L^2 HARMONICS FORMS ON NON COMPACT MANIFOLDS.

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Theses notes aimed to give an insight into some links between L^2 cohomology, L^2 harmonics forms, the topology and the geometry of complete Riemannian manifolds. This is not a survey but a choice of few topics in a very large subject.

The first part can be regard as an introduction ; we define the space of L^2 harmonics forms, of L^2 cohomology. We recall the theorems of Hodge and de Rham on compact Riemannian manifolds. However the reader is assumed to be familiar with the basic of Riemannian geometry and with Hodge theory.

According to J. Roe ([55]) and following the classification of Von Neumann algebra, we can classify problems on L^2 harmonics forms in three types. The first one (type I) is the case where the space of harmonics L^2 forms has finite dimension, this situation is the nearest to the case of compact manifolds. The second (type II) is the case where the space of harmonics L^2 forms has infinite dimension but where we have a "renormalized" dimension for instance when a discrete group acts cocompactly by isometry on the manifold; a good reference is the book of W. Lueck ([47]) and the seminal paper of M. Atiyah ([4]). The third type (type III) is the case where no renormalization procedure is available to define a kind of dimension of the space L^2 harmonics forms. Here we consider only the type I problems and at the end of the first part, we will prove a result of J. Lott which says that the finiteness of the dimension of the space of L^2 harmonics forms depends only on the geometry at infinity.

Many aspects[1] of L^2 harmonics forms will not be treated here : for instance we will not describe the important problem of the L^2 cohomology of locally symmetric spaces, and also we will not speak on the pseudo differential approach developped by R. Melrose and his school. However the reader will find at the end of this first chapter a list of some interesting results on the topological interpretation of the space of L^2 harmonics forms.

In the second chapter, we are interested in the space of harmonic L^2 $1-$forms. This space contains the differential of harmonic functions with L^2 gradient. We will not speak of the endpoint result of A. Grigory'an ([32, 31]) but we have include a study of P.Li and L-F. Tam ([39]) and of A. Ancona ([1]) on non parabolic ends. In this chapter, we will also study the case of Riemannian surfaces where this space depends only on the complex structure.

The last chapter focuses on the L^2 cohomology of conformally compact manifolds. The result is due to R. Mazzeo ([48]) and the proof present here is the one

[1] almost all in fact !

of N. Yeganefar ([65]) who used an integration by parts formula due to H.Donnelly and F.Xavier ([24]).

Contents

1. A short introduction to L^2 cohomology

In this first chapter, we introduce the main definitions and prove some preliminary results.

1.1. **Hodge and de Rham 's theorems.**

1.1.1. *de Rham 's theorem.* Let M^n be a smooth manifold of dimension n, we denote by $C^\infty(\Lambda^k T^*M)$ the space of smooth differential $k-$forms on M and by $C^\infty_0(\Lambda^k T^*M)$ the subspace of $C^\infty(\Lambda^k T^*M)$ formed by forms with compact support; in local coordinates $(x_1, x_2, ..., x_n)$, an element $\alpha \in C^\infty(\Lambda^k T^*M)$ has the following expression

$$\alpha = \sum_{I=\{i_1<i_2<...<i_k\}} \alpha_I dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k} = \sum_I \alpha_I dx_I$$

where α_I are smooth functions of $(x_1, x_2, ..., x_n)$. The exterior differentiation is a differential operator

$$d : C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M),$$

locally we have

$$d\left(\sum_I \alpha_I dx_I\right) = \sum_I d\alpha_I \wedge dx_I.$$

This operator satisfies $d \circ d = 0$, hence the range of d is included in the kernel of d.

Definition 1.1. The k^{th} de Rham's cohomology group of M is defined by

$$H^k_{dR}(M) = \frac{\left\{\alpha \in C^\infty(\Lambda^k T^*M), d\alpha = 0\right\}}{dC^\infty(\Lambda^{k-1}T^*M)}.$$

These spaces are clearly diffeomorphism invariants of M, moreover the deep theorem of G. de Rham says that these spaces are isomorphic to the real cohomology group of M, there are in fact homotopy invariant of M :

Theorem 1.2.

$$H^k_{dR}(M) \simeq H^k(M, \mathbb{R}).$$

From now, we will suppress the subscript dR for the de Rham's cohomology. We can also define the de Rham's cohomology with compact support.

Definition 1.3. The k^{th} de Rham's cohomology group with compact support of M is defined by

$$H^k_0(M) = \frac{\left\{\alpha \in C^\infty_0(\Lambda^k T^*M), d\alpha = 0\right\}}{dC^\infty_0(\Lambda^{k-1}T^*M)}.$$

These spaces are also isomorphic to the real cohomology group of M with compact support. When M is the interior of a compact manifold $\overline{M}$ with compact boundary $\partial\overline{M}$

$$M = \overline{M} \setminus \partial\overline{M},$$

then $H^k_0(M)$ is isomorphic to the relative cohomology group of $\overline{M}$:

$$H^k_0(M) = H^k(\overline{M}, \partial\overline{M}) := \frac{\left\{\alpha \in C^\infty(\Lambda^k T^*\overline{M}), d\alpha = 0, \iota^*\alpha = 0\right\}}{\left\{d\beta, \beta \in C^\infty(\Lambda^{k-1}T^*\overline{M}) \text{ and } \iota^*\beta = 0\right\}}$$

where $\iota : \partial\overline{M} \to \overline{M}$ is the inclusion map.

1.1.2. *Poincaré duality.* When we assume that M is oriented [2] the bilinear map

$$\begin{aligned} H^k(M) \times H^{n-k}_0(M) &\to \mathbb{R} \\ ([\alpha], [\beta]) &\mapsto \int_M \alpha \wedge \beta := I([\alpha], [\beta]) \end{aligned}$$

is well defined, that is to say $I([\alpha], [\beta])$ doesn't depend on the choice of representatives in the cohomology classes $[\alpha]$ or $[\beta]$ (this is an easy application of the Stokes formula). Moreover this bilinear form provides an isomorphism between $H^k(M)$ and $\left(H^{n-k}_0(M)\right)^*$. In particular when $\alpha \in C^\infty(\Lambda^k T^*M)$ is closed ($d\alpha = 0$) and satisfies that

$$\forall [\beta] \in H^{n-k}_0(M), \ \int_M \alpha \wedge \beta = 0$$

then there exists $\gamma \in C^\infty(\Lambda^{k-1}T^*M)$ such that

$$\alpha = d\gamma.$$

[2] It is not a serious restriction we can used cohomology with coefficient in the orientation bundle.

