12}\varphi_k, V_{12}\varphi_l >)_{1 \le k,l \le m} \tag{1.8}$$

for $\Im z > 0$. Under the condition (1.6), assume in addition that V is dilation analytic. The function $z \to \chi_1(\lambda, z) = \det(A_1(z) - \lambda)$ extends holomorphically in z from the upper half complex plane into a small complex neighborhood of e. Let $\{\lambda_1, \cdots, \lambda_k\}$ be the set of zeros of the function $\lambda \to \chi(\lambda, e)$, and ν_l the multiplicity of λ_l, $\sum_{l=1}^k \nu_l = m$. We show that for some $\epsilon_0 > 0$ and $\beta_0 > 0$ small enough, there exists $C > 0$ such that for $\beta \in]0, \beta_0]$, the eigenvalues and resonances of $P(\beta)$ in $D(e, \epsilon_0)$ are located in

$$\cup_{l=1}^k D(e - \beta^2\lambda_l, C|\beta|^{2+2/\nu_l}),$$

and the total multiplicity of eigenvalues and resonances of $P(\beta)$ in $D(e-\beta^2\lambda_l, C|\beta|^{2+2/\nu_l})$ is equal to ν_l. See Theorem 4.3. In particular, if for some l

$$\lambda_l \text{ is not real,} \tag{1.9}$$

the eigenvalues of $P(\beta)$ are absent in a small disk $D(e - \beta^2\lambda_l, \epsilon_0\beta^2)$, $\epsilon_0 > 0$ small, and there are ν_l resonances, $z_k(\beta)$, of $P(\beta)$ given by

$$z_k(\beta) = e - \beta^2\lambda_l + O(|\beta|^{2+2/\nu_l}). \tag{1.10}$$

Note that the imaginary part of the limiting value, $A_1(e + i0)$, of the matrix $A_1(z)$ at e exists and is positive, and that the condition (1.9) implies $\Im\lambda_l > 0$. If the matrix $A_1(e+i0)$ has no real eigenvalues, $P(\beta)$ has no eigenvalues in $[e-\epsilon_0, e+\epsilon_0]$ and its resonances in $D(e, \epsilon_0)$ are all given by (1.10). Note that the condition (1.9) can be compared with (1.5) and the Fermi golden rule assumption used in [6] in the study of Pauli-Fierz operators in quantum field theory.

When the potential V is not dilation analytic, the problem is more subtle and the condition $\rho_0 > \frac{1}{2}$ is needed. A detailed analysis enables us to give in Section 3 a necessary and sufficient condition on the existence or non-existence of embedded eigenvalues of the perturbed operator $P(\beta)$ (Theorem 3.6), which implies the absence of embedded eigenvalues of $P(\beta)$ in $[e - \epsilon_0, e + \epsilon_0]$ if the matrix $A_1(e + i0)$ has no real eigenvalues. To study the perturbation of the threshold eigenvalue zero of P_0, we need $\rho_0 > 1$ and $\rho_1 > 2$. If zero is an eigenvalue of P_2, we assume that $\rho_1 > 2$ and that zero is a regular point of P_1 in the sense of Jensen-Kato [13]; and if zero is an eigenvalue of P_1, we need $\rho_1 > 3$ and assume that zero is in resolvent of P_2 and is not a resonance of P_1 in the sense of [13]. The spectral properties of $P(\beta)$ near 0 are then determined by a Hermitian matrix A_0 defined similarly as above with $z = 0$. We prove that the positive eigenvalues of A_0 give rise to discrete eigenvalues of $P(\beta)$, while, under some additional conditions, its negative ones generate resonances. See Theorems 3.7, 3.8 and 4.5. As an application, we show that if $P_0 \geq 0$ and if zero is an eigenvalue of P_1 or P_2 with multiplicity m, $P(\beta)$ has m strictly negative eigenvalues near zero, so long as V_{12} is non zero and $\beta \neq 0$ small enough. See the example given at the end of Section 3.

Perturbation of embedded eigenvalues has a long history. Here, we only mention that this subject is studied in abstract setting in [10, 11], for second order differential operators in [2, 3, 20] and for Hamiltonians of quantum field theory in [5, 6]. Two-channel type operators present some particularities and are studied more recently in [14] in abstract setting. The main concern of [14] is perturbation of eigenvalues embedded at the thresholds. Under the positive definiteness assumption of some operator, they proved for the threshold 0 that there are no eigenvalues of the perturbed operator in an interval like $]-C\beta^2, \delta_0[$. They also proved the absence of eigenvalues outside the thresholds if some reduced operator is positive definite. Clearly, the positive definiteness of $\Im A_1(e+i0)$ implies that $A_1(e+i0)$ has no real eigenvalues. Resonances are not studied in [14]. Perturbation of two-cluster threshold resonance of N-body Schrödinger operators is studied in [24].

The plan of this work is as follows. In Section 2, we study the decay of eigenfunctions associated with eigenvalues in $]0, E_0[$. Sections 3 and 4 are devoted to weak perturbation. In Section 3, we study the perturbation of embedded eigenvalues in $[0, E_0[$ of the unperturbed operator and give conditions on the existence and non-existence of embedded eigenvalues of the perturbed operator. The resonances are studied in Section 4. To study the resonances generated by eigenvalue zero of P_0, we have to study meromorphic extension of dilated operators in weighted spaces. In Section 5, we prove that if the off-diagonal part of V decays more slowly than its diagonal part, then E_0 is not an accumulating point of eigenvalues of $P = P(1)$.

Notation. For $s \in \mathbb{R}$ and $k \in \mathbb{Z}$, we denote by $L^{2,s}$ and $H^{k,s}$ the weighted-L^2 and weighted-Sobolev spaces $L^2(\langle x\rangle^{2s}dx)$ and $H^k(\langle x\rangle^{2s}dx)$, respectively, and by $\mathcal{L}(s; s')$ and $\mathcal{L}(k, s; k', s')$ the space of bounded operators from $L^{2,s}$ to $L^{2,s'}$ and from $H^{k,s}$ to $H^{k',s'}$, respectively. By the multiplicity of an eigenvalue, we mean in this work its algebraic multiplicity. Resonance at the threshold is taken in the sense of Jensen-Kato [13] which is not a pole, but an essential singularity of the resolvent.

2. Decay of eigenfunctions

In this section, the coupling constant β plays no role and is fixed to be 1. Let $P = P(1)$. Under the conditions (1.6) and (1.7), by methods of spectral analysis for N-body Schrödinger operators ([3, 7]), one can show that there is no embedded eigenvalue of P in $]E_0, \infty[$. However, the same arguments can not be used for $E \in]0, E_0[$, due to the particular structure of two-channel operators.

Lemma 2.1. *Under the condition (1.6), the point spectrum $\sigma_{pp}(P)$ of P is discrete in $]0, E_0[$.*

Proof. To adapt the well-known Mourre's method to two-channel type operators under the condition (1.6), we only work with the first equation. For $A_0 = (x \cdot \nabla + \nabla \cdot x)/(2i)$, the following Mourre's estimate holds for any $\lambda > 0$:

$$iE_\lambda(P_1)[P_1, A_0]E_\lambda(P_1) \geq E_\lambda(P_1)(\lambda + K)E_\lambda(P_1) \tag{2.1}$$

where $E_\lambda(P_1)$ denotes the spectral projection of P_1 onto the interval $]\lambda - \delta, \lambda + \delta[$ with $0 < \delta < \lambda/2$, and K is a compact operator. If $\mu \in]0, E_0[$ is an accumulating point of $\sigma_{pp}(P) \cap]0, E_0[$, one can find a sequence of normalized eigenfunctions $u_j = (v_j, w_j)$ with $Pu_j = \lambda_j u_j$ with $\lambda_j \to \mu$ and u_j converges weakly to zero. u_j is bounded in $H^2(\mathbb{R}^d; \mathbb{C}^2)$. We claim that

$$\liminf_j \|E_\mu(P_1)v_j\| \geq c_0 > 0. \tag{2.2}$$

In fact, by the first equation of the system

$$\begin{cases} (P_1 - \lambda_j)v_j + V_{12}w_j &= 0 \\ (P_2 - \lambda_j)w_j + V_{21}v_j &= 0 \end{cases} \tag{2.3}$$

one deduces that

$$\|(1 - E_\mu(P_1))v_j\| = \|(P_1 - \lambda_j)^{-1}(1 - E_\mu(P_1))V_{12}w_j\| \to 0, \quad j \to \infty.$$

So, if $\liminf_j \|E_\mu(P_1))v_j\| \to 0$, we can extract a subsequence of $\{v_j\}$, still denoted by $\{v_j\}$, such that $\|v_j\| \to 0$. Then, $\|w_j\| \to 1$ and $w_j \rightharpoonup 0$. The second equation of (2.3) shows that for $\epsilon_0 > 0$ such that $-E_0 + \mu + \epsilon_0 < 0$, there is some $j_0 \in \mathbb{N}$ such that

$$< (-\Delta + V_2)w_j, w_j > \leq (-E_0 + \mu + \epsilon_0)\|w_j\|^2,$$

for all $j \geq j_0$. This is impossible because $-\Delta + V_2$ can only have a finite number of eigenvalues in $] - \infty, -E_0 + \mu + \epsilon_0]$. We can now apply standard argument to obtain a contradiction. On one hand, one has

$$< iE_\mu(P_1)[P_1, A_0]E_\mu(P_1)v_j, v_j > \;=\; 2\Im < A_0E_\mu(P_1)v_j, E_\mu(P_1)V_{12}w_j > \to 0,$$

since $\|A_0E_\mu(P_1)v_j\|$ is bounded and $\|E_\mu(P_1)V_{12}w_j\| \to 0$. On the other hand, (2.1) for $\lambda = \mu$ implies that

$$\lim_{j\to\infty} < iE_\mu(P_1)[P_1, A_0]E_\mu(P_1)v_j, v_j > \geq \mu \liminf \|E_\mu(P_1)v_j\|^2 \geq \mu c_0^2 > 0.$$

This contradiction proves that μ can not be an accumulating point of $\sigma_{pp}(P) \cap]0, E_0[$. ■

To study the decay of eigenfunctions associated with eigenvalues in $]0, E_0[$, we need an additional condition on V_1

$$(x \cdot \nabla_x)^j V_1 \text{ is } -\Delta\text{-bounded for some } m \geq 2 \text{ and for } 2 \leq j \leq m. \tag{2.4}$$

The idea used in the proof of the part (c) of the following result is due to Nicolas Lerner.

Theorem 2.2. *Assume (1.6), (1.7) with $\rho_0 > \frac{1}{2}$. Let $\lambda \in]0, E_0[$ be an eigenvalue of P and $u = (u_1, u_2) \in H^2(\mathbb{R}^d; \mathbb{C}^2)$ an associated eigenfunction.*

(a). Assume (2.4) for some $m \geq 2$. Then $u \in L^{2,s}$ for any $s < m - 2 + \rho_0$.

(b). Assume (2.4) for all m and $\rho_0 \geq 1$. Then, for any $0 < \gamma < \sqrt{E_0 - \lambda}$, $e^{\gamma|x|}u \in L^2(\mathbb{R}^d; \mathbb{C}^2)$.

(c). Assume (2.4) for all m, $\rho_1 \geq 1$ and that $V_{12}(x) = \overline{V_{21}(x)}$ is of compact support. Then u_1 is of compact support.

Proof. Under the conditions of (a), it is well-known that the positive eigenvalues of P_1 are absent and the limiting absorption principle holds at any positive energy $E > 0$. Let $u = (u_1, u_2)$. Then, $(P - \lambda)u = 0$ can be written as

$$\left\{ \begin{array}{rcl} (P_1 - \lambda)u_1 + V_{12}u_2 & = & 0 \\ (P_2 - \lambda)u_2 + V_{21}u_1 & = & 0 \end{array} \right. \tag{2.5}$$

Let $R_j(z) = (P_j - z)^{-1}$, $z \notin \sigma(P_j)$. Since $\lambda > 0$ is not an eigenvalue of P_1 and $u_1 \in L^2$, one has

$$\lim_{\epsilon \to 0} \epsilon R_1(\lambda \pm i\epsilon)u_1 = 0.$$

It follows that

$$\left\{ \begin{array}{l} u_1 = -R_1(\lambda \pm i0)(V_{12}u_2) \\ (P_2 - \lambda)u_2 - K(\lambda)u_2 = 0 \end{array} \right.$$

where

$$K(\lambda) = V_{21}R_1(\lambda + i0)V_{12} \tag{2.6}$$

is a compact operator on L^2, since $\rho_0 > 1/2$. In particular, we do not need to distinguish the two boundary values $R_1(\lambda \pm i0)(V_{12}u_2)$ and can simply write $R_1(\lambda)(V_{12}u_2) = R_1(\lambda \pm i0)(V_{12}u_2)$. We need the following result which is our basic tool to prove the decay of eigenfunctions. See also the proof of Theorem 3.6.

Lemma 2.3. *Under the conditions of Theorem 2.2 (a), one has*

$$\|\langle x\rangle^{s-1}R_1(\lambda)(V_{12}u_2)\| \leq C_s\|\langle x\rangle^s V_{12}u_2\| \tag{2.7}$$

for any $1/2 < s < m - 1$ such that $\langle x\rangle^s V_{12}u_2 \in L^2$.

