A COUNTER EXAMPLE TO THE BUELER'S CONJECTURE.

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Abstract. We give a counter example to a conjecture of E. Bueler stating the equality between the DeRham cohomology of complete Riemannian manifold and a weighted L^2 cohomology where the weight is the heat kernel.

1. Introduction

1.1. **Weighted L^2 cohomology :** We first describe weighted L^2 cohomology and the Bueler's conjecture. For more details we refer to E. Bueler's paper ([2] see also [5]).

Let (M,g) be a complete Riemannian manifold and $h \in C^\infty(M)$ be a smooth function, we introduce the measure μ :

$$d\mu(x) = e^{2h(x)} d\operatorname{vol}_g(x)$$

and the space of L^2_μ differential forms :

$$L^2_\mu(\Lambda^k T^*M) = \{\alpha \in L^2_{loc}(\Lambda^k T^*M), \|\alpha\|^2_\mu := \int_M |\alpha|^2(x) d\mu(x) < \infty\}.$$

Let $d^*_\mu = e^{-2h} d^* e^{2h}$ be the formal adjoint of the operator $d : C^\infty_0(\Lambda^k T^*M) \to L^2_\mu(\Lambda^{k+1}T^*M)$. The k^{th} space of (reduced) L^2_μ cohomology is defined by :

$$\mathbb{H}^k_\mu(M,g) = \frac{\{\alpha \in L^2_\mu(\Lambda^k T^*M), d\alpha = 0\}}{\overline{dC^\infty_0(\Lambda^{k-1}T^*M)}} = \frac{\{\alpha \in L^2_\mu(\Lambda^k T^*M), d\alpha = 0\}}{\overline{d\mathcal{D}^{k-1}_\mu(d)}}$$

where we take the L^2_μ closure and $\mathcal{D}^{k-1}_\mu(d)$ is the domain of d, that is the space of forms $\alpha \in L^2_\mu(\Lambda^{k-1}T^*M)$ such that $d\alpha \in L^2_\mu(\Lambda^k T^*M)$. Also if $\mathcal{H}^k_\mu(M) = \{\alpha \in L^2_\mu(\Lambda^k T^*M), d\alpha = 0, d^*_\mu\alpha = 0\}$ then we also have $\mathcal{H}^k_\mu(M) \simeq \mathbb{H}^k_\mu(M)$. Moreover if the manifold is compact (without boundary) then the celebrate Hodge-deRham theorem tells us that these two spaces are isomorphic to $H^k(M,\mathbb{R})$ the real cohomology groups of M. Concerning complete Riemannian manifold, E. Bueler had made the following interesting conjecture [2] :

Conjecture : *Assume that (M,g) is a connected oriented complete Riemannian manifold with Ricci curvature bounded from below. And consider for $t > 0$ and $x_0 \in M$, the heat kernel $\rho_t(x,x_0)$ and the heat kernel measure $d\mu(x) = \rho_t(x,x_0) d\operatorname{vol}_g(x)$, then 0 is an isolated eigenvalue of the self adjoint operator $dd^*_\mu + d^*_\mu d$ and for any k we have an isomorphism :*

$$\mathcal{H}^k_\mu(M) \simeq H^k(M,\mathbb{R}).$$

E. Bueler had verified this conjecture in degree $k = 0$ and according to [3] it also hold in degree $k = \dim M$. About the topological interpretation of some weighted

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L^2 cohomology, there is results of Z.M .Ahmed and D. Strook and more optimal results of N. Yeganefar ([1],[5]). Here we will show that we can not hope more :

Theorem 1.1. *In any dimension n, there is a connected oriented manifold M, such that for any complete Riemannian metric on M and any smooth positive measure μ, the natural map :*

$$\mathcal{H}^k_\mu(M) \to H^k(M,\mathbb{R})$$

is not surjective for $k \neq 0, n$.

Actually the example is simple take a compact surface S of genus $g \geq 2$ and

$$\Gamma \simeq \mathbb{Z} \to \hat{S} \to S$$

be a cyclic cover of S and in dimension n, do consider $M = \mathbb{T}^{n-2} \times \hat{S}$ the product of a $(n-2)$ torus with $\hat{S}$.

2. An technical point : the growth of harmonic forms :

We consider here a complete Riemannian manifold (M^n, g) and a positive smooth measure $d\mu = e^{2h} d\,\mathrm{vol}_g$ on it.