1.1.3. *L^2 cohomology.* 1.1.3 .*a) The operator d^*.* We assume now that M is endowed with a Riemannian metric g, we can define the space $L^2(\Lambda^k T^*M)$ whose elements have locally the following expression

$$\alpha = \sum_{I=\{i_1<i_2<...<i_k\}} \alpha_I dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}$$

where $\alpha_I \in L^2_{loc}$ and globally we have

$$\|\alpha\|^2_{L^2} := \int_M |\alpha(x)|^2_{g(x)} d\operatorname{vol}_g(x) < \infty.$$

The space $L^2(\Lambda^k T^*M)$ is a Hilbert space with scalar product :

$$\langle \alpha, \beta \rangle = \int_M (\alpha(x), \beta(x))_{g(x)} d\operatorname{vol}_g(x).$$

We define the formal adjoint of d :

$$d^* : C^\infty(\Lambda^{k+1}T^*M) \to C^\infty(\Lambda^k T^*M)$$

by the formula

$$\forall \alpha \in C_0^\infty(\Lambda^{k+1}T^*M) \text{ and } \beta \in C_0^\infty(\Lambda^k T^*M),$$
$$\langle d^*\alpha, \beta \rangle = \langle \alpha, d\beta \rangle \ .$$

When ∇ is the Levi-Civita connexion of g, we can give local expressions for the operators d and d^* : let $(E_1, E_2, ..., E_n)$ be a local orthonormal frame and let $(\theta^1, \theta^2, ..., \theta^n)$ be its dual frame :

$$\theta^i(X) = g(E_i, X)$$

then

$$d\alpha = \sum_{i=1}^n \theta^i \wedge \nabla_{E_i}\alpha, \tag{1.1}$$

and

$$d^*\alpha = -\sum_{i=1}^n \operatorname{int}_{E_i}(\nabla_{E_i}\alpha), \tag{1.2}$$

where we have denote by int_{E_i} the interior product with the vector field E_i.
1.1.3 .*b) L^2 harmonic forms.* We consider the space of L^2 closed forms :

$$Z_2^k(M) = \{\alpha \in L^2(\Lambda^k T^*M),\ d\alpha = 0\}$$

where it is understood that the equation $d\alpha = 0$ holds weakly that is to say

$$\forall \beta \in C_0^\infty(\Lambda^{k+1}T^*M),\ \langle \alpha, d^*\beta \rangle = 0.$$

That is we have :

$$Z_2^k(M) = \left(d^* C^\infty(\Lambda^{k+1}T^*M)\right)^\perp ,$$

hence $Z_2^k(M)$ is a closed subspace of $L^2(\Lambda^k T^*M)$. We can also define

$$\begin{aligned}\mathcal{H}^k(M) &= \left(d^* C_0^\infty(\Lambda^{k+1}T^*M)\right)^\perp \cap \left(d C_0^\infty(\Lambda^{k-1}T^*M)\right)^\perp \\ &= Z_2^k(M) \cap \{\alpha \in L^2(\Lambda^k T^*M),\ d^*\alpha = 0\} \\ &= \{\alpha \in L^2(\Lambda^k T^*M),\ d\alpha = 0 \text{ and } d^*\alpha = 0\}.\end{aligned}$$

Because the operator $d+d^*$ is elliptic, we have by elliptic regularity : $\mathcal{H}^k(M)\subset C^\infty(\Lambda^kT^*M)$. We also remark that by definition we have

$$\forall\alpha\in C_0^\infty(\Lambda^{k-1}T^*M),\ \forall\beta\in C_0^\infty(\Lambda^{k+1}T^*M)$$
$$\langle d\alpha,d^*\beta\rangle=\langle dd\alpha,\beta\rangle=0$$

Hence

$$dC_0^\infty(\Lambda^{k-1}T^*M)\perp d^*C_0^\infty(\Lambda^{k+1}T^*M)$$

and we get the Hodge-de Rham decomposition of $L^2(\Lambda^kT^*M)$

$$L^2(\Lambda^kT^*M)=\mathcal{H}^k(M)\oplus\overline{dC_0^\infty(\Lambda^{k-1}T^*M)}\oplus\overline{d^*C_0^\infty(\Lambda^{k+1}T^*M)}\ , \tag{1.3}$$

where the closures are taken for the L^2 topology. And also

$$\mathcal{H}^k(M)\simeq\frac{Z_2^k(M)}{\overline{dC_0^\infty(\Lambda^{k-1}T^*M)}}. \tag{1.4}$$

1.1.3 *.c) L^2 cohomology:* We also define the (maximal) domain of d by

$$\mathcal{D}^k(d)=\{\alpha\in L^2(\Lambda^kT^*M),\ d\alpha\in L^2\}$$

that is to say $\alpha\in\mathcal{D}^k(d)$ if and only if there is a constant C such that

$$\forall\beta\in C_0^\infty(\Lambda^{k+1}T^*M),\ |\langle\alpha,d^*\beta\rangle|\le C\|\beta\|_2.$$

In that case, the linear form $\beta\in C_0^\infty(\Lambda^{k+1}T^*M)\mapsto\langle\alpha,d^*\beta\rangle$ extends continuously to $L^2(\Lambda^{k+1}T^*M)$ and there is $\gamma=:d\alpha$ such that

$$\forall\beta\in C_0^\infty(\Lambda^{k+1}T^*M),\ \langle\alpha,d^*\beta\rangle=\langle\gamma,\beta\rangle.$$

We remark that we always have $d\mathcal{D}^{k-1}(d)\subset Z_2^k(M)$.

Definition 1.4. We define the k^{th} space of reduced L^2 cohomology by

$$H_2^k(M)=\frac{Z_2^k(M)}{\overline{d\mathcal{D}^{k-1}(d)}}.$$

The k^{th} space of non reduced L^2 cohomology is defined by

$${}^{nr}H_2^k(M)=\frac{Z_2^k(M)}{d\mathcal{D}^{k-1}(d)}.$$

These two spaces coincide when the range of $d:\mathcal{D}^{k-1}(d)\to L^2$ is closed; the first space is always a Hilbert space and the second is not necessary Hausdorff. We also have $C_0^\infty(\Lambda^{k-1}T^*M)\subset\mathcal{D}^{k-1}(d)$ hence we always get a surjective map :

$$\mathcal{H}^k(M)\to H_2^k(M)\to\{0\}.$$

In particular any class of reduced L^2 cohomology contains a smooth representative.
1.1.3 *.d) Case of complete manifolds.* The following result is due to Gaffney ([28], see also part 5 in [64]) for a related result)