Proof. Recall the following microlocal resolvent estimates proved in [12] for smooth symbol-like potentials, and in [21] for potentials satisfying (1.6) and (2.4):

$$\|\langle x\rangle^{s-1}b_{\mp}(x, D)R_1(\lambda \pm i0)\langle x\rangle^{-s}\| \leq C, \quad \frac{1}{2} < s < m - 1, \tag{2.8}$$

where $b_\pm(x, D)$ are bounded pseudo-differential operators with symbols $b_\pm$ supported in $\{(x, \xi); \pm x \cdot \xi > -\mu_0|x|\}$ for some $\mu_0 > 0$ depending only on $\lambda > 0$. Let $\chi \in C_0^\infty(\mathbb{R})$ which is equal to 1 in a neighborhood of λ. Then,

$$\|\langle x\rangle^s(1 - \chi(P_1))R_1(\lambda)(V_{12}u_2)\| \leq C_s\|\langle x\rangle^s V_{12}u_2\|$$

and $\chi(P_1) = \chi(-\Delta) + R$ with $R \in \mathcal{L}(s, s + \rho_1)$ for any s, $|s| \leq m$. See Lemma A.1 in [21]. On the support of $\chi(|\xi|^2)$, we can construct a partition of unity

$$b_+(x, \xi) + b_-(x, \xi) = 1$$

with $b_\pm$ bounded symbols satisfying the support properties needed in (2.8). Applying (2.8) to

$$\chi(-\Delta)R_1(\lambda)(V_{12}u_2) = (b_-(x,D)\chi(-\Delta)R_1(\lambda+i0) + b_+(x,D)\chi(-\Delta)R_1(\lambda-i0))V_{12}u_2,$$

we obtain

$$\|\langle x\rangle^{s-1} f(-\Delta)R_1(\lambda)(V_{12}u_2)\| \le C_s\|\langle x\rangle^s V_{12}u_2\| \tag{2.9}$$

for any $1/2 < s < m-1$. This proves

$$\|\langle x\rangle^{s-1}R_1(\lambda)(V_{12}u_2)\| \le C_s\{\|\langle x\rangle^s V_{12}u_2\| + \|\langle x\rangle^{s-1-\rho_1}R_1(\lambda)(V_{12}u_2)\|\} \tag{2.10}$$

for any $1/2 < s < m-1$. For each fixed $s > \frac{1}{2}$, repeatedly applying (2.10) k times with k the smallest integer such that $s-1-k\rho_1 < -\frac{1}{2}$, we obtain (2.7). ■

Let us continue the proof of (a). For a Lipschitz function, F, on $\mathbb{R}^d$, one has

$$\begin{aligned}
&\|\nabla(e^F\phi)\|^2 + \int_{\mathbb{R}^d}(E_0+V_2-\lambda-|\nabla F|^2)|e^F\phi|^2dx \\
&= \Re\left\{< e^F[(P_2-\lambda-K(\lambda))\phi, e^F\phi> + < e^{2F}(K(\lambda)\phi),\phi>\right\},
\end{aligned} \tag{2.11}$$

for $\phi \in C_0^\infty$. By an argument of density, we deduce for bounded Lipschitz function F and for $u_2 \in H^1(\mathbb{R}^d)$ such that $(P_2-\lambda)u_2 = K(\lambda)u_2$ that

$$\|\nabla(e^F u_2)\|^2 + \int_{\mathbb{R}^d}(E_0+V_2-\lambda-|\nabla F|^2)|e^F u_2|^2dx = \Re\left\{\int_{\mathbb{R}^d} e^{2F}(K(\lambda)u_2)\overline{u_2}\,dx\right\}. \tag{2.12}$$

Take $F = F_{s,\sigma}$ as

$$F = \ln\left(\frac{1+\langle x\rangle}{1+\langle x\rangle/\sigma}\right)^s, \quad s>0, \sigma>1.$$

We can check that $|\nabla F(x)| \le s/(1+\langle x\rangle)$. So for any $\epsilon > 0$, one has for $R \ge \frac{\epsilon}{s}$

$$|\nabla F(x)| \le \epsilon, \quad |x| > R.$$

Set $B_R = \{x\in\mathbb{R}^d; |x|<R\}$. Let χ_1 be a smooth cut-off function with $\chi_1(x) = 1$ on B_R; 0 outside B_{2R} and $0 \le \chi_1(x) \le 1$. Since V_2 is $-\Delta$-compact, one has for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\begin{aligned}
\int_{B_R}|V_2||e^F u_2|^2dx &\le \epsilon\int_{\mathbb{R}^d}|\nabla(\chi_1 e^F u_2)|^2dx + C_\epsilon\int_{B_{2R}}|e^F u_2|^2dx \\
&\le \epsilon\int_{\mathbb{R}^d}|\nabla(e^F u_2)|^2dx + C(\epsilon,R)\int_{B_{2R}}|e^F u_2|^2dx.
\end{aligned}$$

Taking $0<\epsilon<1/2$ and $R>1$ large enough such that

$$\sup_{|x|\ge R}|V_2(x)| \le \epsilon^2,$$

il follows from (2.12) that

$$\begin{aligned}
&\frac{1}{2}\|\nabla(e^F u_2)\|^2 + (E_0-\lambda-2\epsilon^2)\int_{\mathbb{R}^d\setminus B_R}|e^F u_2|^2dx \\
&\le (s^2+C(\epsilon,R))\int_{B_{2R}}|e^F u_2|^2dx + \Re\left\{\int_{\mathbb{R}^d} e^{2F}(K(\lambda)u_2)\overline{u_2}\,dx\right\}.
\end{aligned} \tag{2.13}$$

Since u is normalized and $\lambda < E_0$, taking $\epsilon > 0$ suitably small, we obtain an Agmon-type energy estimate

$$\|\nabla(e^F u_2)\|^2 + \|e^F u_2\|^2 \le C' + \Re\left\{\int_{\mathbb{R}^d} e^{2F}(K(\lambda)u_2)\overline{u_2}\, dx\right\}. \tag{2.14}$$

Here C' is independent of σ. An direct calculation shows that $\sup e^{2F}$ is bounded uniformly in σ.

Since $\rho_0 > 1/2$, by (2.7), $K(\lambda)u_2 \in L^{2,s_0}$ for $s_0 = 2\rho_0 - 1 > 0$. We can apply (2.14) with $F = F_{s,\sigma}$, $s = s_1 = (2\rho_0 - 1)/2$, $\sigma > 1$. Taking $\sigma \to \infty$, we obtain

$$u_2 \in L^{2,s_1}.$$

With this new property on u_2, applying (2.7) with $s = s_1 + \rho_0$ (if $s_1 + \rho_0 < m - 1$), one sees that $K(\lambda)u_2 \in L^{2,3s_1}$. Then, we can apply (2.14) with $s = 2s_1$ which gives

$$u_2 \in L^{2,2s_1}.$$

We can repeat these arguments and each time, we gain an additional decay of the order s_1. This proves by induction that

$$u_2 \in L^{2,s}, \quad \forall s < m - 1. \tag{2.15}$$

Equation (2.7) shows that $u_1 \in L^{2,s-1+\rho_0}$. This proves (a).

To prove (b), we take

$$F = \ln\left(1 + \gamma\langle x\rangle/\sigma\right)^\sigma, \quad 0 < \gamma < \sqrt{E_0 - \lambda}, \sigma > 1.$$

Making use of the decay properties of u obtained above, similarly to (2.14), one can show

$$\|\nabla(e^F u_2)\|^2 + \|e^F u_2\|^2 \le C(\lambda,\gamma) + \Re\left\{\int_{\mathbb{R}^d} e^{2F}(K(\lambda)u_2)\overline{u_2}\, dx\right\}, \tag{2.16}$$

uniformly in $\sigma > 1$.

We need the following estimate proved in [3]. Let $\xi = (1 + \gamma\langle x\rangle/\sigma)^\sigma$. For any $\nu > 0$, there exists $k > 0$ and K a compact operator such that

$$\|\xi\phi\| \le k\|\langle x\rangle\xi(P_1 - \nu)\phi\| + \|K(\xi\phi)\|, \phi \in C_0^\infty, \tag{2.17}$$

uniformly in $\sigma > 1$. Since $\lambda > 0$ is not an eigenvalue of P_1, one can prove that for some $k_1 > 0$,

$$\|\xi\phi\| \le k_1\|\langle x\rangle\xi(P_1 - \lambda)\phi\|, \quad \phi \in C_0^\infty, \tag{2.18}$$

uniformly in $\sigma > 1$. By an argument of density, (2.18) remains true if $\xi\phi$ and $\langle x\rangle\xi(P_1 - \lambda)\phi$ are both in L^2. By (a), when (2.4) is satisfied for all $m \ge 2$, $u \in L^{2,s}$ for any $s > 0$. We can apply (2.18) and obtain

$$\|e^F R_1(\lambda)(V_{12}u_2)\| \le k_1\|\langle x\rangle e^F V_{12}u_2\| \tag{2.19}$$

uniformly in $\sigma > 1$. Evaluating the integral according to $|x| \le R$ and $|x| > R$, $R > 1$, one obtains from (2.19) that

$$\begin{aligned}\|e^F K(\lambda)u_2\| &\le C\|e^F\langle x\rangle^{-\rho_0}R_1(\lambda)(V_{12}u_2)\| \\ &\le C_R + CR^{-\rho_0}\|e^F R_1(\lambda)(V_{12}u_2)\| \\ &\le C_R + CR^{-\rho_0}\|e^F\langle x\rangle V_{12}u_2\|.\end{aligned}$$

When $\rho_0 \ge 1$, we can take $R >> 1$ so that (2.16) gives

$$\|\nabla(e^F u_2)\|^2 + \|e^F u_2\|^2 \le C'(\lambda,\gamma) \tag{2.20}$$

uniformly in $\sigma > 1$. Taking $\sigma \to \infty$, we obtain the exponential decay of u_2. The same estimate for u_1 can be derived from (2.19) .

(c). The eigenvalue problem $Pu = \lambda u$ with $\lambda \in]0, E_0[$ can be written as

$$\left\{\begin{array}{rcl}(-\Delta + V_1 - E_1)u_1 + V_{12}u_2 &=& 0\\ (-\Delta + V_2 - E_2)u_2 + V_{21}u_1 &=& 0\end{array}\right. \tag{2.21}$$

where $u = (u_1, u_2)$ and $E_1 = \lambda > 0$, $E_2 = \lambda - E_0 < 0$. We need the following estimate (Proposition 14.7.1, [9]):

$$\||x|^{2+\sigma}(\Delta + E_1)w\|^2_{L^2(dx)} \ge 4(\sigma+1)E_1\||x|^{\sigma+1}w\|^2_{L^2(dx)}, \quad w \in C_0^\infty(\mathbb{R}^d). \tag{2.22}$$

for any $\sigma > -1$. Let $R_0 > 0$ be such that supp $V_{12} \subset \{x; |x| < R_0\}$. Choose $\chi \in C^\infty$ with supp $\chi \subset \{|x| > R_1\}$, $R_1 > R_0$ and $\chi(x) = 1$ for $|x| \ge R_2 > R_1$. By Theorem 2.2 (b), $u \in H^2 \cap L^{2,s}$ for any $s > 0$. By an argument of limit, we can apply (2.22) to χu_1 for any $\sigma > 0$. Since $\chi V_{12} = 0$, (2.21) implies that

$$(\Delta + E_1)(\chi u_1) = V_1\chi u_1 + [\Delta, \chi]u_1.$$

Applying (2.22), we obtain

$$\sigma'\||x|^{\sigma+1}\chi u_1\|^2 \le \||x|^{\sigma+2}[\Delta,\chi]u_1\|^2 + \||x|^{\sigma+2}\chi V_1 u_1\|^2 \tag{2.23}$$

where $\sigma' = 4(\sigma+1)E_1$. Since $V_1(x) = O(|x|^{-1})$ on supp χ,

$$\||x|^{\sigma+2}\chi V_1 u_1\|^2 \le C\||x|^{\sigma+1}\chi u_1\|^2$$

with $C > 0$ independent of σ. This gives

$$(\sigma' - C)\||x|^{\sigma+1}\chi u_1\|^2 \le \||x|^{\sigma+2}[\Delta,\chi]u_1\|^2.$$

Take σ large enough so that $\sigma' > C$. Let $R_3 > R_2$. One has

$$(\sigma' - C)\int_{|x|>R_3} ||x|^{\sigma+1-\frac{d}{2}}u_1|^2 dx \le \int_{R_1\le|x|\le R_2} ||x|^{\sigma+2-\frac{d}{2}}[\Delta,\chi]u_1|^2 dx \tag{2.24}$$

or

$$(\sigma' - C)R_3^{2(\sigma+1)}\int_{|x|>R_3} |u_1|^2 dx \le R_2^{2(\sigma+2)}\int_{R_1\le|x|\le R_2} |[\Delta,\chi]u_1|^2 dx \tag{2.25}$$

which implies

$$(\sigma' - C)\left(\frac{R_3}{R_2}\right)^{2(\sigma+1)}\int_{|x|>R_3} |u_1|^2 dx \le R_2^2\int_{R_1\le|x|\le R_2} |[\Delta,\chi]u_1|^2 dx. \tag{2.26}$$

Note that the right hand side of the above estimate is independent of σ. Let $R_3 > R_2 > R_1 > R_0$ be fixed. Taking $\sigma \to \infty$, we obtain $\int_{|x|>R_3} |u_1|^2 dx = 0$, which implies that supp $u_1 \subset \{|x| \le R_3\}$ for any $R_3 > R_0$. ■

It is well-known ([7]) that if an eigenfunction, v, of an N-body Schrödinger operator decays super-exponentially: $\forall \gamma > 0$, $e^{\gamma|x|}v \in L^2$, then $v = 0$. Theorem 2.2 (c) shows that the first component of an eigenfunction of two-channel type Schrödinger operators may decay super-exponentially. Since the reduced scalar equation satisfied by the first component, u_1, of an eigenfunction u of two-channel operators is not local, we can not conclude $u_1 = 0$ even if it is of compact support.

3. Weak perturbation of embedded eigenvalues

Throughout this Section, we assume the conditions (1.6), (1.7) with $\rho_0 > \frac{1}{2}$ and (2.4) with $m = 2$. We want to study the embedded eigenvalues of P_0 under weak off-diagonal perturbation. Consider the operator $P(\beta)$ in the form

$$P(\beta) = P_0 + \beta \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix}$$

where $\beta > 0$ is a small parameter. For $\Im z > 0$, the equation $(P(\beta) - z)u = v$ is equivalent with

$$\begin{cases} (P_1 - z)u_1 + \beta V_{12} u_2 &= v_1 \\ (P_2 - z)u_2 + \beta V_{21} u_1 &= v_2 \end{cases}$$

The solution of the above system can be written as

$$\begin{cases} u_1 &= Q_1(z,\beta)v_1 - \beta Q_1(z,\beta)V_{12}R_2(z)v_2 \\ u_2 &= -\beta R_2(z)V_{21}Q_1(z,\beta)v_1 + R_2(z)(1 + \beta^2 V_{21}Q_1(z,\beta)V_{12}R_2(z))v_2 \end{cases}$$

where $R_j(z) = (P_j - z)^{-1}$, $j = 1, 2$, and

$$Q_1(z,\beta) = R_1(z)(1 - \beta^2 K_2(z)R_1(z))^{-1} \tag{3.1}$$

$$K_2(z) = V_{12}R_2(z)V_{21}. \tag{3.2}$$

One obtains

$$R(z,\beta) = \begin{pmatrix} Q_1(z,\beta) & -\beta Q_1(z,\beta)V_{12}R_2(z) \\ -\beta R_2(z)V_{21}Q_1(z,\beta) & R_2(z)(1 + \beta^2 V_{21}Q_1(z,\beta)V_{12}R_2(z)) \end{pmatrix} \tag{3.3}$$

where $R(z,\beta) = (P(\beta) - z)^{-1}$.