Proposition 2.1. *Let $o \in M$ be a fixed point, for $x \in M$, let $r(x) = d(o,x)$ be the geodesic distance between o and x, $R(x)$ be the maximum of the absolute value of sectional curvature of planes in T_xM and define $m(R) = \max_{r(x)\leq R}\{|\nabla dh|(x) + R(x)\}$. There is a constant C_n depending only of the dimension such that if $\alpha \in \mathcal{H}^k_\mu(M)$ then on the ball $r(x) \leq R$:*

$$e^{h(x)}|\alpha|(x) \leq C_n \frac{e^{C_n m(2R)R^2}}{\sqrt{\mathrm{vol}(B(o,2R))}} \|\alpha\|_\mu.$$

Proof.– If we let $\theta(x) = e^{h(x)}\alpha(x)$ then θ satisfies the equation :

$$(dd^* + d^*d)\theta + |dh|^2\theta + 2\nabla dh(\theta) - (\Delta h)\theta = 0.$$

where the Hessian of h acts on k forms by :

$$\nabla dh(\theta) = \sum_{i,j} \theta_j \wedge \nabla dh(e_i, e_j)\mathrm{int}_{e_j}\theta,$$

where $\{e_i\}_i$ is a local orthonormal frame and $\{\theta_i\}_i$ is the dual frame. If $\mathcal{R}$ is the curvature operator of (M,g), the Bochner-Weitzenböck formula tells us that $(dd^* + d^*d)\theta = \nabla^*\nabla\theta + \mathcal{R}(\theta)$. Hence, the function $u(x) = |\theta|(x)$ satisfies (in the distribution sense)the subharmonic estimate :

$$\Delta u(x) \leq C_n(R(x) + |\nabla dh|(x))u(x). \tag{1}$$

Now according to L. Saloff-Coste (theorem 10.4 in [4]), on $B(o,2R) = \{r(x) < 2R\}$ the ball of radius $2R$, we have a Sobolev inequality : $\forall f \in C^\infty_0(B(o,2R))$

$$\|f\|^2_{L^{\frac{2\nu}{\nu-2}}} \leq C_n \frac{R^2 e^{c_n\sqrt{k_R}R}}{\Big(\mathrm{vol}(B(o,2R))\Big)^{2/\nu}} \|df\|^2_{L^2} \tag{2}$$

where $-k_R < 0$ is a lower bound for the Ricci curvature on the ball $B(o,4R)$ and $\nu = \max(3,n)$. With (1) and (2), the Moser iteration scheme implies that for $x \in B(o,R)$,

$$u(x) \leq C_n \frac{e^{C_n m(2R)R^2}}{\sqrt{\mathrm{vol}(B(o,2R))}} \|u\|_{L^2(B(o,2R)}.$$

From which we easily infer the desired estimate. □

3. Justification of the example and further comments

3.1. **Justification.** Now, we consider the manifold $M = \mathbb{T}^{n-2} \times \hat{S}$ which is a cyclic cover of $\mathbb{T}^{n-2} \times S$; let γ be a generator of the covering group. We assume M is endowed with a complete Riemannian metric and a smooth measure $d\mu = e^{2h} d\,\mathrm{vol}_g$. For every $k \in \{1, ..., n-1\}$ we have a k-cycle c such that $\gamma^l(c) \cap c = \emptyset$ for any $l \in \mathbb{Z} \setminus \{0\}$ and a closed k-form ψ with compact support such that $\int_c \psi = 1$ and such that $\left(\mathrm{support}\,\psi\right) \cap \left(\mathrm{support}\,(\gamma^l)^*\psi\right) = \emptyset$ for any $l \in \mathbb{Z} \setminus \{0\}$. Let $a = (a_p)_{p\in\mathbb{N}}$ be a non zero sequence of real number : then the k-form $\psi_a = \sum_{p\in\mathbb{N}} a_p (\gamma^p)^* \psi$ represents a non zero k cohomology class of M, indeed $\int_{\gamma^p c} \psi_a = a_p$. We define $R_p = max\{r(\gamma^l(x)),\ x \in c,\ l = 0, ..., p\}$, then if the deRham cohomology class of ψ_a contains $\alpha \in \mathcal{H}^k_\mu(M)$, then according to (2.1), we will have $|a_p| = \left|\int_{\gamma^p c} \alpha\right| \leq M_p \|\alpha\|_\mu$; where

$$M_p = \mathrm{vol}_g(\gamma^p(c)) C_n \frac{e^{C_n m(2R_p) R_p^2}}{\sqrt{\mathrm{vol}(B(o, 2R_p))}} \max_{r(x)\leq R_p} e^{-h(x)}.$$

As a consequence, for the sequence defined by $a_p = (M_p + 1)2^p$, ψ_a can not be represented by a element of $\mathcal{H}^k_\mu(M)$.

3.2. **Further comments.** Our counter example doesn't exclude that this conjecture hold for a complete Riemannian metric with bounded curvature, positive injectivity radius on the interior of a compact manifold with boundary.

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