Lemma 1.5. *Assume that g is a complete Riemannian metric then*

$$\overline{d\mathcal{D}^{k-1}(d)}=\overline{dC_0^\infty(\Lambda^{k-1}T^*M)}.$$

Proof. We already know that $dC_0^\infty(\Lambda^{k-1}T^*M) \subset \mathcal{D}^{k-1}(d)$, moreover using a partition of unity and local convolution it is not hard to check that if $\alpha \in \mathcal{D}^{k-1}(d)$ has compact support then we can find a sequence $(\alpha_l)_{l\in\mathbb{N}}$ of smooth forms with compact support such that

$$\|\alpha_l - \alpha\|_{L^2} + \|d\alpha_l - d\alpha\|_{L^2} \leq 1/l.$$

So we must only prove that if $\alpha \in \mathcal{D}^{k-1}(d)$ then we can build a sequence $(\alpha_N)_N$ of elements of $\mathcal{D}^{k-1}(d)$ with compact support such that

$$L^2 - \lim_{N\to\infty} d\alpha_N = d\alpha.$$

We fix now an origin $o \in M$ and denote by $B(o,N)$ the closed geodesic ball of radius N and centered at o, because (M,g) is assumed to be complete we know that $B(o,N)$ is compact and

$$M = \cup_{N\in\mathbb{N}} B(o,N).$$

We consider $\rho \in C_0^\infty(\mathbb{R}_+)$ with $0 \leq \rho \leq 1$ with support in $[0,1]$ such that

$$\rho = 1 \text{ on } [0,1/2]$$

and we define

$$\chi_N(x) = \rho\left(\frac{d(o,x)}{N}\right). \tag{1.5}$$

Then χ_N is a Lipschitz function and is differentiable almost everywhere and

$$d\chi_N(x) = \rho'\left(\frac{d(o,x)}{N}\right) dr$$

where dr is the differential of the function $x \mapsto d(o,x)$. Let $\alpha \in \mathcal{D}^{k-1}(d)$ and define

$$\alpha_N = \chi_N \alpha,$$

the support of α_N is included in the ball of radius N and centered at o hence is compact. Moreover we have

$$\|\alpha_N - \alpha\|_{L^2} \leq \|\alpha\|_{L^2(M\setminus B(o,N/2))}$$

hence

$$L^2 - \lim_{N\to\infty} \alpha_N = \alpha$$

Moreover when $\varphi \in C_0^\infty(\Lambda^{k+1}T^*M)$ we have

$$\begin{aligned}
\langle \alpha_N, d^*\varphi\rangle &= \langle \alpha, \chi_N d^*\varphi\rangle \\
&= \langle \alpha, d^*(\chi_N\varphi)\rangle + \langle \alpha, \mathrm{int}_{\overrightarrow{grad}\,\chi_N}\varphi\rangle \\
&= \langle \chi_N d\alpha + d\chi_N \wedge \alpha, \varphi\rangle
\end{aligned}$$

Hence $\alpha_N \in \mathcal{D}^{k-1}(d)$ and

$$d\alpha_N = \chi_N d\alpha + d\chi_N \wedge \alpha.$$

But for almost all $x \in M$, we have $|d\chi_N|(x) \leq \|\rho'\|_{L^\infty}/N$ hence

$$\begin{aligned}
\|d\alpha_N - d\alpha\|_{L^2} &\leq \frac{\|\rho'\|_{L^\infty}}{N}\|\alpha\|_{L^2} + \|\chi_N d\alpha - d\alpha\|_{L^2} \\
&\leq \frac{\|\rho'\|_{L^\infty}}{N}\|\alpha\|_{L^2} + \|d\alpha\|_{L^2(M\setminus B(o,N/2))}\,.
\end{aligned}$$

Hence we have build a sequence α_N of elements of $\mathcal{D}^{k-1}(d)$ with compact support such that $L^2 - \lim_{N\to\infty} d\alpha_N = d\alpha$. $\square$

A corollary of this lemma (1.5) and of (1.4) is the following :

Corollary 1.6. *When (M,g) is a complete Riemannian manifold then the space of harmonic L^2 forms computes the reduced L^2 cohomology :*

$$H_2^k(M) \simeq \mathcal{H}^k(M).$$

With a similar proof, we have another result :

Proposition 1.7. *When (M,g) is a complete Riemannian manifold then*

$$\mathcal{H}^k(M) = \{\alpha \in L^2(\Lambda^k T^*M), (dd^* + d^*d)\alpha = 0\}.$$

Proof. Clearly we only need to check the inclusion :

$$\{\alpha \in L^2(\Lambda^k T^*M), (dd^* + d^*d)\alpha = 0\} \subset \mathcal{H}^k(M).$$

We consider again the sequence of cut-off functions χ_N defined previously in (1.5). Let $\alpha \in L^2(\Lambda^k T^*M)$ satisfying $(dd^* + d^*d)\alpha = 0$ by elliptic regularity we know that α is smooth. Moreover we have :

$$\begin{aligned}
\|d(\chi_N\alpha)\|^2_{L^2} &= \int_M \left[|d\chi_N \wedge \alpha|^2 + 2\langle d\chi_N \wedge \alpha, \chi_N d\alpha\rangle + \chi_N^2|d\alpha|^2\right] d\,\mathrm{vol}_g \\
&= \int_M \left[|d\chi_N \wedge \alpha|^2 + \langle d\chi_N^2 \wedge \alpha, d\alpha\rangle + \chi_N^2|d\alpha|^2\right] d\,\mathrm{vol}_g \\
&= \int_M \left[|d\chi_N \wedge \alpha|^2 + \langle d(\chi_N^2\alpha), d\alpha\rangle\right] d\,\mathrm{vol}_g \\
&= \int_M \left[|d\chi_N \wedge \alpha|^2 + \langle \chi_N^2\alpha, d^*d\alpha\rangle\right] d\,\mathrm{vol}_g
\end{aligned}$$

Similarly we get :

$$\|d^*(\chi_N\alpha)\|^2_{L^2} = \int_M \left[|\operatorname{int}_{\overrightarrow{grad}\,\chi_N} \alpha|^2 + \langle \chi_N^2\alpha, dd^*\alpha\rangle\right] d\,\mathrm{vol}_g$$

Summing these two equalities we obtain :

$$\|d^*(\chi_N\alpha)\|^2_{L^2} + \|d^*(\chi_N\alpha)\|^2_{L^2} = \int_M |d\chi_N|^2|\alpha|^2 d\,\mathrm{vol}_g \leq \frac{\|\rho'\|^2_{L^\infty}}{N^2}\int_M |\alpha|^2 d\,\mathrm{vol}_g\,.$$

Hence when N tends to ∞ we obtain

$$\|d^*\alpha\|^2_{L^2} + \|d^*\alpha\|^2_{L^2} = 0.$$

$\square$

This proposition has the consequence that on a complete Riemannian manifold harmonic L^2 functions are closed hence locally constant. Another corollary is that the reduced L^2 cohomology of the Euclidean space is trivial[3] :

Corollary 1.8.

$$H_2^k(\mathbb{R}^n) = \{0\}.$$

[3]This can also be proved with the Fourier transform.

Proof. On the Euclidean space $\mathbb{R}^n$ a smooth k form α can be expressed as

$$\alpha = \sum_{I=\{i_1<i_2<...<i_k\}} \alpha_I dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}$$

and α will be a L^2 solution of the equation

$$(dd^* + d^*d)\alpha = 0$$

if and only if all the functions α_I are harmonic and L^2 hence zero because the volume of $\mathbb{R}^n$ is infinite. □

Remark 1.9. When (M, g) is not complete, we have not necessary equality between the space $\mathcal{H}^k(M)$ (whose elements are sometimes called harmonics fields) and the space of the L^2 solutions of the equation $(dd^* + d^*d)\alpha = 0$. For instance, on the interval $M = [0, 1]$ the space $\mathcal{H}^0(M)$ is the space of constant functions, whereas L^2 solutions of the equation $(dd^* + d^*d)\alpha = 0$ are affine. More generally, on a smooth compact connected manifold with smooth boundary endowed with a smooth Riemannian metric, then again $\mathcal{H}^0(M)$ is the space of constant functions, whereas the space $\{f \in L^2(M), d^*df = 0\}$ is the space of harmonic L^2 function; this space is infinite dimensionnal when $\dim M > 1$.