Proposition 3.1. *Let $0 < a < b < E_0$ and*

$$\mathcal{D} = \{z = E + i\epsilon \in \mathbb{C}; E \in [a,b] \text{ with dist } (E, \sigma(P_2)) \ge \epsilon_0, \epsilon \in]0,1]\}$$

for some $\epsilon_0 > 0$. Then there exists $\beta_0 > 0$ such that if $\rho_0 > \frac{1}{2}$ and $0 < \beta \le \beta_0$, $R(z,\beta) : L^{2,s} \times L^2 \to H^{2,-s} \times H^2$, $s > \frac{1}{2}$, extends continuously to $\overline{\mathcal{D}}$ and is uniformly bounded in β.

Proof. The limiting absorption principle for P_1 says that $R_1(z)$ defined for $\Im z > 0$ extends continuously for $z \in \overline{\mathcal{D}}$ as operator in $\mathcal{L}(s,-s)$, $s > \frac{1}{2}$. If $\rho_0 > \frac{1}{2}$ and $\frac{1}{2} < s < \rho_0$,

$$\beta^2 K_2(z) R_1(z) : L^{2,s} \to L^{2,2\rho_0 - s} \subset L^{2,s}$$

is uniformly bounded by $C\beta^2$. In particular, for β_0 small, $1 - \beta^2 K_2(z) R_1(z) : L^{2,s} \to L^{2,s}$ is invertible with uniformly bounded inverse. This shows that $Q_1(z,\beta) = R_1(z)(1 - \beta^2 K_2(z) R_1(z))^{-1} : L^{2,s} \to H^{2,-s}$ is uniformly bounded and extends continuously to $\overline{\mathcal{D}}$. ■

Let e be an eigenvalue with multiplicity m of P_2 in $]0, E_0[$ such that $\mathrm{d}(e, \sigma(P_2)\setminus\{e\}) \geq 2\epsilon_0$, $\epsilon_0 > 0$, and φ_i, $i = 1, \cdots, m$, orthonormal eigenfunctions of P_2 associated with e. Let $\mathcal{D}_+(e,\epsilon_0) = \{z = E + i\epsilon \in \mathbb{C}; \epsilon \in]0,1], |\nu - E| \leq \epsilon_0\}$. We want to construct the inverse of a Grushin problem associated to $P(\beta)$ on $\mathcal{D}_+(e,\epsilon_0)$. Let Π denote the orthogonal projection of L^2 onto the eigenspace of P_2 associated with e. Denote $\Pi' = 1 - \Pi$ and $P_2' = \Pi' P_2 \Pi'$. Set

$$R'(z,\beta) = \begin{pmatrix} Q_1'(z,\beta) & -\beta Q_1'(z,\beta) V_{12} R_2'(z) \\ -\beta R_2'(z) V_{21} Q_1'(z,\beta) & R_2'(z)(1 + \beta^2 V_{21} Q_1'(z,\beta) V_{12} R_2'(z)) \end{pmatrix} \tag{3.4}$$

where $R_2'(z) = (P_2' - z)^{-1}\Pi'$ and

$$\begin{aligned} Q_1'(z,\beta) &= R_1(z)(1 - \beta^2 K_2'(z) R_1(z))^{-1}, && (3.5)\\ K_2'(z) &= V_{12} R_2'(z) V_{21}. && (3.6) \end{aligned}$$

For $z \in \mathcal{D}_+(e,\epsilon_0)$, one has

$$(P(\beta) - z) R'(z,\beta) = \begin{pmatrix} 1 & 0 \\ \beta \Pi V_{21} Q_1'(z,\beta) & \Pi' - \beta^2 \Pi V_{21} Q_1'(z,\beta) V_{12} R_2'(z) \end{pmatrix}.$$

Let us study the Grushin problem associated to the operator

$$\mathcal{P}(z,\beta) = \begin{pmatrix} P(\beta) - z & \mathcal{E}_+^0 \\ \mathcal{E}_-^0 & 0 \end{pmatrix} : H^2(\mathbb{R}^d;\mathbb{C}^2) \times \mathbb{C}^m \to L^2(\mathbb{R}^d;\mathbb{C}^2) \times \mathbb{C}^m, \tag{3.7}$$

where

$$\begin{aligned} \mathcal{E}_+^0 : \mathbb{C}^m \ni (c_1, \cdots, c_m) &\longrightarrow (0, \sum_{j=1}^m c_j \varphi_j) \in H^2(\mathbb{R}^d;\mathbb{C}^2), \\ \mathcal{E}_-^0 : L^2(\mathbb{R}^d;\mathbb{C}^2) \ni (u,v) &\longrightarrow (< \varphi_1, v >, \cdots, < \varphi_m, v >) \in \mathbb{C}^m. \end{aligned}$$

Let

$$\begin{aligned} B &= \begin{pmatrix} 0 & 0 \\ \beta \Pi V_{21} Q_1'(z,\beta) & -\beta^2 \Pi V_{21} Q_1'(z,\beta) V_{12} R_2'(z) \end{pmatrix}, \\ C &= (P(\beta) - z)\mathcal{E}_+^0. \end{aligned}$$

Then, the inverse $\mathcal{R}(z,\beta)$ of $\mathcal{P}(z,\beta)$ is given by

$$\begin{aligned}\mathcal{R}(z,\beta) &= \begin{pmatrix} R'(z)(1-B) & \mathcal{E}_+^0 - R'(z)(1-B)C \\ \mathcal{E}_-^0(1-B) & -\mathcal{E}_-^0(1-B)C \end{pmatrix} \\ &\equiv \begin{pmatrix} \mathcal{E}(z,\beta) & \mathcal{E}_+(z,\beta) \\ \mathcal{E}_-(z,\beta) & \mathcal{E}_{+-}(z,\beta) \end{pmatrix}.\end{aligned}$$

In particular,

$$\mathcal{E}_{+-}(z,\beta) = \mathcal{E}_{-,2}^0(z - e + \beta^2 V_{21} Q_1'(z,\beta) V_{12}) \mathcal{E}_{+,2}^0 \tag{3.8}$$

where $\mathcal{E}_{+,2}^0 : \mathbb{C}^m \ni (c_1, \cdots, c_m) \to v = \sum_{j=1}^m c_j \varphi_j$ and $\mathcal{E}_{-,2}^0$ is its adjoint. From (3.8), il follows that

$$R(z,\beta) = \mathcal{E}(z,\beta) - \mathcal{E}_+(z,\beta)\mathcal{E}_{+-}(z,\beta)^{-1}\mathcal{E}_-(z), \quad \Im z > 0. \tag{3.9}$$

See also [8, 19, 22].

Lemma 3.2. *For $s > \frac{1}{2}$, $Q_1'(E+i0,\beta)$ exists in $\mathcal{L}(L^{2,s}; L^{2,-s})$ for $s > \frac{1}{2}$ and is Hölder continuous in E with $|E-e| \le \epsilon_0$, uniformly in $\beta \in]0,\beta_0]$.*

Proof. By the assumption on e, $K_2'(z) = V_{12}R_2'(z)V_{21}$ is continuous in $z \in \overline{\mathcal{D}_+(e,\epsilon_0)}$ as operator from L^2 to $L^{2,2\rho_0}$. Since $R_1(E+i0) : L^{2,s} \to L^{2,-s}$ exists and is Hölder continuous in $E > 0$ if $s > \frac{1}{2}$,

$$K_2'(z)R_1(z) : L^{2,s} \to L^{2,2\rho_0 - s} \subset L^{2,s}$$

is uniformly bounded and the limiting value $K_2'(E)R_1(E+i0)$ exists in $\mathcal{L}(L^{2,s}; L^{2,-s})$ for $\frac{1}{2} < s < \rho_0$ and $|E-e| \le \epsilon_0$. In particular, for $0 < \beta \le \beta_0$ with β_0 small enough $1 - \beta^2 K_2(E)R_1(E+i0) : L^{2,s} \to L^{2,s}$ is invertible and

$$Q_1(E+i0,\beta) = R_1(E+i0)(1 - \beta^2 K_2(E)R_1(E+i0))^{-1} : L^{2,s} \to L^{2,-s}$$

exists and is Hölder continuous in E with $|E-e| \le \epsilon_0$, uniformly in β. ■

Corollary 3.3. *Let $\beta_0 > 0$ be small and $s > \frac{1}{2}$. The operators*

$$\begin{aligned}\mathcal{E}(z,\beta) : &\quad L^{2,s} \times L^2 \to H^{2,-s}(\mathbb{R}^d) \times H^2(\mathbb{R}^d) \\ \mathcal{E}_+(z,\beta) : &\quad \mathbb{C}^m \to H^{2,-s}(\mathbb{R}^d) \times H^2(\mathbb{R}^d) \\ \mathcal{E}_-(z,\beta) : &\quad L^{2,s}(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to \mathbb{C}^m \\ \mathcal{E}_{+-}(z,\beta) : &\quad \mathbb{C}^m \to \mathbb{C}^m\end{aligned}$$

are holomorphic in z with $\Im z > 0$ and extend continuously in $z \in \overline{\mathcal{D}_+(e,\epsilon_0)}$ uniformly with respect to $\beta \in]0,\beta_0]$.

Definition *We call $E \in \mathbb{R}$ a generalized eigenvalue of $P(\beta)$ if equation $(P(\beta)-E)u = 0$ admits a non trivial solution $u \in L^{2,-s}(\mathbb{R}^d; \mathbb{C}^2)$ for any $s > \frac{1}{2}$ if $E \neq 0, E_0$, and for any $s > 1$ if $E = 0$ or E_0.*

Theorem 3.4. *Let $\epsilon_0 > 0$ and $\beta_0 > 0$ be small.*

(a). Let $E \in [e-\epsilon_0, e+\epsilon_0]$. Then, E is a generalized eigenvalue of $P(\beta)$ for $\beta \in]0, \beta_0]$ if and only if

$$\det \mathcal{E}_{+-}(E + i0, \beta) = 0. \tag{3.10}$$

(b). Let

$$A(z) = (< R_1(z)V_{12}\varphi_i, V_{12}\varphi_j >)_{1\leq i,j\leq m}, \quad z \in \overline{\mathcal{D}_+(e, \epsilon_0)}. \tag{3.11}$$

Assume that the matrix $A(e+i0)$ has no real eigenvalues. Then $P(\beta)$ has no eigenvalues in $[e - \epsilon_0, e + \epsilon_0]$ for any $\beta \in]0, \beta_0]$. The resolvent $R(z, \beta) : L^{2,s} \times L^2 \to H^{2,-s} \times H^2$, $s > \frac{1}{2}$, extends continuously to $\overline{\mathcal{D}_+(e, \epsilon_0)}$ and satisfies

$$\|R(E + i0, \beta)\|_{\mathcal{L}(s;-s)} \leq C_s\beta^{-2} \tag{3.12}$$

uniformly in $|E - e| \leq \epsilon_0$ and $0 < \beta \leq \beta_0$.

Proof. (a). From the relations $\mathcal{P}(z)\mathcal{R}(z, \beta) = 1$ on L^2 and $\mathcal{R}(z, \beta)\mathcal{P}(z) = 1$ on H^2, Corollary 3.3 shows that these equations extend continuously up to $[e - \epsilon_0, e + \epsilon_0]$ in appropriately weighted spaces. One obtains

$$(P(\beta) - z)\mathcal{E}_+(z, \beta) + \mathcal{E}_+^0\mathcal{E}_{+-}(z, \beta) \;=\; 0 \tag{3.13}$$

$$\mathcal{E}_-(z, \beta)(P(\beta) - z) + \mathcal{E}_{+-}(z, \beta)\mathcal{E}_-^0 \;=\; 0 \tag{3.14}$$

for $z \in \overline{D(e, \epsilon_0)}$. If $E \in [e-\epsilon_0, e+\epsilon_0]$ is an generalized eigenvalue of $P(\beta)$ and $w = (u, v) \neq 0$ a generalized eigenfunction associated with E, (3.14) shows that $\mathcal{E}_{+-}(E+i0, \beta)\mathcal{E}_-^0 w = 0$. Since $\mathcal{E}(E+i0, \beta)(P(\beta)-E)+\mathcal{E}_+(E+i0, \beta)\mathcal{E}_-^0 = 1$ as operator from $H^{2,-s}$ to $H^{2,-s}$ for any $s > \frac{1}{2}$, it follows that $\mathcal{E}_+(E + i0, \beta)\mathcal{E}_-^0 w = w$ and $\mathcal{E}_-^0 w = (< v, \varphi_1 >, \cdots, < v, \varphi_m >) \neq 0$. Therefore, $\det \mathcal{E}_{+-}(E + i0, \beta) = 0$. Conversely, if $\det \mathcal{E}_{+-}(E + i0, \beta) = 0$ and $c \in \mathbb{C}^m \setminus \{0\}$ is in the kernel of $\mathcal{E}_{+-}(E+i0, \beta)$, (3.13) shows that $(P(\beta)-E)w = 0$ where $w = \mathcal{E}_+(E+i0, \beta)c \in H^{2,-s} \times H^2$ for any $s > \frac{1}{2}$. Since $\mathcal{E}_-^0 w = \mathcal{E}_-^0\mathcal{E}_+(E+i0, \beta)c = c \neq 0$, w is non zero. Therefore, E is a generalized eigenvalue of $P(\beta)$.