1.1.3*e) Case of compact manifolds* The Hodge's theorem says that for compact manifold cohomology is computed with harmonic forms :

Theorem 1.10. *If M is a compact Riemannian manifold without boundary then*

$$H_2^k(M) \simeq \mathcal{H}^k(M) \simeq H^k(M).$$

When M is the interior of a compact manifold $\overline{M}$ with compact boundary $\partial\overline{M}$ and when g extends to $\overline{M}$ (hence g is incomplete) a theorem of P. Conner ([19]) states that

$$H_2^k(M) \simeq H^k(M) \simeq \mathcal{H}^k_{abs}(\overline{M})$$

where

$$\mathcal{H}^k_{abs}(M) = \{\alpha \in L^2(\Lambda^k T^*M),\ d\alpha = d^*\alpha = 0 \text{ and } \operatorname{int}_{\vec{\nu}} \alpha = 0 \text{ along } \partial\overline{M}\}$$

and $\vec{\nu} : \partial\overline{M} \to T\overline{M}$ is the inward unit normal vector field. In fact when $K \subset M$ is a compact subset of M with smooth boundary and if g is a complete Riemannian metric on M then for $\Omega = M \setminus K$, we also have the equality

$$H_2^k(\Omega) \simeq \mathcal{H}^k_{abs}(\Omega)$$

where if $\vec{\nu} : \partial\Omega \to TM$ is the inward unit normal vector field, we have also denoted

$$\mathcal{H}^k_{abs}(\Omega) = \{\alpha \in L^2(\Lambda^k T^*\Omega),\ d\alpha = d^*\alpha = 0 \text{ and } \operatorname{int}_{\vec{\nu}} \alpha = 0 \text{ along } \partial\Omega\}. \tag{1.6}$$

1.2. **Some general properties of reduced L^2 cohomology.**

1.2.1. *a general link with de Rham's cohomology.* We assume that (M, g) is a complete Riemannian manifold, the following result is due to de Rham (theorem 24 in [21])

Lemma 1.11. *Let $\alpha \in Z_2^k(M) \cap C^\infty(\Lambda^k T^*M)$ and suppose that α is zero in $H_2^k(M)$ that is there is a sequence $\beta_l \in C_0^\infty(\Lambda^{k-1}T^*M)$ such that*

$$\alpha = L^2 - \lim_{l\to\infty} d\beta_l$$

then there is $\beta \in C^\infty(\Lambda^{k-1}T^*M)$ *such that*

$$\alpha = d\beta.$$

In full generality, we know nothing about the behavior of β at infinity.

Proof. We can always assume that M is oriented, hence by the Poincaré duality (1.1.2), we only need to show that if $\psi \in C_0^\infty(\Lambda^{n-k}T^*M)$ is closed then

$$\int_M \alpha \wedge \psi = 0 \ .$$

But by assumption,

$$\begin{aligned}\int_M \alpha \wedge \psi &= \lim_{l\to\infty} \int_M d\beta_l \wedge \psi \\ &= \lim_{l\to\infty} \int_M d(\beta_l \wedge \psi) \text{ because } \psi \text{ is closed} \\ &= 0.\end{aligned}$$

Hence the result. □

This lemma implies the following useful result which is due to M. Anderson ([2]):

Corollary 1.12. *There is a natural injective map*

$$\mathrm{Im}\left(H_0^k(M) \to H^k(M)\right) \to H_2^k(M).$$

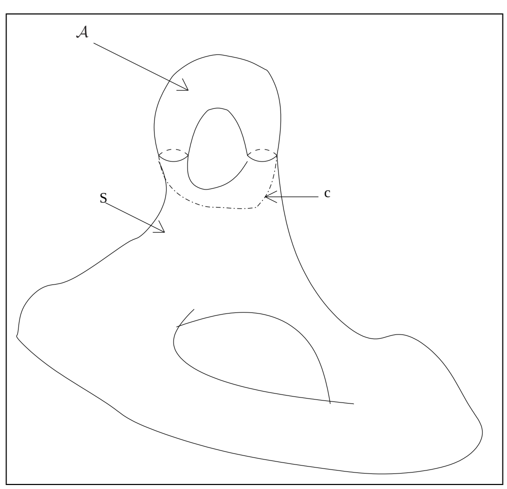
Proof. As a matter of fact we need to show that if $\alpha \in C_0^\infty(\Lambda^k T^*M)$ is closed and zero in the reduced L^2 cohomology then it is zero in usual cohomology: this is exactly the statement of the previous lemma (1.11). □

1.2.2. *Consequence for surfaces.* These results have some implications for a complete Riemannian surface (S, g) :

i) If the genus of S is infinite then the dimension of the space of L^2 harmonic 1−forms is infinite.

ii) If the space of L^2 harmonics 1−forms is trivial then the genus of S is zero and S is diffeomorphic to a open set of the sphere.

As a matter of fact, a handle of S is a embedding $f : \mathbb{S}^1 \times [-1, 1] \to S$ such that if we denote $\mathcal{A} = f(\mathbb{S}^1 \times [-1, 1])$ then $S \setminus \mathcal{A}$ is connected.



We consider now a function ρ on $\mathbb{S}^1 \times [-1,1]$ depending only on the second variable such that
$$\rho(\theta,t) = \begin{cases} 1 & \text{when } t > 1/2 \\ 0 & \text{when } t < -1/2 \end{cases}$$
then $d\rho$ is a $1-$form with compact support in $\mathbb{S}^1 \times]-1,1[$ and we can extend $\alpha = \left(f^{-1}\right)^* d\rho$ to all S ; we obtain a closed 1-form also denoted by α which has compact support in $\mathcal{A}$. Moreover because $S \setminus \mathcal{A}$ is connected, we can find a continuous path $c\,:\,[0,1] \to S \setminus \mathcal{A}$ joining $f(1,1)$ to $f(1,-1)$; we can defined the loop γ given by
$$\gamma(t) = \begin{cases} f(1,t) & \text{for } t \in [-1,1] \\ c(t-1) & \text{for } t \in [1,2]. \end{cases}$$
It is easy to check that
$$\int_\gamma \alpha = 1 \ ,$$
hence α is not zero in $H^1(S)$. A little elaboration from this argument shows that
$$\operatorname{genus}(S) \leq \dim \operatorname{Im}\left(H^1_0(M) \to H^1(M)\right).$$

1.3. **Lott's result.** We will now prove the following result due to J.Lott ([44]):

Theorem 1.13. *Assume that (M_1, g_1) and (M_2, g_2) are complete oriented manifold of dimension n which are isometric at infinity that is to say there are compact sets $K_1 \subset M_1$ and $K_2 \subset M_2$ such that $(M_1 \setminus K_1, g_1)$ and $(M_2 \setminus K_2, g_2)$ are isometric. Then for $k \in [0,n] \cap \mathbb{N}$*
$$\dim \mathcal{H}^k(M_1, g_1) < \infty \Leftrightarrow \dim \mathcal{H}^k(M_2, g_2) < \infty.$$

We will give below the proof of this result, this proof contains many arguments which will be used and refined in the next two lectures. In view of the Hodge-de Rham theorem and of J. Lott's result, we can ask the following very general questions :

(1) What are the geometry at infinity iensuring the finiteness of the dimension of the spaces $H^k_2(M)$?