(b). One has

$$Q_1'(z, \beta) = R_1(z) + \beta^2 R_1(z)K_2'(z)R_1(z)(1 - \beta^2 K_2'(z)R_1(z))^{-1}.$$

By Lemma 3.2,

$$Q_1'(z, \beta) = R_1(z) + O(\beta^2), \quad \beta \to 0,$$

in $\mathcal{L}(L^{2,s}, L^{2,-s})$ for $\frac{1}{2} < s < \rho_0$, uniformly in $z = E + i\epsilon \in \mathcal{D}_+$. One has

$$\begin{aligned}\mathcal{E}_{+-}(E + i0, \beta) \;&=\; E - e + \beta^2(A(E + i0) + O(\beta^2)) \\ &=\; E - e + \beta^2(A(e + i0) + o(1) + O(\beta^2))\end{aligned}$$

uniformly in E near e. Here $o(1) = A(E + i0) - A(e + i0) \to 0$ as $E \to e$. Suppose now $A(e + i0)$ has no real eigenvalues. Then, there exists $c_0 > 0$ such that

$$\|(\lambda + A(e + i0))^{-1}\| \leq c_0, \quad \forall\lambda \in \mathbb{R}.$$

Set $E = e + \beta^2 E_1(\beta), E_1(\beta) \in \mathbb{R}$. One sees that

$$\mathcal{E}_{+-}(E + i0, \beta) = \beta^2(E_1(\beta) + A(e + i0))[1 + (E_1(\beta) + A(e + i0))^{-1}(o(1) + O(\beta^2))]$$

is an invertible matrix for E near e and β small enough. This proves that $\det \mathcal{E}_{+-}(E+i0,\beta) \neq 0$. Part (a) shows that $P(\beta)$ has no generalized eigenvalues, therefore no eigenvalues in the usual sense, in $[e-\epsilon_0, e+\epsilon_0]$ for ϵ_0 small enough. The above argument shows also that $\|\mathcal{E}_{+-}(E+i0,\beta)^{-1}\| = O(\beta^{-2})$, $\beta \to 0$. (3.12) follows from (3.9). ■

Corollary 3.5. *Let $0 < a < b < E_0$ et let $\{E_1, \cdots, E_N\}$ denote the discrete spectrum of P_2 in $[a,b]$. Denote by m_j the multiplicity of E_j and by $\varphi_k^{(j)}$, $1 \le k \le m_j$ orthonormal eigenfunctions of P_2 associated with E_j. Assume that the matrices*

$$A_j = (< R_1(E_j+i0)V_{12}\varphi_k^{(j)}, V_{12}\varphi_l^{(j)} >)_{1\le k,l\le m_j} \text{ have no real eigenvalues} \tag{3.15}$$

for $j = 1, \cdots, N$. Then there exists $\delta_0 > 0$ and $\beta_0 > 0$ such that $P(\beta)$ has no eigenvalues embedded in $[a-\delta_0, b+\delta_0]$ for all $\beta \in]0,\beta_0]$.

Proof. Let $0 < \delta_0 < dist(\sigma(P_2) \setminus \{E_1, \cdots, E_N\}, [a,b])$ be small enough. Then, $[a-\delta_0, b+\delta_0] \subset I_0 \cup (\cup_{j=1}^N I_j)$ where $I_0 = \{E \in [a-\delta_0, b+\delta_0]; dist(E, \sigma(P_2) \ge \epsilon_0\}$ and $I_j = \{E; |E-E_j| \le \epsilon_0\}$ with $\epsilon_0 > 0$ small enough. We need only to apply Proposition 3.1 to $P(\beta)$ over I_0 and Theorem 3.4 (b) with $e = E_j$ over I_j for $j = 1, \cdots, N$. ■

From the proof of Theorem 3.4, one sees that if E is a generalized eigenvalue of $P(\beta)$ near e, then $\dim\ker(P(\beta)-E) = \dim\ker \mathcal{E}_{+-}(E+i0,\beta)$. One may expect that generalized eigenvalues of $P(\beta)$ outside the thresholds 0 and E_0 are always eigenvalues of $P(\beta)$ in the usual sense. Indeed, an argument used in [1] allows to prove that if V is short range (i.e., $V(x) = O(|x|^{-1-\epsilon})$, $\epsilon > 0$), then $E > 0$ is a generalized eigenvalue if and only if E is an eigenvalue of $P(\beta)$ in the usual sense. Here we prove the following

Theorem 3.6. *With the notation of Theorem 3.4, let $E \in [e-\epsilon_0, e+\epsilon_0]$ and $\beta \in]0,\beta_0]$. Assume (2.4) with $m = 3$. Then, E is an eigenvalue of $P(\beta)$ if and only if $\det \mathcal{E}(E+i0,\beta) = 0$. If $E \in [e-\epsilon_0, e+\epsilon_0]$ is an eigenvalue of $P(\beta)$, its multiplicity is equal to $\dim\ker \mathcal{E}_{+-}(E+i0,\beta)$.*

Proof. By Theorem 3.4 (a), it is sufficient to prove that if $\det \mathcal{E}(E+i0,\beta) = 0$, E is an eigenvalue of $P(\beta)$. The proof of Theorem 3.4 (a) shows that if $c \in \ker \mathcal{E}_{+-}(E+i0,\beta)$ with $c = (c_1, \cdots, c_m) \in \mathbb{C}^m \setminus \{0\}$, $u = \mathcal{E}_+(E+i0,\beta)c$ is a generalized eigenfunction of $P(\beta)$ associated with E. We want to prove that $u \in H^2$. Let $\psi = \sum_{j=1}^m c_j\varphi_j$. Then,

$$\Im < V_{12}\psi, Q_1'(E+i0,\beta)V_{12}\psi >= \Im < c, \mathcal{E}_{+-}(E+i0,\beta)c >= 0. \tag{3.16}$$

Note that $\Im Q_1'(z,\beta) = \frac{1}{2i}(Q_1'(z,\beta) - Q_1'(\overline{z},\beta)$ is given by

$$\Im Q_1'(z,\beta) = Q_1'(\overline{z},\beta)(\Im z + \beta^2 \Im K_2'(z))Q_1'(z,\beta).$$

Since $\Im K_2'(z) \ge 0$ for $\Im z > 0$, one has $\Im Q_1'(z,\beta) \ge 0$ for $\Im z > 0$. It follows from Lemma 3.2 that $< f, Q_1'(E+i0,\beta)f >\ge 0$ for any $f \in L^{2,s}$, $s > \frac{1}{2}$. (3.16) implies then $\Im Q_1'(E+i0,\beta)V_{12}\psi = 0$, or equivalently,

$$Q_1'(E+i0,\beta)V_{12}\psi = Q_1'(E-i0,\beta)V_{12}\psi \tag{3.17}$$

in $L^{2,-s}$ for any $s > \frac{1}{2}$. Note that

$$Q_1'(E \pm i0,\beta) = R_1(E \pm i0) + \beta^2 R_1(E \pm i0)K_2'(E)Q_1'(E \pm i0,\beta).$$

Making use of (2.8), one has

$$b_{\mp}(x,D)R_1(E\pm i0)V_{12}\psi \in L^{2,s}, \quad \forall\, s<1, \quad \text{ and}$$

$$\|\langle x\rangle^{s-1}b_{\mp}(x,D)R_1(E\pm i0)K_2'(E)\langle x\rangle^{-s+2\rho_0}\| \le C, \quad \forall\, \frac{1}{2}<s<2,$$

where $b_\pm$ are bounded symbols with support properties of (2.8). Using (3.17), we can decompose

$$Q_1'(E+i0,\beta)V_{12}\psi = (b_-(x,D)Q_1'(E+i0,\beta)+b_+(x,D)Q_1'(E-i0,\beta)+RQ_1'(E+i0,\beta))V_{12}\psi$$

with $R\in\mathcal{L}(s,s+\rho_1)$, for $|s|\le 3$. By the argument used in the proof of Lemma 2.3, we obtain that

$$\|\langle x\rangle^s Q_1'(E+i0,\beta)V_{12}\psi\| \le C_s(1+\|\langle x\rangle^{s-s_0}Q_1'(E+i0,\beta)V_{12}\psi\|)$$

for any $-\frac{1}{2}<s<1$, where $s_0=\min\{\rho_1, 2\rho_0-1\}$. Since $s_0>0$, we can repeatedly apply the above estimate to obtain that $Q_1'(E\pm i0,\beta)V_{12}\psi\in L^{2,s}$ for any $s<1$. Note that $u=\mathcal{E}_+(E+i0)c=\mathcal{E}_+^0c-R'(E+i0)(C-BC)c$ with

$$(C-BC)c=(\beta V_{12}\psi, \beta^2\Pi V_{21}Q_1'(E+i0,\beta)V_{12}\psi).$$

Since $R_2'(E)\Pi=0$, using (3.4), we can calculate that

$$\mathcal{E}_+(E+i0)c=(-\beta Q_1'(E+i0,\beta)V_{12}\psi,\ \psi+\beta^2R_2'(E)V_{21}Q_1'(E+i0,\beta)V_{12}\psi)$$

From the above estimate on $Q_1'(E+i0,\beta)V_{12}\psi$, one deduces easily that $u\in H^2$. Therefore, E is an eigenvalue of $P(\beta)$ in the usual sense. The other assertions in Theorem 3.6 are immediate. ■

The method presented above can be applied to studying the perturbation of eigenvalues of P_0 at the thresholds 0 and E_0. See [14] for results on the non-existence of perturbed eigenvalues in various situations. Here we give a result on the location of eigenvalues of $P(\beta)$ near 0. In the remaining part of this section, we assume that $d\ge 3$ and $\rho_1>2$.

When zero is a regular point (*i.e.*, neither eigenvalue nor resonance) of P_1 in the sense of [13], $R_1(z)$ defined for $\Im z>0$ extends continuously up to the real axis near zero as operator from $L^{2,s}$ to $L^{2,-s}$ for any $s>1$ and the limiting value $R_1(0)$ of $R_1(z)$ at 0 is symmetric as unbounded operator in L^2. See [13], and also [23] for a unified treatment in a geometric setting for $d\ge 2$. We have the following

Theorem 3.7. *Assume that $\rho_0>1$, $\rho_1>2$ and that zero is a regular point of P_1 and an eigenvalue of multiplicity m of P_2. Let φ_j, $j=1,\cdots,m$, be associated orthonormal eigenfunctions of P_2. Suppose the matrix*

$$A_1=(<R_1(0)V_{12}\varphi_i, V_{12}\varphi_j>)_{1\le i,j\le m} \tag{3.18}$$

has k strictly positive eigenvalues, λ_j, $1\le j\le k$, counted with their multiplicity. Then, there exists $\epsilon_0>0$ and $\beta_0>0$ such that for $\beta\in]0,\beta_0]$, $P(\beta)$ has at least k eigenvalues, counted with their multiplicity, $\lambda_j(\beta)$, in $]-\epsilon_0,0[$ verifying

$$\lambda_j(\beta)=-\lambda_j\beta^2+o(\beta^2), \quad 1\le j\le k. \tag{3.19}$$

In particular, if A_1 is positive definite, $P(\beta)$ has exactly m eigenvalues in $]-\epsilon_0, \epsilon_0[$ given by (3.19) with $k = m$.

Proof. Under the conditions of Theorem 3.7, Corollary 3.3 holds for $e = 0$ and $s > 1$. The same arguments as those used for the proof of Theorem 3.4 (a) show that for $E \in]-\epsilon_0, \epsilon_0[$, $(P(\beta) - E)u = 0$ has a solution $u \in H^{2,-s}$ for any $s > 1$ if and only if $\det \mathcal{E}_{+-}(E + i0, \beta) = 0$. Recall that for $\epsilon_0 > 0$ small enough, the spectrum of P_1 in $[-\epsilon_0, 0[$ is void and that $R_1(E + i0) = R_1(E)$ is symmetric for $E \in [-\epsilon_0, 0]$. By (3.8) with $e = 0$, one has for $E \in [-\epsilon_0, 0]$

$$\mathcal{E}_{+-}(E + i0, \beta) = E + \beta^2 A(E) + \beta^4 B(E, \beta)$$

where

$$\begin{aligned} A(E) &= (< R_1(E)V_{12}\varphi_i, V_{12}\varphi_j >)_{1\leq i,j\leq m} \quad \text{and} \\ B(E, \beta) &= (< R_1(E)K_2'(E)Q_1'(E, \beta)V_{12}\varphi_i, V_{12}\varphi_j >)_{1\leq i,j\leq m} \end{aligned}$$

are Hermitian matrices. Let $\lambda_0 > 0$ be an eigenvalue of A_1 with multiplicity m_0. Since $B(E, \beta)$ is continuous in E and analytic in β^2 near 0, the finite dimensional perturbation theory affirms that for each $E \in [-\epsilon_0, 0]$, there are m_0 eigenvalues, $\mu_j(E, \beta)$, of $A(E) + \beta^2 B(E, \beta)$, counted according to their multiplicity, which are analytic in β^2 near 0 verifying

$$\mu_j(E, \beta) = \mu_j(E) + \sum_{k=1}^{\infty} \beta^{2k}\mu_{jk}(E), \quad j = 1, \cdots, m_0, \tag{3.20}$$

where $\mu_j(E)$ are eigenvalues of $A(E)$ converging to λ_0 as $E \to 0$. In particular, for $\beta_0 > 0$ small enough, $f_j(E, \beta) = E + \beta^2\mu_j(E, \beta)$ verifies $f_j(-\epsilon_0, \beta) < 0$ and $f_j(0, \beta) = \lambda_0 + O(\beta^2) > 0$. Since $f_j(E, \beta)$ is continuous and real, it has at least one zero $E = \lambda_j(\beta)$ in $]-\epsilon_0, 0[$ verifying

$$\lambda_j(\beta) = -\beta^2\mu_j(\lambda_j(\beta), \beta).$$

(3.20) gives

$$\lambda_j(\beta) = -\beta^2(\lambda_0 + o(1)), \quad \beta \to 0. \tag{3.21}$$

Let $v_j(E, \beta)$ be orthonormal eigenvectors of $A(E) + \beta^2 B(E, \beta)$ associated with the eigenvalues $\mu_j(E, \beta)$, $j = 1, \cdots, m_0$. Then $u_j(\beta) = v_j(\lambda_j(\beta), \beta)$ is in the kernel of $\mathcal{E}_{+-}(\lambda_j(\beta)+i0, \beta)$, $j = 1, \cdots, m_0$. When $\lambda_j(\beta) = \lambda_k(\beta)$, $u_j(\beta)$ and $u_k(\beta)$ are orthogonal. As in the proof of Theorem 3.4, one can show that these $u_j(\beta)$'s give rise to m_0 linearly independent generalized eigenfunctions of $P(\beta)$ associated with generalized eigenvalues $\lambda_j(\beta)$, $j = 1, \cdots, m_0$. Notice that the interval $]-\epsilon_0, 0[$ is contained in spectral gap of P_1 and P_2 and that for $z = E \in]-\epsilon_0, 0[$, the inverse of Grushin problem constructed before exists as operators in L^2. In particular, since $\lambda_j(\beta) < 0$, $\mathcal{E}(\lambda_j(\beta)+i0, \beta)u_j(\beta) \in H^2$ is an eigenfunction of $P(\beta)$ in the usual sense. Therefore, from a positive eigenvalue of multiplicity m_0 of A_1, we can construct m_0 independent eigenfunctions of $P(\beta)$. The first part of Theorem 3.7 is proved. By (3.9), $P(\beta)$ has at most m eigenvalues in $]-\epsilon_0, \epsilon_0[$, counted according to their multiplicity. Thus, if A_0 is positive definite, all these eigenvalues are given by (3.19) with $k = m$. ■

When 0 is an exceptional point of P_1 (an eigenvalue and/or a resonance of P_1), to be simple, we assume that $0 \notin \sigma(P_2)$. For $d \geq 3$, $G_0 = \lim_{z\to 0, \Im z>0}(-\Delta - z)^{-1}$ exists as operator from $H^{-1,s}$ to $H^{1,-s}$ for any $s > 1$. In the following, take $s \in]1, \min\{\rho_0, \rho_1/2\}[$, $\rho_0 > 1$, $\rho_1 > 2$. The kernel, $\mathcal{K}$, of $(1 + G_0V_1)$ in $H^{1,-s}(\mathbb{R}^d)$ is spanned by the resonant states and eigenfunctions of P_1 associated with zero. Let $\mathcal{F}$ denote the range of $(1+G_0V_1)$ in $H^{1,-s}(\mathbb{R}^d)$. By the Fredholm theory, one can show that $H^{1,-s}(\mathbb{R}^d) = \mathcal{K} \oplus \mathcal{F}$ and that the restriction of $(1 + G_0V_1)$ on $\mathcal{F}$ is invertible with bounded inverse. Let $m = \dim \mathcal{K}$. One can choose a basis $\varphi_1, \cdots, \varphi_m$ of $\mathcal{K}$ satisfying $< \varphi_i, -V_1\varphi_j >= \delta_{ij}$. Here $< \cdot, \cdot >$ is the L^2-product which is used as $(H^{1,-s}, H^{-1,s})$ dual product. Then the projection Q from $\mathcal{K} \oplus \mathcal{F} \to \mathcal{K}$ is given by

$$Qf = \sum_{j=1}^{m} < -V_1\varphi_j, f > \varphi_j, \quad f \in H^{1,-s}.$$

Let $Q' = 1 - Q : H^{1,-s} \to H^{1,-s}$. One can show that for $|z|$ small and $\Im z > 0$,

$$T_1' = (Q'(1 + R_0(z)V_1)Q')^{-1}Q' \tag{3.22}$$

exists and is uniformly bounded in $\mathcal{L}(1, -s; 1, -s)$ for any $1 < s < \rho_1/2$. Write $P(\beta)$ as

$$P(\beta) = \mathcal{P}_0 + V_\beta, \text{ where } \mathcal{P}_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & P_2 \end{pmatrix}, \quad V_\beta = \begin{pmatrix} V_1 & \beta V_{12} \\ \beta V_{21} & 0 \end{pmatrix}.$$

Set $\mathcal{R}_0(z) = (\mathcal{P}_0 - z)^{-1}$ and $\mathcal{H}^r = L^{2,r} \oplus L^2$, $r \in \mathbb{R}$. As map from $\mathcal{H}^s$ to $\mathcal{H}^{-s}$, $\mathcal{R}_0(z)$ defined for $\Im z > 0$ extends continuously up to the real axis near 0 for any $s > 1$. One has

$$R(z, \beta) = (1 + \mathcal{R}_0(z)V_\beta)^{-1}\mathcal{R}_0(z), \quad \Im z > 0.$$

Set

$$W(z, \beta) = 1 + \mathcal{R}_0(z)V_\beta = \begin{pmatrix} 1 + R_0(z)V_1 & \beta R_0(z)V_{12} \\ \beta R_2(z)V_{21} & 1 \end{pmatrix}$$

and

$$U'(z) = \begin{pmatrix} T_1' & 0 \\ 0 & 1 \end{pmatrix}.$$

On $\mathcal{H}^{-s}$, one has

$$W(z, \beta)U'(z) = \begin{pmatrix} Q' & 0 \\ 0 & 1 \end{pmatrix} + B \tag{3.23}$$

with

$$B = \begin{pmatrix} B_{11} & \beta R_0(z)V_{12} \\ \beta R_2(z)V_{21}T_1' & 0 \end{pmatrix} \tag{3.24}$$

where $B_{11} = Q(1 + R_0(z)V_1)T_1' = Q(R_0(z) - G_0)V_1)T_1'$. As operator from $L^{2,-s}$ to $L^{2,-s}$, one has

$$B_{11} = O(|z|^\epsilon), \text{ if } s > 1, \rho_1 > 2; \quad B_{11} = O(|z|^{1/2}), \text{ if } s > 3/2, \rho_1 > 3. \tag{3.25}$$

Consider the Grushin problem associated with the operator

$$\mathcal{W} = \begin{pmatrix} W(z, \beta) & \epsilon_+^0 \\ \epsilon_-^0 & 0 \end{pmatrix} : \mathcal{H}^{-s} \oplus \mathbb{C}^m \to \mathcal{H}^{-s} \oplus \mathbb{C}^m \tag{3.26}$$

where

$$\epsilon_+^0: \qquad \mathbb{C}^m \ni c = (c_1, \cdots, c_m) \to (\sum_{j=1}^m c_j\varphi_j, 0) \in \mathcal{H}^{-s} \tag{3.27}$$

$$\epsilon_-^0: \qquad \mathcal{H}^{-s} \ni (u,v) \to (< -V_1\varphi_1, u >, \cdots, < -V_1\varphi_m, u >) \in \mathbb{C}^m. \tag{3.28}$$

Using

$$\mathcal{U} = \begin{pmatrix} U'(z) & \epsilon_+^0 \\ \epsilon_-^0 & 0 \end{pmatrix}$$

as an approximate inverse, one can show $\mathcal{W}$ is invertible and

$$\begin{aligned} \mathcal{W}^{-1} &= \begin{pmatrix} U'(z)(1+B)^{-1} & \epsilon_+^0 - U'(z)(1+B)^{-1}W(z,\beta)\epsilon_+^0 \\ \epsilon_-^0(1+B)^{-1} & -\epsilon_-^0(1+B)^{-1}W(z,\beta)\epsilon_+^0 \end{pmatrix} \\ &\equiv \begin{pmatrix} \epsilon(z,\beta) & \epsilon_+(z,\beta) \\ \epsilon_-(z,\beta) & \epsilon_{+-}(z,\beta) \end{pmatrix}. \end{aligned}$$

$\mathcal{W}^{-1}$ defined for $\Im z > 0$ extends continuously in z up to $\mathbb{R}$ near 0 as operator on $\mathcal{H}^{-s} \oplus \mathbb{C}^m$. By standard arguments in Grushin problems, one obtains that

$$W(z,\beta)^{-1} = \epsilon(z,\beta) - \epsilon_+(z,\beta)\epsilon_{+-}(z,\beta)^{-1}\epsilon_-(z,\beta) \tag{3.29}$$

and the singularities of $W(z,\beta)^{-1}$ is determined by those of $\epsilon_{+-}(z,\beta)^{-1}$. By elementary but lengthy calculations, one has

$$\begin{aligned} \epsilon_{+-}(z,\beta) &= \epsilon_{-,1}^0\{(1-B_{11})[1+\beta^2 R_0V_{12}R_2V_{21}T_1'(1-B_{11})](1+R_0V_1) \\ &\quad -\beta^2(1-B_{11})R_0V_{12}R_2V_{21} + O(\beta^3)\}\epsilon_{+,1}^0 \\ &= \epsilon_{-,1}^0\{(1-B_{11})(1+R_0V_1) - \beta^2R_0V_{12}R_2V_{21} + O(\beta^2|z|^\epsilon + \beta^3)\}\epsilon_{+,1}^0 \end{aligned} \tag{3.30}$$

where $\epsilon_{+,1}^0 : \mathbb{C}^m \ni (c_1, \cdots, c_m) \to \sum_{j=1}^m c_j\varphi_j \in L^{2,-s}$ and $\epsilon_{-,1}^0$ is its formal adjoint. In the above reduction leading to (3.30), we only used $s > 1$. Thus, $\rho_0 > 1$ $\rho_1 > 2$ are sufficient.

We examine $\epsilon_{+-}(z,\beta)$ in the case $d = 3$. The case $d \geq 4$ can be treated similarly. In the case $d = 3$

$$R_0(z) = G_0 + iz^{1/2}G_1 + O(|z|^{1/2+\epsilon}) : L^{2,s} \to L^{2,-s}, s > 3/2, \tag{3.31}$$

for some $\epsilon > 0$. Suppose now that $\rho_1 > 3$ and *zero is an eigenvalue, but not a resonance of* P_1. Using (3.31) and the condition $\rho_1 > 3$, one can show that the matrix elements of $\epsilon_{+-}(z,\beta)$ are given by

$$\begin{aligned} \epsilon_{+-}(z,\beta)_{ij} &= < -V_1\varphi_i, R_0(z)V_1\varphi_j > -\beta^2 < \varphi_i, V_{12}R_2(z)V_{21}\varphi_j > \\ &\quad + O(|z|^{1+\epsilon} + \beta^2|z|^\epsilon + \beta^3) \\ &= -z < \varphi_i, \varphi_j > -\beta^2 < \varphi_i, V_{12}R_2(0)V_{21}\varphi_j > \\ &\quad + O(|z|^{1+\epsilon} + \beta^2|z|^\epsilon + \beta^3) \end{aligned} \tag{3.32}$$

uniformly in $|z|$ and β near 0 with $\Im z > 0$. Remark that for $\lambda < 0$ and λ near 0, the boundary value $\epsilon_{+-}(\lambda,\beta)$ of $\epsilon_{+-}(z,\beta)$ at λ is self-adjoint. Let ϕ_j, $j = 1, \cdots, m$ be an orthonormal basis of the zero eigenspace of P_1. Let M be the transfer matrix from $\{\phi_1, \cdots, \phi_m\}$ to $\{\varphi_1, \cdots, \varphi_m\}$ and let $\tilde{\epsilon}_{+-}(\lambda, \beta) = (M^{-1})^* \epsilon_{+-}(\lambda, \beta) M^{-1}$. Then one has

$$\tilde{\epsilon}_{+-}(\lambda, \beta) = -\lambda - \beta^2 A_2 + O(|\lambda|^{1+\epsilon}) + O(\beta^2 |\lambda|^\epsilon) + O(\beta^3). \tag{3.33}$$

where

$$A_2 = (< \phi_i, V_{12} R_2(0) V_{21} \phi_j >)_{i,j}$$

By the perturbation theory of self-adjoint matrices, one can show as in the proof of Theorem 3.7 that if $\lambda_0 > 0$ is an eigenvalue of A_2 with multiplicity k, then for $\beta > 0$ small enough there exists $\lambda^{(j)}(\beta) = -\beta^2 \lambda_0 + o(\beta^2)$, $j = 1, \cdots$, and $v_j \in \mathbb{C}^m$ linearly independent such that $\epsilon_{+-}(\lambda^{(j)}(\beta), \beta) v_j = 0$. Using the inverse of the Grushin problem, one has: $u_j = \epsilon_+(\lambda^{(j)}(\beta), \beta) v_j \in \mathcal{H}^{-s}$ and $(P(\beta) - \lambda^{(j)}(\beta)) u_j = 0$, $j = 1, \cdots, k$. Since $\lambda^{(j)}(\beta) < 0$, we can derive from *a priori* energy estimate and the ellipticity of $-\Delta$ that $u_j \in H^2$. Thus $P(\beta)$ has at least k eigenvalues verifying $\lambda(\beta) = -\beta^2 \lambda_0 + o(\beta^2)$. Summing up, we have proved the following

Theorem 3.8. *Assume that $d = 3$, $\rho_0 > 1$ and $\rho_1 > 3$. Suppose that zero is an eigenvalue of multiplicity m, but not a resonance, of P_1 and that $0 \in \rho(P_2)$. Let ϕ_j, $j = 1, \cdots, m$ be orthonormal eigenfunctions of P_1 associated with zero. Assume that the matrix*

$$A_2 = (< R_2(0) V_{21} \phi_i, V_{21} \phi_j >)_{1 \le i,j \le m} \tag{3.34}$$

has k strictly positive eigenvalues, ν_j, $1 \le j \le k$, counted according to their multiplicity. Then, there exists $\epsilon_0 > 0$ and $\beta_0 > 0$ such that for $\beta \in]0, \beta_0]$, $P(\beta)$ has at least k eigenvalues, $\nu_j(\beta)$, counted according to their multiplicity, in $] - \epsilon_0, 0[$ verifying

$$\nu_j(\beta) = -\nu_j \beta^2 + o(\beta^2), \quad 1 \le j \le k. \tag{3.35}$$

In particular, if A_2 is positive definite, $P(\beta)$ has exactly m eigenvalues in $] - \epsilon_0, \epsilon_0[$ all given by (3.35) with $k = m$.