   Within a class of Riemannian manifold having the same geometry at infinity :

(2) What are the links of the spaces of reduced L^2 cohomology $H^k_2(M)$ with the topology of M and with the geometry "at infinity" of (M, g) ?

There is a lot of articles dealing with these questions, I mention some of them :

(1) In the pioneering article of Atiyah-Patodi-Singer ([5]), the authors considered manifold with cylindrical end : that is to say there is a compact K of M such that $M \setminus K$ is isometric to the Riemannian product $\partial K \times]0,\infty[$. Then they show that the dimension of the space of L^2-harmonic forms is finite ; and that these spaces are isomorphic to the image of the relative cohomology in the absolute cohomology. These results were used by Atiyah-Patodi-Singer in order to obtain a formula for the signature of compact Manifolds with boundary.
(2) In [48, 50], R. Mazzeo and R.Phillips give a cohomological interpretation of the space $\mathcal{H}^k(M)$ for geometrically finite real hyperbolic manifolds. These manifolds can be compactified. They identify the reduced L^2 cohomology with the cohomology of smooth differential forms satisfying certain boundary conditions.

(3) The solution of the Zucker's conjecture by L.Saper-M.Stern and E.Looijenga ([46],[58]) shows that the spaces of L^2 harmonic forms on Hermitian locally symmetric space with finite volume are isomorphic to the middle intersection cohomology of the Baily-Borel-Satake compactification of the manifold. An extension of this result has been given by A.Nair and L.Saper ([52],[56]). Moreover recently, L. Saper obtains the topological interpretation of the reduced L^2 cohomology of any locally symmetric space with finite volume ([57]). In that case the finiteness of the dimension of the space of L^2 harmonics forms is due to A. Borel and H.Garland ([8]).
(4) According to Vesentini ([63]) if M is flat outside a compact set, the spaces $\mathcal{H}^k(M)$ are finite dimensional. J. Dodziuk asked about the topological interpretation of the space $\mathcal{H}^k(M)$ ([22]). In this case, the answer has been given in [15].
(5) In a recent paper ([33]) Tamás Hausel, Eugenie Hunsicker and Rafe Mazzeo obtain a topological interpretation of the L^2 cohomology of complete Riemannian manifold whose geometry at infinity is fibred boundary and fibred cusp (see [49, 62]). These results have important application concerning the Sen's conjecture ([34, 59]).
(6) In [45], J. Lott has shown that on a complete Riemannian manifold with finite volume and pinched negative curvature, the space of harmonic L^2 forms has finite dimension. N. Yeganefar obtains the topological interpretation of these spaces in two cases, first when the curvature is enough pinched ([65]) and secondly when the metric is Kähler ([66]).

Proof of J. Lott's result. We consider (M, g) a complete oriented Riemannian manifold and $K \subset M$ a compact subset with smooth boundary and we let $\Omega = M \setminus K$ be the exterior of K, we are going to prove that

$$\dim H_2^k(M) < \infty \Leftrightarrow \dim H_2^k(\Omega) < \infty \ ;$$

this result clearly implies Lott's result.

The co boundary map $b : H_2^k(\Omega) \to H^{k+1}(K, \partial K)$ is defined as follow: let $c = [\alpha] \in H_2^k(\Omega)$ where α is a smooth representative of c, we choose $\bar{\alpha} \in C^\infty(\Lambda^k T^*M)$ a smooth extension of α, then $d\bar{\alpha}$ is a closed smooth form with support in K and if $\iota : \partial K \to K$ is the inclusion we have $\iota^*(d\bar{\alpha}) = 0$. Some standard verifications show that

$$b(c) = [d\bar{\alpha}] \in H^{k+1}(K, \partial K)$$

is well defined, that is it doesn't depend of the choice of $\alpha \in c$ nor on the smooth extension of α.

The inclusion map $j_\Omega : \Omega \to M$ induced a linear map (the restriction map)

$$[j_\Omega^*] : H_2^k(M) \to H_2^k(\Omega).$$

Lemma 1.14. *We always have*

$$\ker b = \operatorname{Im}[j_\Omega^*].$$

Proof of lemma 1.14. First by construction we have $b \circ [j_\Omega^*] = 0$ hence we only need to prove that

$$\ker b \subset \operatorname{Im}[j_\Omega^*].$$

Let $c \in \ker b$ and let $\alpha \in Z_2^k(\Omega)$ a smooth representative of c, we know that α has a smooth extension $\bar{\alpha}$ such that $d\bar{\alpha}$ is zero in $H^{k+1}(K, \partial K)$. That is to say

there is a smooth $k-$ form $\beta \in C^\infty(\Lambda^k T^*K)$ such that
$$d\bar{\alpha} = d\beta \text{ on } K \text{ and } \iota^*\beta = 0$$
We claim that the L^2 form $\tilde{\alpha}$ defined by
$$\tilde{\alpha} = \begin{cases} \bar{\alpha} - \beta & \text{on } K \\ \alpha & \text{on } \Omega \end{cases}$$
is weakly closed. As a matter of fact, we note $\vec{\nu} : \partial K \to TM$ the unit normal vector field pointing into Ω and $\iota : \partial K \to M$ the inclusion, then let $\varphi \in C_0^\infty(\Lambda^{k+1} T^*M)$ with the Green's formula, we obtain
$$\begin{aligned}\int_M (\tilde{\alpha}, d^*\varphi) &= \int_K (\bar{\alpha} - \beta, d^*\varphi) + \int_\Omega (\alpha, d^*\varphi) \\ &= -\int_{\partial K} (\iota^*(\bar{\alpha} - \beta), \operatorname{int}_{\vec{\nu}} \varphi) d\sigma + \int_{\partial K} (\iota^*\bar{\alpha}, \operatorname{int}_{\vec{\nu}} \varphi) d\sigma \\ &= 0.\end{aligned}$$
We clearly have $j_\Omega^*\tilde{\alpha} = \alpha$, hence $c = [\alpha] \in \operatorname{Im}[j_\Omega^*]$. □