Theorem 3.8 can be compared with the results of [14] in abstract setting. If specified to the present case, their results say that if the matrix A_2 is positive definite and if the orthogonal projection onto the zero eigenspace is continuous from $H^{-1,s}$ to $H^{-1,s}$ for some appropriate s, then $P(\beta)$ has no eigenvalues in some small interval of the form $] - \delta_1 \beta^2, \delta_0[$. Note that in many problems arising from quantum physics, it is important to prove the existence of eigenvalues. The following example may be of interest in this connexion.

Example. Assume that $V_{12} = \overline{V_{21}}$ is non-trivial *i.e.*, $V_{12}(x) \neq 0$ for x in some non-empty open set, and that $\rho_0 > 1, \rho_1 > 2$. Suppose that $P_1 \ge 0$ and that 0 is a regular point for P_1 (which is satisfied if $V_1 \ge 0$). Let 0 be an eigenvalue of P_2 with multiplicity m. The matrix A_1 defined by (3.18) is then positive. We claim that it is in fact positive definite. In fact, if zero is an eigenvalue of A_1, let $c = (c_1, \cdots, c_m) \neq 0$ be an associated eigenvector: $A_1 c = 0$. Set $\psi = \sum c_j V_{12} \varphi_j$. Then, ψ is rapidly decreasing in x and $< R_1(0)\psi, \psi >= 0$. Since $< R_1(0)u, u >\ge 0$ for any $u \in L^{2,s}$, $s > 1$, we obtain $R_1(0)\psi = 0$ in $L^{2,-s}$. It follows that $\psi = \sum c_j V_{12} \varphi_j = 0$, or $v(x) = \sum c_j \varphi_j(x) = 0$ in a non-trivial open set where $V_{12}(x)$ is non zero. Since v is an eigenfunction of P_2 associated with the eigenvalue zero, the unique continuation theorem for P_2 ([9]) implies that $v = 0$. This is impossible, because $c \neq 0$ and $\{\varphi_1, \cdots, \varphi_m\}$ are orthonormal. This proves that all eigenvalues of A_1 are strictly positive. Theorem 3.7 implies that for $\epsilon_0 > 0$ and $\beta_0 > 0$ small enough, the operator

$$P(\beta) = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} + \beta \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix}$$

has exactly m eigenvalues $\lambda_j(\beta)$ in $]-\epsilon_0, \epsilon_0[$ verifying $\lambda_j(\beta) = -\beta^2(\lambda_j + o(1))$ as $\beta \to 0$, where $\lambda_j > 0$, $j = 1, \cdots, m$, are eigenvalues of the matrix A_1. Similarly, if $P_2 = -\Delta + V_2 + E_0 > 0$ and if 0 is an eigenvalue of multiplicity m, but not a resonance of P_1, and $\rho_0 > 1$, $\rho_1 > 3$, one can show as above that the matrix A_2 is positive definite for any non-trivial V_{12} and Theorem 3.8 can be applied. We conclude that for $\beta > 0$ small enough, $P(\beta)$ has exactly m negative eigenvalues in $]-\epsilon_0, \epsilon_0[$ satisfying (3.35).

The method developed above can also be applied to study the case where zero is resonance, but not an eigenvalue of P_1.

4. Resonances generated by the channel interaction

It is natural to expect that the eigenvalues of P_0 in $]0, E_0[$ dissolve into resonances of $P(\beta)$ when the potential is analytic. Here we use the elementary analytic dilation method of [4] to study these resonances. See [8, 18, 19] for more advanced resonance theory.

Let $U(\theta) : f \to e^{-d\theta/2} f(e^{-\theta}x), f \in L^2(\mathbb{R}^d), \theta \in \mathbb{R}$. We keep the notation of Section 3 and denote by $X(\theta)$ the operator obtained from an operator X by dilation: $X(\theta) = U(\theta)^{-1} X U(\theta)$. Assume that V_i, $i = 1, 2$, and V_{ij} are dilation analytic, that is, there exists some $\alpha > 0$ such that $V_i(\theta) = V_i(x, \theta)$, $V_{ij}(\theta) = V_{ij}(x, \theta)$ defined for θ real extend to holomorphic families of $-\Delta$-compact operators for $\theta \in \mathbb{C}$ with $|\Im\theta| < \alpha$. Then, $P_j(\theta)$ and $P(\beta, \theta)$ are holomorphic families of type (A) in the sense of Kato. Resonances are complex eigenvalues of dilated operators for $\theta \in \mathbb{C}$ with $0 < \Im\theta < \alpha$. In particular, it is well-known that

$$\begin{aligned}
\sigma_{\text{ess}}(P_1(\theta)) &= e^{-2\theta}\mathbb{R}_+, \ \sigma_{\text{ess}}(P_2(\theta)) = E_0 + e^{-2\theta}\mathbb{R}_+ \\
\sigma_{\text{d}}(P_1(\theta)) &\subset \sigma_{\text{pp}}(P_1) \cup \{z \in \mathbb{C}; -2\Im\theta < \arg z < 0\}, \\
\sigma_{\text{d}}(P_2(\theta)) &\subset \sigma_{\text{pp}}(P_2) \cup \{z \in \mathbb{C}; -2\Im\theta < \arg(z - E_0) < 0\}.
\end{aligned}$$

Since P_1 has no positive eigenvalues, for $\Im\theta > 0$, the spectrum of $P_0(\theta)$ in $\{z \in \mathbb{C}; 0 < \Re z < E_0, -2\Im\theta < \arg z \leq 0, \arg(z - E_0) < -2\Im\theta\}$ is discrete and $\sigma_{\text{d}}(P_0(\theta)) \cap]0, E_0[= \sigma_d(P_2) \cap]0, E_0[$. Let e be an eigenvalue of P_2 in $]0, E_0[$ such that $dist(e, \sigma(P_2) \setminus \{e\}) = 2\epsilon_0 > 0$. For $\Im\theta > 0$ small enough, e is the only point of $\sigma(P_0(\theta))$ in $\mathcal{D}(e, \delta_0) = \{z \in \mathbb{C}; |\Re z - e| < \epsilon_0, -\delta_0 < \Im z < 1\}$, where $\delta_0 > 0$ depends on e and $\Im\theta$. For $\epsilon > 0$ small enough and $\theta \in \mathbb{C}$ with $|\Im\theta| < \alpha$, we can define the spectral projection associated to e by

$$\Pi(\theta) = -\frac{1}{2\pi i} \int_{|z-e|=\epsilon} (P_2(\theta) - z)^{-1} \, dz.$$

$\theta \to \Pi(\theta)$ is a holomorphic family of projections and $\Pi(0) = \Pi$. Therefore, rank $\Pi(\theta)$ is constant and is equal to m for all θ as above. It is clear that $R_2'(z,\theta) = U(\theta)^{-1}R_2'(z)U(\theta)$ defined for θ real and z in a complex neighborhood of e can be extended to a holomorphic family of bounded operators on L^2 in z near e and $\theta \in \mathbb{C}$ with $|\Im\theta| < \alpha$. By (3.9), we have for $\Im z > 0$ and $\theta \in \mathbb{R}$,

$$R(z;\beta,\theta) = \mathcal{E}(z,\beta,\theta) - \mathcal{E}_+(z,\beta,\theta)\mathcal{E}_{+-}(z,\beta,\theta)^{-1}\mathcal{E}_-(z,\beta,\theta), \tag{4.1}$$

where $\mathcal{E}(z,\beta,\theta)$, $\mathcal{E}_\pm(z,\beta,\theta)$ and $\mathcal{E}_{+-}(z,\beta,\theta)$ are the dilated operators.

Proposition 4.1. *Let $\mathcal{E}_1(z,\beta,\theta) = \mathcal{E}_+(z,\beta,\theta)\mathcal{E}_{+-}(z,\beta,\theta)^{-1}\mathcal{E}_-(z,\beta,\theta)$.*

(a). $\mathcal{E}(z,\beta,\theta)$ defined for z with $\Im z > 0$ and $\theta \in \mathbb{R}$ extends to a holomorphic family of bounded operators in (θ, z) with $0 < \Im\theta < \alpha$ and $z \in \mathcal{D}(e,\delta_0)$.

(b). $\mathcal{E}_1(z,\beta,\theta)$ defined for z with $\Im z > 0$ and $\theta \in \mathbb{R}$ extends meromorphically in (θ, z) with $0 < \Im\theta < \alpha$ and $z \in \mathcal{D}(e,\delta_0)$. $\mathcal{E}_1(z,\beta,\theta)$ has a pole at $z_0 \in \mathcal{D}(e,\delta_0)$ if and only if $\mathcal{F}(z_0,\beta,\theta) = 0$, where

$$\mathcal{F}(z,\beta,\theta) = \det \Pi(\theta)(z - e + \beta^2 V_{21}(\theta)Q_1'(z,\beta,\theta)V_{12}(\theta))\Pi(\theta). \tag{4.2}$$

Proof. (a). Since $P_1(\theta)$ has no spectrum near e, $R_1(z,\theta)$ has a holomorphic extension in (θ, z) with $0 < \Im\theta < \alpha$ and $z \in \mathcal{D}(e,\delta_0)$ where $\delta_0 = \delta_0(\Im\theta) > 0$. For $\beta \in \mathbb{C}$ with $|\beta| < \beta_0$, $\beta_0 > 0$ small enough, one sees that as bounded operator-valued function,

$$Q_1'(z,\beta,\theta) = R_1(z,\theta)(1 - \beta^2 V_{12}(\theta)R_2'(z,\theta)V_{21}(\theta)R_1(z,\theta))^{-1}$$

is holomorphic in (z,β,θ) with $|\beta| < \beta_0$, $0 < \Im\theta < \alpha$ and $z \in \mathcal{D}(e,\delta_0)$. Using the formula

$$\begin{aligned} \mathcal{E}(z,\beta,\theta) &= R'(z,\beta,\theta) \\ &\times \begin{pmatrix} 1 & 0 \\ \beta\Pi(\theta)V_{21}(\theta)Q_1'(z,\beta,\theta) & 1 + \beta^2\Pi(\theta)V_{21}(\theta)Q_1'(z,\beta,\theta)V_{12}(\theta)R_2'(z,\theta) \end{pmatrix}, \end{aligned} \tag{4.3}$$

$\mathcal{E}(z,\beta,\theta)$ defined for z with $\Im z > 0$ and $\theta \in \mathbb{R}$ extends to a holomorphic family in (θ, z) with $0 < \Im\theta < \alpha$ and $z \in \mathcal{D}(e,\delta_0)$.

(b). Note that

$$\mathcal{E}_0(z,\beta,\theta) = \Pi(\theta)(z - e + \beta^2 V_{21}(\theta)Q_1'(z,\beta,\theta)V_{12}(\theta))\Pi(\theta)$$

is of finite rank and is holomorphic in (β,θ,z) for β near 0 in $\mathbb{C}$, θ and z as above. Therefore, its inverse as operator on the range of $\Pi(\theta)$ is meromorphic in the same region. The meromorphic extension of $\mathcal{E}_1(z,\theta)$ follows from the expression

$$\mathcal{E}_1(z,\beta,\theta) = (1 - \mathcal{E}(z,\beta,\theta)(P(\beta,\theta) - z))\Pi(\theta)\mathcal{E}_0(z,\beta,\theta)^{-1}\Pi(\theta)(1 - B(\theta)).$$

As in the proof of Theorem 3.4 (a), one can show that $z_0 \in \mathcal{D}(e,\delta_0)$ is a pole of $z \to \mathcal{E}_1(z,\beta,\theta)$ if and only if $\mathcal{F}(z_0,\beta,\theta) = 0$. ■

It follows from Proposition 4.1 that $R(z,\beta,\theta)$ has a meromorphic extension for (θ, z) with $0 < \Im\theta < \alpha$ and $z \in \mathcal{D}(e,\delta_0)$ and z_0 is a pole of $R(z,\beta,\theta)$ in $\mathcal{D}(e,\delta_0)$ if and only if $\mathcal{F}(z_0,\beta,\theta) = 0$. If $\Im z > 0$ and $\Im\theta = \Im\theta'$, $\mathcal{E}_0(z,\beta,\theta)$ and $\mathcal{E}_0(z,\beta,\theta')$ are unitarily equivalent. Consequently, $\det\mathcal{E}_0(z,\beta,\theta) = \det\mathcal{E}_0(z,\beta,\theta')$. Since $\mathcal{E}_0(z,\beta,\theta)$ is holomorphic in (θ, z), $\mathcal{F}(z,\beta,\theta)$ is independent of θ for $z \in \mathcal{D}(e,\delta_0)$ and gives a holomorphic extension of $\mathcal{F}(z,\beta,0)$ from the upper half-complex plane into $\mathcal{D}(e,\delta_0)$. Consequently, we obtain the following

Corollary 4.2. *For any $0 < \Im\theta < \alpha$ fixed, $z \to \mathcal{F}(z,\beta,0)$ defined for $\Im z > 0$ extends holomorphically into $z \in \mathcal{D}(e,\delta_0)$ and $z(\beta) \in \mathcal{D}(e,\delta_0)$ is a pole of $z \to R(z,\beta,\theta)$ if and only if $\mathcal{F}(z(\beta),\beta,0) = 0$.*

Theorem 4.3. *Assume (1.6) and that V_i, and V_{ij} are dilation analytic. Let*

$$A(z) = (< R_1(z)V_{12}\varphi_i, V_{12}\varphi_j >)_{1\le i,j\le m}, \quad \Im z > 0. \tag{4.4}$$

Then $\mathcal{F}_0(\lambda,z) = \det(A(z) - \lambda)$, $\lambda \in \mathbb{C}$, has a holomorphic extension in $z \in \mathcal{D}(e,\delta_0)$. Let $\lambda_1, \cdots, \lambda_k$ denote the zeros of $\lambda \to \mathcal{F}_0(\lambda,e)$ and ν_j the multiplicity of λ_j, $\sum_{j=1}^k \nu_j = m$. Let $\Im\theta > 0$ be fixed. For β small enough, $R(z,\beta,\theta)$ has in all m poles, counted according to their multiplicity, in $\mathcal{D}(e,\delta_0)$ and for each j, there are ν_j ones verifying $z(\beta) = e - \beta^2\lambda_j + O(\beta^{2+\frac{2}{\nu_j}})$.