Now because $H^{k+1}(K, \partial K)$ has finite dimension, we know that
$$\dim \operatorname{Im}[j_\Omega^*] < \infty \Leftrightarrow \dim H_2^k(\Omega) < \infty.$$
Hence we get the implication :
$$\dim H_2^k(M) < \infty \Rightarrow \dim H_2^k(\Omega) < \infty.$$
To prove the reverse implication, we consider the reduced L^2 cohomology of Ω relative to the boundary $\partial\Omega = \partial K$. We introduce
$$\begin{aligned} Z_2^k(\Omega, \partial\Omega) &= \left\{\alpha \in L^2(\Lambda^k T^*\Omega), \text{ such that } \forall \varphi \in C_0^\infty(\Lambda^k T^*\overline{\Omega}), \langle \alpha, d^*\varphi\rangle = 0\right\} \\ &= \left(d^* C_0^\infty(\Lambda^k T^*\overline{\Omega})\right)^\perp. \end{aligned}$$
We remark here that the elements of $C_0^\infty(\Lambda^k T^*\overline{\Omega})$ have compact support in $\overline{\Omega}$ in particular their support can touch the boundary; in fact a smooth L^2 closed $k-$form belongs to $Z_2^k(\Omega, \partial\Omega)$ if and only its pull-back by ι is zero. This is a consequence of the integration by part formula
$$\int_\Omega (\alpha, d^*\varphi) d\operatorname{vol}_g = \int_\Omega (d\alpha, \varphi) d\operatorname{vol}_g + \int_{\partial\Omega} (\iota^*\alpha, \operatorname{int}_{\vec{\nu}} \varphi) d\sigma$$
where $d\sigma$ is the Riemannian volume on $\partial\Omega$ induced by the metric g and $\vec{\nu} : \partial\Omega \to T\Omega$ is the unit inward normal vector field.

We certainly have $dC_0^\infty(\Lambda^k T^*\Omega) \subset Z_2^k(\Omega, \partial\Omega)$ and we define
$$H_2^k(\Omega, \partial\Omega) = \frac{Z_2^k(\Omega, \partial\Omega)}{\overline{dC_0^\infty(\Lambda^k T^*\Omega)}} .$$
These relative (reduced) L^2 cohomology space can be defined for every Riemannian manifold with boundary. In fact, these relative (reduced) L^2 cohomology spaces also have an interpretation in terms of harmonics forms :

$$H_2^k(\Omega, \partial\Omega) \simeq \mathcal{H}_{rel}^k(\Omega) \tag{1.7}$$
where
$$\mathcal{H}_{rel}^k(\Omega) = \Big\{\alpha \in L^2(\Lambda^k T^*\Omega), d\alpha = d^*\alpha = 0 \text{ and } \iota^*\alpha = 0\Big\}.$$

There is a natural map : the extension by zero map :

$$e\,:\,H_2^k(\Omega,\partial\Omega)\to H_2^k(M)\,.$$

When $\alpha\in L^2(\Lambda^kT^*\Omega)$ we define $e(\alpha)$ to be α on Ω and zero on K, $e\,:\,L^2(\Lambda^kT^*\Omega)\to L^2(\Lambda^kT^*M)$ is clearly a bounded map moreover

$$e\Big(dC_0^\infty(\Lambda^kT^*\Omega)\Big)\subset dC_0^\infty(\Lambda^kT^*M),$$

hence by continuity of e : $e\Big(\overline{dC_0^\infty(\Lambda^kT^*\Omega)}\Big)\subset\overline{dC_0^\infty(\Lambda^kT^*M)}$. Moreover when $\alpha\in Z_2^k(\Omega,\partial\Omega)$ then $e(\alpha)\in Z_2^k(M)$: because if $\varphi\in C_0^\infty(\Lambda^{k+1}T^*M)$ then

$$\langle e(\alpha),d^*\alpha\rangle=\int_\Omega(\alpha,d^*(j_\Omega^*\varphi))d\operatorname{vol}_g$$

but $(j_\Omega^*\varphi)\in C_0^\infty(\Lambda^kT^*\overline{\Omega})$ hence this integral is zero by definition of $Z_2^k(\Omega,\partial\Omega)$.

Let $j_K\,:\,K\to M$ the inclusion map, it induces as before a linear map

$$[j_K^*]\,:\,H_2^k(M)\to H_2^k(K)\simeq H^k(K).$$

We always have $[j_K^*]\circ e=0$ hence

$$\operatorname{Im}e\subset\ker[j_K^*]\ .$$

In fact as before

Lemma 1.15. *We have the equality*

$$\operatorname{Im}e=\ker[j_K^*].$$

Proof of the lemma 1.15. If $c\in\ker[j_K^*]$ and $\alpha\in L^2(\Lambda^kT^*M)$ is a smooth representative of c (for instance α is L^2 and harmonic). By definition we know that there is $\beta\in C^\infty(\Lambda^{k-1}T^*K)$ such that

$$j_K^*\alpha=d\beta$$

Now consider $\bar\beta$ any smooth extension of β with compact support. We clearly have

$$[\alpha-d\bar\beta]=[\alpha]=c\text{ in }H_2^k(M).$$

Moreover by construction

$$\alpha-d\bar\beta=e(j_\Omega^*(\alpha-d\bar\beta))$$

If we verify that $j_\Omega^*(\alpha-d\bar\beta)\in Z_2^k(\Omega,\partial\Omega)$ we have finish the proof of the equality the lemma 1.15. In fact, this verification is straightforward. Let $\varphi\in C_0^\infty(\Lambda^kT^*\overline{\Omega})$ and consider $\bar\varphi$ any smooth extension of φ :

$$\int_\Omega(j_\Omega^*(\alpha-d\bar\beta),d^*\varphi)d\operatorname{vol}_g=\int_M(\alpha-d\bar\beta,d^*\bar\varphi)d\operatorname{vol}_g=0\ .$$

□

Again since $H^k(K)$ have finite dimension, the kernel of $[j_K^*]$ have finite codimension in $H_2^k(M)$ and we have obtain the implication

$$\dim H^k(\Omega,\partial\Omega)<\infty\Rightarrow\dim H_2^k(M)<\infty.$$

In order to conclude, we use the Hodge star operator; because our manifold is oriented, the Hodge star operator is an isometry which exchanges k forms and $(n-k)$ forms :

$$\star\,:\,\Lambda^kT_x^*M\to\Lambda^{n-k}T_x^*M.$$

This operator satisfies the following properties

$$\star \circ \star = \pm \operatorname{Id}$$

where the sign depends on the degree. Moreover we have : $d^* = \pm \star d \star$. Hence the Hodge star operator maps the space $\mathcal{H}^k(M)$ to $\mathcal{H}^{n-k}(M)$. Moreover it is also a good exercise to check that in our setting :

$$\iota^*(\star\alpha) = \pm \operatorname{int}_{\vec{\nu}} \alpha.$$

Hence the Hodge star operator maps the space $\mathcal{H}^k_{abs}(\Omega)$ to $\mathcal{H}^{n-k}_{rel}(\Omega)$.