Proof. For $\Im z > 0$, the eigenvalues of $A(z)$ are the same as those of the operator $\Pi V_{21}R_1(z)V_{12}\Pi$ on the range of Π. Therefore,

$$\mathcal{F}_0(\lambda,z) = \det(A(z) - \lambda) = \det \Pi(V_{21}R_1(z)V_{12} - \lambda)\Pi.$$

By dilation analyticity of V, one sees as above that for fixed θ with $\Im\theta > 0$, $\mathcal{E}_0(z,\beta,\theta)$ extends holomorphically in (z,β) for β near 0 and $z \in \mathcal{D}(e,\delta_0)$, satisfying

$$\mathcal{E}_0(z,\beta,\theta) = \Pi(\theta)(z - e + \beta^2 V_{21}(\theta)R_1(z,\theta)V_{12}(\theta) + O(\beta^4))\Pi(\theta). \tag{4.5}$$

It follows that

$$\mathcal{F}(z,\beta,\theta) = \beta^2(\mathcal{F}_0((e-z)\beta^{-2}, z, \theta) + O(\beta^2))$$

where

$$\mathcal{F}_0(\lambda,z,\theta) = \det \Pi(\theta)(V_{21}(\theta)R_1(z,\theta)V_{12}(\theta) - \lambda)\Pi(\theta). \tag{4.6}$$

Note that $\mathcal{F}_0(\lambda,z,\theta)$ defined for $\Im z > 0$ and $\theta \in \mathbb{R}$ extends holomorphically in (z,θ) for $0 < \Im\theta < \alpha$ and $z \in \mathcal{D}(e,\delta_0)$ and is independent of θ. It gives a natural holomorphic extension of $\mathcal{F}_0(\lambda,z)$ from $\Im z > 0$ into $\mathcal{D}(e,\delta_0)$. Let $F(w,\beta) = \beta^{-2}\mathcal{F}(e - \beta^2 w, \beta, 0)$. Then,

$$F(w,\beta) = \det(A(e - \beta^2 w) - w) + O(\beta^2), \tag{4.7}$$

uniformly in w with $|w - e\beta^{-2}| < \delta_0|\beta|^{-2}$. Remark that w is a zero of $F(w,0)$ if and only if it is an eigenvalue of $A(e)$. Let λ_j, $j = 1, \cdots, k$, be eigenvalues of $A(e)$ with multiplicity ν_j. In a small neighborhood of λ_j,

$$F(w,0) = (w - \lambda_j)^{\nu_j} G_j(w),$$

where $G_j(w)$ is holomorphic and $G_j(w) \neq 0$ for z near λ. Since

$$F(w,\beta) = (w - \lambda_j)^{\nu_j} G_j(w) + O(\beta^2)$$

for w in any β-independent compact domain, Rouché's Theorem shows that for β small enough, the equation $F(w,\beta) = 0$ in w has exactly ν_j solutions (counted according to their multiplicity), $w_{l,j}(\beta)$, near λ_j satisfying $w_{l,j}(\beta) = \lambda_j + O(|\beta|^{2/\nu_j})$. By (4.7), all zeros of $z \to \mathcal{F}(z,\beta,0) = \beta^2 F((e-z)\beta^{-2}, \beta)$ in $\mathcal{D}(e,\delta_0)$ are of the form $z_{l,j}(\beta) = e - \beta^2\lambda_j + O(|\beta|^{2+2/\nu_j})$, , $1 \le j \le k$, $1 \le l \le \nu_j$. According to Corollary 4.2, these $z_{l,j}(\beta)$ are all the poles of $R(z,\beta,\theta)$ in $\mathcal{D}(e,\delta_0)$. ∎

As a consequence of Theorem 4.3, if the matrix $A(e)$ has no real eigenvalues, all the poles of $z \to R(z, \beta, \theta)$ in $\mathcal{D}(e, \delta_0)$ are resonances of $P(\beta)$. By following the analyticity in β, one can show the resonances of $P(\beta)$ near an eigenvalue, λ_0, of multiplicity ν of the matrix $A(e)$ are given by Puiseux series of the form

$$z(\beta) = e - \beta^2 \lambda_0 + \sum_{k=1}^{\infty} C_k \beta^{2+\frac{2k}{\nu}}.$$

This is to compare with the scalar case (see [10, 11]) where the similar Puiseux series for resonances are of the form $e + \sum_{k=1}^{\infty} B_k \beta^{\frac{k}{\nu}}$. This difference may be explained in noticing that the poles of the resolvent of two channel operators are formally determined by the operator-valued function $z \to (P_2 - z - \beta^2 V_{21} R_1(z) V_{12})^{-1}$.

Concerning the resonances generated by eigenvalues embedded at the thresholds, we only give a result corresponding to Theorem 3.7. The following lemma may be known and we only sketch its proof.

Lemma 4.4. *Assume V_1 is dilation analytic, $\rho_1 > 2$ and $d \geq 3$. Then 0 is not an accumulating point of the resonances of P_1, that is, for any θ with $0 < \Im\theta < \alpha$, there exists $\epsilon_0 > 0$ such that there is no resonance of P_1 in $\{z \in \mathbb{C}; -2\Im\theta < \arg z < 0, 0 < \Re z < \epsilon_0\}$.*

Proof. Under the conditions of Lemma, $F(\theta) = U(\theta)^{-1} G_0 V_1 U(\theta)$ defined for θ real extends to a holomorphic family of compact operators in $\mathcal{L}(-s, -s)$, $1 < s < \rho_1/2$, for $\theta \in \mathbb{C}$ with $|\Im\theta| < \alpha$. The dimension, $m \geq 0$, of kernel of $1 + F(\theta)$ is independent of θ. Note that $(-e^{-2\theta}\Delta - z)^{-1}$ is uniformly bounded in $\mathcal{L}(s, -s)$, $s > 1$, for $z \in \mathcal{D}_0$, where

$$\mathcal{D}_0 = \{z; \Im(e^{2\theta} z) > 0,\ |\Re z| < \delta_0\}$$

for some $\delta_0 > 0$ small enough. Studying a Grushin problem for $1 + (-e^{-2\theta}\Delta - z)^{-1} V_1(\theta)$ constructed on the kernel of $1 + F(\theta)$ for $z \in \mathcal{D}_0$, one can show that $z \to (1 + (-e^{-2\theta}\Delta - z)^{-1} V_1(\theta))^{-1}$ has at most m poles in $\mathcal{D}_0$ as operator from $L^{2,s}$ to $L^{2,-s}$, $s > 1$. The same is true for $z \to (P_1(\theta) - z)^{-1} = (1 + (-e^{-2\theta}\Delta - z)^{-1} V_1(\theta))^{-1} (-e^{-2\theta}\Delta - z)^{-1}$. In particular, there are no resonances of P_1 in $\{z \in \mathbb{C}; -2\Im\theta < \arg z < 0,\ 0 < \Re z < \epsilon_0\}$ if $\epsilon_0 > 0$ is taken small enough. ■

To study the resonances of $P(\beta)$ near 0 induced by the zero eigenvalue of P_2, we keep the notation and conditions of Theorem 3.7 and assume in addition that $d = 3$, $\rho_1 > 3$ and that V is dilation analytic. The case $d \geq 4$ can be studied in the same way. Since zero is a regular point of P_1, as operators from $L^{2,s}$ to $L^{2,-s}$, $s > \frac{3}{2}$, one has the asymptotic expansions

$$\begin{aligned}
(-\Delta - z)^{-1} &= G_0 + i\sqrt{z} G_1 + o(|z|^{\frac{1}{2}}), && (4.8)\\
R_1(z) &= R_1^{(0)} + i\sqrt{z} R_1^{(1)} + o(|z|^{\frac{1}{2}}), \quad \text{with} && (4.9)\\
R_1^{(0)} &= G_0(1 + V_1 G_0)^{-1}, \quad R_1^{(1)} = (1 + G_0 V_1)^{-1} G_1 (1 + V_1 G_0)^{-1}
\end{aligned}$$

for $\Im z > 0$ and $\Re z$ small. Here the branch of $\sqrt{z}$ is chosen such that its imaginary part is positive when $\Im z > 0$. Set $a = (a_1, \cdots, a_m) \in \mathbb{C}^m$ where

$$a_j = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}^3} ((1 + V_1 G_0)^{-1} V_{12} \varphi_j)(x)\, dx, j = 1, \cdots, m. \tag{4.10}$$

Let $a^\perp$ denote the subspace of $\mathbb{C}^m$ orthogonal to a. a is identified with an $m \times 1$ matrix.

Theorem 4.5. *Under the conditions of Theorem 3.7, assume in addition that $d = 3$, $\rho_1 > 3$ and that V is dilation analytic. Let λ_0 be a negative eigenvalue of multiplicity k of the matrix A_1 defined by (3.18). Assume that*

$$E_0 \cap a^\perp = \{0\} \tag{4.11}$$

where E_0 is the eigenspace of A_1 associated with λ_0. Then, there exists $\epsilon > 0$ and $\beta_0 > 0$ such that for $\beta \in]0, \beta_0]$, $P(\beta)$ has exactly k resonances, $z_j(\beta)$, in a small disk $\{|z + \lambda_0 \beta^2| < \epsilon \beta^2\}$ verifying

$$z_j(\beta) = -\lambda_0 \beta^2 - i c_j \beta^3 + o(\beta^3) \tag{4.12}$$

with $c_j > 0$ for $1 \le j \le k$.

Proof. Although 0 is always an essential singularity of $R_1(z, \theta)$, Lemma 4.4 and the condition on P_1 imply that for $0 \le \Im\theta < \alpha$, $z \to R_1(z, \theta)$ has no pole for $z \in \mathcal{D}_0$ and is uniformly bounded in $\mathcal{L}(s, -s)$ for $s > 1$. Therefore for $\beta > 0$ small, we can make the same reduction as in Theorem 4.3 with $e = 0$ and obtain (4.1) for $z \in \mathcal{D}_0$. The same proof as Corollary 3.3 shows that for $s > 1$

$$\begin{aligned}
\mathcal{E}(z, \beta, \theta) : &\quad L^{2,s} \times L^2 \to H^{2,-s}(\mathbb{R}^d) \times H^2(\mathbb{R}^d) \\
\mathcal{E}_+(z, \beta, \theta) : &\quad \mathbb{C}^m \to H^{2,-s}(\mathbb{R}^d) \times H^2(\mathbb{R}^d) \\
\mathcal{E}_-(z, \beta, \theta) : &\quad L^{2,s}(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to \mathbb{C}^m \\
\mathcal{E}_{+-}(z, \beta, \theta) : &\quad \mathbb{C}^m \to \mathbb{C}^m
\end{aligned}$$

are uniformly bounded in $z \in \mathcal{D}_0$ and β near 0 for each fixed θ, and are holomorphic in (θ, z) for θ with $\Im\theta > 0$ and $z \in \mathcal{D}_0$ for each fixed β. The proof of Theorem 4.3 shows that as operator in $\mathcal{L}(s, -s)$ with $s > 1$, z_0 is a pole of $z \to (P(\theta) - z)^{-1}$ in $\mathcal{D}_0$ if and only if z_0 is a zero of $\mathcal{F}(z, \beta, \theta) = \det \Pi(\theta)(z + \beta^2 V_{21}(\theta) Q_1'(z, \beta, \theta) V_{12}(\theta))\Pi(\theta)$. Since $\mathcal{F}(z, \beta, \theta) = \mathcal{F}(z, \beta, 0)$ for $\Im z > 0$, the zeros of $\mathcal{F}(z, \beta, \theta)$ and $\mathcal{F}(z, \beta, 0)$ in $\mathcal{D}_0$ are the same. By (4.9),

$$\mathcal{F}(z, \beta, 0) = \det\{z + \beta^2(A_1 + i z^{1/2} a a^* + o(|z|^{1/2})) + O(|\beta|^4)\}$$

for $z \in \mathcal{D}_0$ and β small. Now let $\lambda_0 < 0$ be an eigenvalue of multiplicity k of A_1. Since A_1 and aa^* are self-adjoint, $A_1 + i z^{1/2} a a^*$ has k eigenvalues, $\lambda_j(z)$, near λ_0 which are holomorphic in the parameter $\kappa = i z^{1/2}$ near 0. One has

$$\lambda_j(z) = \lambda_0 + i z^{1/2} \nu_j + O(|z|), \quad z \to 0. \tag{4.13}$$

From (4.11), one deduces that $\nu_j = | < a, u_0 > |^2 > 0$, where u_0 is an eigenvector of A_1 associated with λ_0. Since $z \to \mathcal{F}(z, \beta, 0)$ is holomorphic and verifies $\mathcal{F}(z, \beta, 0) =$

$\det\{z + \beta^2(A_1 + iz^{1/2}aa^*)\} + o(\beta^2|z|^{\frac{1}{2}})$ in $\mathcal{D}_0$, it can be shown that in a small disk $\{|z+\lambda_0\beta^2| < \epsilon\beta^2\}$, $\mathcal{F}(z,\beta,0)$ has k zeros $z_j(\beta)$, $j = 1, \cdots, k$, verifying

$$z_j(\beta) = -\lambda_0\beta^2 - i\nu_j\sqrt{-\lambda_0}\beta^3 + o(\beta^3)$$

as $\beta \to 0$. (4.12) is proved with $c_j = \nu_j\sqrt{-\lambda_0} > 0$. ■

When (4.11) fails, one has to study high order terms in β of the function $z \to \mathcal{F}(z,\beta,0)$ to see if its zeros near $-\lambda_0\beta^2$ are real.

5. Absence of embedded eigenvalues near E_0

In this section, we give a result in the case where the off-diagonal part is stronger at the infinity compared with the diagonal part of the potential. Let $\beta = 1$ and $P = P(1)$. In addition to (1.6), (1.7) with $\rho_0 > \frac{1}{2}$ and $\rho_j > 1$, $j = 1, 2$, and (2.4) on V_1 with $m = 3$, we assume that outside some compact of $\mathbb{R}^d$, for some constants $0 < c_1 < c_2$

$$c_1\langle x\rangle^{-\rho_0} \le |V_{12}(x)| \le c_2\langle x\rangle^{-\rho_0}, \tag{5.1}$$

and

$$|V_2(x)| \le C_0|V_{12}(x)|^2, \quad C_0 < \frac{1}{E_0}. \tag{5.2}$$

We also need the following condition to control some commutators:

$$|\partial_x^\alpha V_{12}(x)| \le C\langle x\rangle^{-\rho_0-|\alpha|}, \quad |\alpha| \le 2 \tag{5.3}$$

for x outside some compact. Note that V_{12} may be complex-valued and P_0 may have an infinite number of eigenvalues embedded in $]0, E_0[$ accumulating at E_0 from below. The following result shows that most of them disappear under the off-diagonal perturbation.