But our last result says that

$$\dim \mathcal{H}^k_{rel}(\Omega) < \infty \Rightarrow \dim \mathcal{H}^k_2(M) < \infty.$$

Hence using the Hodge star operator we get

$$\dim \mathcal{H}^{n-k}_{abs}(\Omega) < \infty \Rightarrow \dim \mathcal{H}^{n-k}_2(M) < \infty.$$

That is $\dim H^{n-k}_2(\Omega) < \infty \Rightarrow \dim H^{n-k}_2(M) < \infty$. It is now clear that we have prove Lott's results. □

Remarks 1.16. i) The reader can verify that using forms with coefficients in the orientation bundle, we can removed the orientability condition.

ii) These two properties are in fact a heritage of the following two exact sequences for the de Rham cohomology:

$$... \to H^k(K, \partial K) \to H^k(M) \to H^k(\Omega) \to H^{k+1}(K, \partial K) \to ...$$

$$... \to H^k(\Omega, \partial\Omega) \to H^k(M) \to H^k(K) \to H^{k+1}(\Omega, \partial\Omega) \to ...$$

1.4. **Some bibliographical hints.** We recommend the reading of the classical book of G. de Rham [21] or W. Hodge [35]. For a modern treatment of the de Rham's theorem a very good reference is the book of Bott-Tu [9] . The Hodge theorem for compact manifold with boundary have been proved by P.E. Conner [19]. Other proofs of the Hodge-de Rham theorem can be found in other classical book (for instance in the book of Griffith-Harris (chapter 0 section 6 in [27]) or in the book of M. Taylor (chap. 5 in [60]). We should also mentioned a sheaf theoretical proof of the Hodge-de Rham theorem by N. Telemann [61] . About the general feature on non compact manifold, you can read the paper of J. Dodziuk [22], the now classical paper of J. Cheeger [18], the first section of the paper by J. Lott [44] or look at [13] and also read the beautiful paper of M. Anderson [2]. The paper of J. Brüning and M. Lesch deals with an abstract approach about the identification between the space of L^2 harmonic form and L^2 cohomology [10]. For a first approach on the L^2 cohomology of symmetric space, the paper of S. Zucker [67] is very nice, the first parts of the survey of W. Casselman is also instructive [17].

2. Harmonics L^2 1− forms

In this second lecture, we survey some results between the space of harmonics L^2 1− forms, cohomology with compact support and the geometry of ends.

2.1. **Ends.**

2.1.1. *Definitions.* When $U \subset M$ is a subset of an open manifold M we say that U is *bounded* if $\overline{U}$ is a compact subset of M, when U is not bounded we say that U is *unbounded*. When g is complete Riemannian metric on M, then $U \subset M$ is bounded if and only if there is some $R > 0$ and $o \in M$ such that $U \subset B(o, R)$.

Let M be a smooth manifold, we say that M has only one end if for any compact subset $K \subset M$, $M \setminus K$ has only one unbounded connected component.

For instance, when $n \geq 2$, the Euclidean space $\mathbb{R}^n$ has only one end.

More generaly, we say that M has k ends (where $k \in \mathbb{N}$) if there is a compact set $K_0 \subset M$ such that for every compact set $K \subset M$ containing K_0, $M \setminus K$ has exactly k unbounded connected components.

A compact manifold is a manifold with zero end. For instance, $\mathbb{R}$ or $\mathbb{R} \times \mathbb{S}^1$ have two ends. The topology at infinity of manifold with only one end can be very complicated.

2.1.2. *Number of ends and cohomology.* Here M is a smooth non compact connected manifold.

Lemma 2.1. *If M has at least two ends, then*

$$H^1_0(M) \neq \{0\}.$$

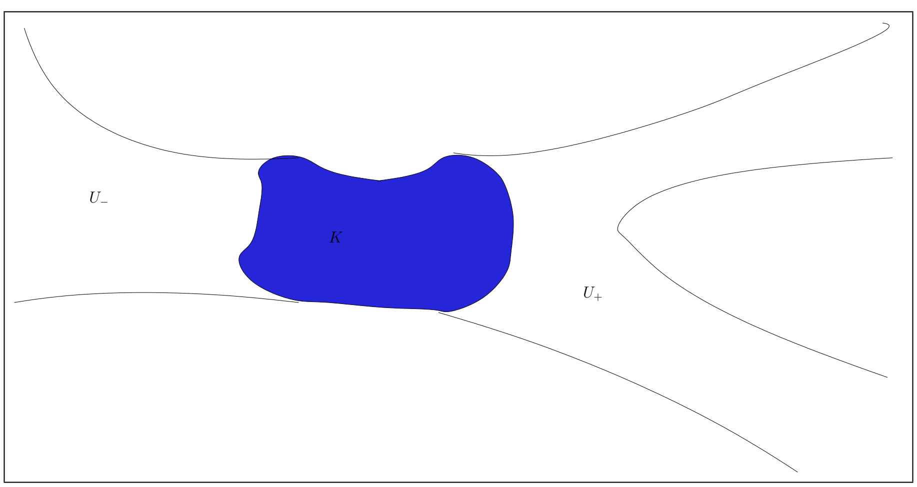
In fact, if M has at least k ends then

$$\dim H^1_0(M) \geq k - 1.$$

Proof. If M has more than two ends, then we can find a compact set $K \subset M$ such that

$$M \setminus K = U_- \cup U_+$$

where U_-, U_+ are unbounded and $U_- \cap U_+ = \emptyset$.



Let $u \in C^\infty(M)$ such that $u = \pm 1$ on $U_\pm$, then clearly $\alpha = du$ is a closed $1-$form with compact support. If the cohomology class of α in $H^1_0(M)$ is trivial, then we find some $f \in C^\infty_0(M)$ such that $du = \alpha = df$. But M is connected hence we have a constant c such that $u = f + c$. We look at this equation outside the support of f and on $U_\pm$ we find that $c = \pm 1$. $\square$

In fact there is a weak reciprocal to this result (see proposition 5.2 in [16]) :

Proposition 2.2. *If M^n is an open manifold having one end, and if every twofold normal covering of M has also one end, then*

$$H_0^1(M,\mathbb{Z}) = \{0\}.$$

In particular, $H_0^1(M) = \{0\}$ and if furthermore M is orientable, then

$$H_{n-1}(M,\mathbb{Z}) = \{0\}.$$

2.2. $H_0^1(M)$ **versus** $H_2^1(M)$**.** From now we assume that (M,g) is a complete Riemannian manifold.

2.2.1. *An easy case.*

Lemma 2.3. *Assume that M has only one end, then*

$$\{0\} \to H_0^1(M) \to H_2^1(M).$$

Proof. Let $\alpha \in C_0^\infty(T^*M)$ be a closed $1-$form which is zero in $H_2^1(M)$. By the result (1.11), we know that there is a smooth function $f \in C^\infty(M)$ such that

$$\alpha = df.$$

But $M \setminus \operatorname{supp}\alpha$ has only one unbounded connected component U, hence on U $df = 0$ hence there is a constant c such that $f = c$ on U. Now by construction the function $f-c$ has compact support and $\alpha = d(f-c)$. Hence α is zero in $H_0^1(M)$ $\square$

In fact the main purpose of this lecture is to go from geometry to topology : we want to find geometrical conditions insuring that this map is injective.