Theorem 5.1. *Under the above assumptions on V, E_0 is not an accumulating point of eigenvalues of P.*

Proof. Suppose by absurd that there is an infinite number of eigenvalues $\lambda_j \in]0, E_0[$, $j = 0, 1, 2\cdots$, such that

$$0 < \lambda_0 \le \lambda_1 \le \cdots, \quad \lambda_j \to E_0, \quad j \to \infty.$$

Let $u_j = (v_j, w_j)$ be associated eigenfunctions. Using the Mourre's estimate for P_1 at E_0, one can show as in the proof of Lemma 2.1 that $\lim_{j\to\infty} v_j = 0$. We can thus normalize u_j by $\|w_j\| = 1$ and $w_j \rightharpoonup 0$ as $j \to \infty$. Equation $(P - \lambda_j)u_j = 0$ is equivalent with

$$(P_2(\lambda_j)w_j = 0, \quad v_j = -R_1(\lambda_j)(V_{12}w_j), \tag{5.4}$$

where $P_2(\lambda_j) = P_2 - K(\lambda_j) - \lambda_j$ and

$$\begin{aligned} R_1(\lambda_j)(V_{12}w_j) &= R_1(\lambda_j \pm i0)(V_{12}w_j), \\ K(\lambda_j)w_j &= V_{21}R_1(\lambda_j)(V_{12}w_j). \end{aligned}$$

Remark that as in Section 2, the choice of sign in $R_1(\lambda_j \pm i0)$ is irrelevant here. This fact combined with microlocal resolvent estimates of [22] allows us to estimate various terms below resulting from $R_1(\lambda_j)(V_{12}w_j)$. Since the argument is the same as that used in the proof of Lemma 2.3, we will not repeat it here. Let $\epsilon_0 = \min\{\rho_1 - 1, 2\rho_0 - 1\}$. One has

$$\begin{aligned} 0 &= < P_2(\lambda_j)w_j, w_j > \\ &= \|\nabla w_j\|^2 + (E_0 - \lambda_j) + < V_2 w_j, w_j > - < K(\lambda_j)w_j, w_j > \\ &> \|\nabla w_j\|^2 + < V_2 w_j, w_j > - < K(\lambda_j)w_j, w_j > . \end{aligned} \tag{5.5}$$

Since

$$- < K(\lambda_j)w_j, w_j > = < v_j, V_{12}w_j > = \lambda_j \|v_j\|^2 - < v_j, P_1 v_j >,$$

one gets

$$\|\nabla w_j\|^2 + < V_2 w_j, w_j > + \lambda_j \|v_j\|^2 - < v_j, P_1 v_j > < 0 \tag{5.6}$$

for all j. The main task is to estimate $\|v_j\|^2$ and $< v_j, P_1 v_j >$ uniformly in j large.

To estimate $\|v_j\|^2$, take $\chi \in C_0^\infty(] - \delta, \delta[)$ with $0 \le \chi \le 1$ and $\chi(s) = 1$ on $[-\delta/2, \delta/2]$, $0 < \delta < \lambda_0/2$. One has

$$\begin{aligned} \|v_j\|^2 &\ge \|\chi(P_1)v_j\|^2 = \|\chi(P_1)R_1(\lambda)(V_{12}w_j)\|^2 \\ &\ge \frac{1}{|\lambda_j - \delta|^2}\|\chi(P_1)(V_{12}w_j)\|^2 \\ &= \frac{1}{|\lambda_j - \delta|^2}(\|V_{12}w_j\|^2 - | < (1 - \chi(P_1)^2)(V_{12}w_j), V_{12}w_j > |). \end{aligned} \tag{5.7}$$

Since $1 - \chi^2$ is equal to 0 over $[-\delta/2, \delta/2]$, from the relation

$$(1 - \chi(P_1)^2)P_1^{-1} = (1 - \chi(P_1)^2)(P_1 + i)^{-1} - i(1 - \chi(P_1)^2)P_1^{-1}(P_1 + i)^{-1}$$

one deduces easily that

$$\|(1 - \chi(P_1)^2)P_1^{-1}\|_{\mathcal{L}(L^2, H^2)} \le \frac{C}{\delta}, \tag{5.8}$$

for some $C > 0$ independent of δ. Making use of the equation $P_2(\lambda_j)w_j = 0$, one obtains

$$\begin{aligned} P_1(V_{12}w_j) &= [-\Delta, V_{12}]w_j + V_{12}P_1 w_j \\ &= [-\Delta, V_{12}]w_j + V_{12}((V_1 - V_2) + K(\lambda_j) - E_0 + \lambda_j)w_j. \end{aligned}$$

Lemma 2.3 shows that

$$\||V_{12}|^2 R_1(\lambda_j)(V_{12}w_j)\| \le C\|\langle x\rangle^{-2\rho_0+1}V_{12}w_j\|$$

uniformly in j. Since $[-\Delta, V_{12}] = O(\langle x\rangle^{-1-\rho_0})\nabla + O(\langle x\rangle^{-2-\rho_0})$, it follows from (5.8) that

$$\begin{aligned} &| < (1 - \chi(P_1)^2)(V_{12}w_j), V_{12}w_j > | \\ &\le \frac{C_1}{\delta^2}(\|\langle x\rangle^{-\rho_0-1}w_j\|^2 + \|\langle x\rangle^{-2\rho_0+1}V_{12}w_j\|^2 + (E_0 - \lambda_j)^2\|(V_{12}w_j)\|^2), \end{aligned}$$

where C_1 is independent of δ and j. Since $\lambda_j \to E_0$, this proves that for any $\epsilon > 0$,

$$| < (1 - \chi(P_1)^2)(V_{12}w_j), V_{12}w_j > | \le \frac{C_1}{\delta^2}(\epsilon\|V_{12}w_j\|^2 + C\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2)$$

for all $j \geq j_0$ large enough. It follows from (5.7) that for $j \geq j_0$,

$$\|v_j\|^2 \geq \frac{1}{|\lambda_j - \delta|^2}\left((1 - \frac{C_1\epsilon}{\delta^2})\|V_{12}w_j\|^2 - C_\delta\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2\right). \tag{5.9}$$

To estimate the term $< P_1v_j, v_j >$ uniformly in j large, remark that for any $\nu > 0$, decomposing $\langle x\rangle^{-\nu}$ into two parts according to whether $|x|$ is sufficiently large or not, and applying (2.8) to the compactly support part with appropriately chosen sign $\pm$, one can show as in Lemma 2.3 that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\|\langle x\rangle^{-\nu}v_j\| \leq \epsilon\|v_j\| + C_\epsilon\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\| \tag{5.10}$$

for all j. Using (5.4), we can write $< v_j, P_1v_j >$ as

$$\begin{aligned} &< v_j, P_1v_j > \qquad (5.11)\\ &= (\lambda_j - E_0)\|v_j\|^2 + < v_j, R_1(\lambda_j)\left\{([V_{12}, \Delta] + V_{12}(V_1 - V_2)\right\}w_j + |V_{12}|^2v_j > . \end{aligned}$$

By (2.8) and (5.10),

$$\begin{aligned} | < v_j, |V_{12}|^2v_j > | &\leq \epsilon\|v_j\|^2 + C_\epsilon\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2,\\ | < v_j, R_1(\lambda_j)V_{12}(V_1 - V_2)w_j > | &\leq \epsilon\|v_j\|^2 + C_\epsilon\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2. \end{aligned}$$

It remains to study the term $< v_j, R_1(\lambda_j)[V_{12}, \Delta]w_j >$. Introduce the cut-offs $\chi_1 \in C^\infty(\mathbb{R}^d)$ with support outside some ball of radius R with R large enough, and $\chi_2 \in C_0^\infty(]E_0 - \delta, E_0 + \delta[)$, $\delta > 0$, which are equal to 1 in a slightly smaller domain. Then for all $j \geq j_0$ large enough such that $\lambda_j > E_0 - \delta/2$, one has

$$\begin{aligned} | < v_j, (1 - \chi_2(P_1))R_1(\lambda_j)[V_{12}, \Delta]w_j > | &\leq \epsilon\|v_j\|^2 + C_{\epsilon,\delta}\|\langle x\rangle^{-1-\rho_0}w_j\|^2,\\ | < v_j, \chi_2(P_1)R_1(\lambda_j)(1 - \chi_1(x))[V_{12}, \Delta]w_j > | &\leq \epsilon\|v_j\|^2 + C_\epsilon\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2. \end{aligned}$$

Consider the term

$$< v_j, \chi_2(P_1)R_1(\lambda_j)\chi_1(x)[V_{12}, \Delta]w_j > .$$

By the assumptions on V_{12}, $V_{12}(x) \neq 0$ for $|x| > R$ and $R > 1$ large enough and

$$[V_{12}, \Delta] = \nabla O(\langle x\rangle^{-1})V_{12} + O(\langle x\rangle^{-2-\rho_0}).$$

One has

$$\chi_1(x)[V_{12}, \Delta]w_j = \chi_1(x)O(\langle x\rangle^{-1})\nabla(P_1 - \lambda_j)v_j + O(\langle x\rangle^{-2-\rho_0})w_j.$$

The term related to $O(\langle x\rangle^{-2-\rho_0})w_j$ is estimated by

$$| < v_j, \chi_2(P_1)R_1(\lambda_j)O(\langle x\rangle^{-2-\rho_0})w_j > | \leq \epsilon\|v_j\|^2 + C_\epsilon\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2.$$

Since $\chi_2(P_1)\nabla$ is bounded and χ_1 is supported far away from 0, one can write

$$\begin{aligned} &\chi_2(P_1)R_1(\lambda_j)\chi_1(x)O(\langle x\rangle^{-1})\nabla(P_1 - \lambda_j)v_j \qquad (5.12)\\ &= \chi_2(P_1)(\chi_1(x)O(\langle x\rangle^{-1})\nabla v_j + R_1(\lambda_j)[\chi_1(x)O(\langle x\rangle^{-1})\nabla, P_1]v_j. \end{aligned}$$

Note that $[\chi_1(x)O(\langle x\rangle^{-1})\nabla, P_1]$ is a second order differential operator with coefficients decaying like $O(|x|^{-2})$. The method of the proof for Lemma 2.3 allows us to show that

$$| < v_j, \chi_2(P_1)R_1(\lambda_j)[\chi_1(x)O(\langle x\rangle^{-1})\nabla, P_1]v_j > | \leq C\|\langle x\rangle^{-\epsilon_0}v_j\|^2,$$

uniformly in j. It follows that

$$\begin{aligned}
&|<v_j,\chi_2(P_1)R_1(\lambda_j)\chi_1(x)O(\langle x\rangle^{-1})\nabla(P_1-\lambda_j)v_j>|\\
\le\ & \epsilon\|v_j\|^2+C_\epsilon\|\langle x\rangle^{-1}v_j\|^2+C\|\langle x\rangle^{-\epsilon_0})v_j\|^2\\
\le\ & 2\epsilon\|v_j\|^2+C'_\epsilon\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2,
\end{aligned}$$

uniformly in j. Summing up, we have proved that for any $\epsilon>0$, there exists C_ϵ and j_0 such that

$$|<v_j,P_1v_j>|\le\epsilon\|v_j\|^2+C_\epsilon\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2,\quad \forall j\ge j_0. \tag{5.13}$$

Since $\lambda_j\to E_0>0$ as $j\to\infty$, choosing appropriately $\epsilon,\delta>0$, we obtain from (5.6), (5.9) and (5.13) that for any $\eta>0$, there exists C_η and j_1 such that

$$\begin{aligned}
0 &> \|\nabla w_j\|^2+<V_2w_j,w_j>+\lambda_j\|v_j\|^2-<v_j,P_1v_j>\\
&\ge \|\nabla w_j\|^2+<V_2w_j,w_j>+(\lambda_j-\epsilon)\|v_j\|^2-C_\epsilon\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2\\
&\ge \|\nabla w_j\|^2+<V_2w_j,w_j>\\
&\quad+\frac{\lambda_j-\epsilon}{|\lambda_j-\delta|^2}\{(1-\frac{C_1\epsilon}{\delta^2})\|V_{12}w_j\|^2-C_\delta\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2\}-C_\epsilon\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2\\
&\ge \|\nabla w_j\|^2+<V_2w_j,w_j>+\frac{1-\eta}{E_0}\|V_{12}w_j\|^2-C_\eta\|\langle x\rangle^{-\epsilon_0-\rho_0}w_j\|^2,
\end{aligned}$$

for all $j\ge j_1$. For $0<\eta<\frac{1}{E_0}-C_0$ small enough, the conditions (5.1) and (5.2) show that

$$Q(x)=V_2(x)+\frac{1-\eta}{E_0}|V_{12}(x)|^2-C_\eta\langle x\rangle^{-2\epsilon_0-2\rho_0}>0 \tag{5.14}$$

for $|x|$ large enough. The above estimate says that

$$<(-\Delta+Q(x))w_j,w_j><0,\quad \forall j\ge j_1. \tag{5.15}$$

This is impossible since by (5.14) the operator $-\Delta+Q(x)$ can only have a finite number of negative eigenvalues, which implies that the space $\{v\in H^2;<(-\Delta+Q(x))v,v><0\}$ is of finite dimension. This proves that E_0 can not be an accumulating point of eigenvalues of P. ■

We expect that if the condition (5.1) is satisfied globally (for all x in $\mathbb{R}^d$), $P(\beta)$ has no eigenvalues in $]0,E_0[$ for $\beta>1$ large enough. But we do not go further in this direction, because this needs to develop other techniques. The results of Sections 3 and 5 show that the off-diagonal perturbation tends to eliminate eigenvalues of P_0 embedded between the two thresholds of two-channel type operators.

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