2.2.2. *Condition involving the spectrum of the Laplacian.*

Proposition 2.4. *Assume that all ends of M have infinite volume* [4] *and assume that there is a $\lambda > 0$ such that*

$$\forall f \in C_0^\infty(M),\ \lambda \int_M f^2 d\operatorname{vol}_g \le \int_M |df|^2 d\operatorname{vol}_g . \tag{2.1}$$

Then

$$\{0\} \to H_0^1(M) \to H_2^1(M).$$

Proof. Let $[\alpha] \in H_0^1(M)$ is mapped to zero in $H_2^1(M)$. Hence there is a sequence (f_k) of smooth functions with compact support on M such that $\alpha = L^2 - \lim df_k$. Since we have the inequality

$$\|df_k - df_l\|_{L^2}^2 \ge \lambda \|f_k - f_l\|_{L^2}^2,$$

and since $\lambda > 0$, we conclude that this sequence (f_k) converges to some $f \in L^2$, so that $\alpha = df$. But α has compact support, hence f is locally constant outside the compact set $\operatorname{supp}(\alpha)$. Since all unbounded connected components of $M \setminus \operatorname{supp}(\alpha)$ have infinite volume and since $f \in L^2$, we see that f has compact support, hence $[\alpha] = [df] = 0$ in $H_0^1(M)$. $\square$

Before giving some comments on the hypothesis on this proposition, let's give a consequence of this injectivity :

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[4] that is ouside every compact subset of M, all unbounded connected components have infinite volume.

Proposition 2.5. *Assume that*

$$\{0\} \to H^1_0(M) \to H^1_2(M),$$

and that M has at least k ends then

$$\dim \left\{ h \in C^\infty(M), \Delta h = 0 \text{ and } dh \in L^2 \right\} \geq k. \tag{2.2}$$

A general formula for the dimension of the space of bounded harmonic function with L^2 gradient has been obtained by A. Grigor'yan ([31]).

Proof. Assume that $k = 2$ (the other cases are similar), there is a compact set $K \subset M$ such that

$$M \setminus K = U_- \cup U_+$$

with U_-, U_+ unbounded and $U_- \cap U_+ = \emptyset$. Let $u \in C^\infty(M)$ such that $u = \pm 1$ on $U_\pm$ we know that $\alpha = du$ is not zero in cohomology with compact support , hence we know that there is a non zero harmonic L^2 $1-$form η such that $[\alpha] = [\eta]$ in $H^1_2(M)$; in particular with (1.11), we can find $v \in C^\infty(M)$ such that

$$du = \alpha = \eta + dv$$

that is if $h = u - v$ then $dh = \eta \in L^2$ and $\Delta h = d^* d(u - v) = d^*\eta = 0$. And because $\eta \neq 0$ we know that h is not the constant function. The linear span of h and of the constant function 1 is of dimension 2 and is include in

$$\dim \left\{ h \in C^\infty(M), \Delta h = 0 \text{ and } dh \in L^2 \right\}.$$

□

Remark 2.6. In the setting of the proposition (2.4), we can give a direct and more classical proof of this inequality : we assume that (M, g) satisfies that the assumption made in proposition (2.4), we are going to show the inequality (2.2). Again we assume that $k = 2$: there is a compact set $K \subset M$ such that

$$M \setminus K = U_- \cup U_+$$

with U_-, U_+ unbounded and $U_- \cap U_+ = \emptyset$. Let $o \in M$ be a fixed point and for large k we have $K \subset B(o, k)$. We consider the solution of the Dirichlet problem :

$$\begin{cases} \Delta u_k = 0 & \text{on } B(o,k) \\ u_k = \pm 1 & \text{on } \partial B(o,k) \cap U_\pm \end{cases}$$

Then u_k is the minimizer of the functional

$$u \mapsto \int_{B(o,k)} |du|^2$$

amongst all functions in $W^{1,2}(B(o,k))$ (functions in $L^2(B(o,k))$ whose derivatives are also in L^2) such that $u = \pm 1$ on $U_\pm \cap \partial B(o,k)$). We extend u_k to all M by setting $u_k = \pm 1$ on $U_\pm \setminus B(o,k)$. Then this extension (also denote by u_k) is the minimizer of the functional

$$u \mapsto \int_M |du|^2$$

amongst all functions in $W^{1,2}_{loc}$ (functions in L^2_{loc} whose derivatives are also in L^2_{loc}) such that $u = \pm 1$ on $U_\pm \setminus B(o,k)$. Hence we always have when $k < l$:

$$\int_M |du_l|^2 \leq \int_M |du_k|^2.$$

Moreover by the maximum principle, we always have

$$-1 \leq u_k \leq 1.$$

After extraction of a subsequence we can assume that uniformly on compact set

$$\lim_{k\to\infty} u_k = u.$$

The function u is then a harmonic function whose value are in $[-1,1]$ and it satisfies

$$\int_M |du|^2 \leq \int_M |du_k|^2 < \infty.$$

We must show that this u is not a constant function. We apply our estimate to the function $u_k - u_l$ where $l \geq k$, then we get

$$\lambda \int_M |u_k - u_l|^2 \leq \int_M |du_k - du_l|^2 \leq 4 \int_M |du_k|^2$$

In particular if we let $l \to \infty$, we find that

$$u_k - u \in L^2$$

that is

$$\int_{U_\pm} |u - (\pm 1)|^2 < \infty.$$

But by hypothesis the volume of $U_\pm$ are infinite hence u cannot be the constant function.

We know make some comments on the assumption of the proposition (2.4) :
On the first condition. We have the following useful criterion :

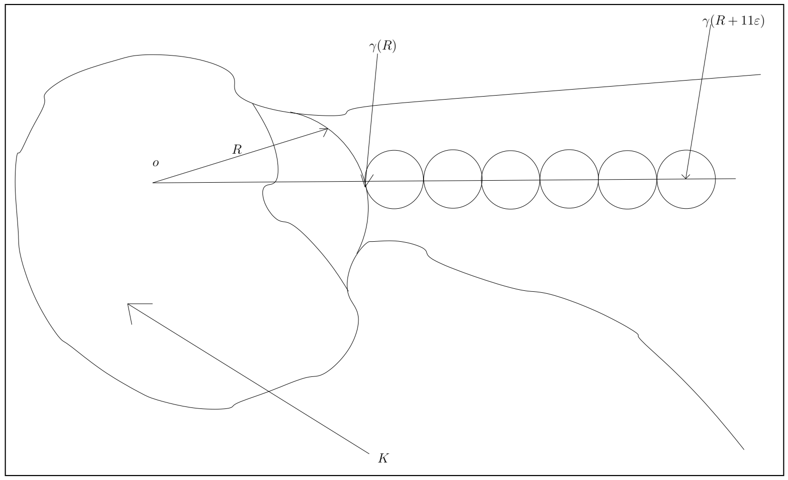
Lemma 2.7. *Assume that there is $v > 0$ and $\varepsilon > 0$ such that for all $x \in M$*

$$\operatorname{vol} B(x, \varepsilon) \geq v$$

then all ends of M have infinite volume.

Proof. We fix a based point $o \in M$ and let $K \subset M$ be a compact set and U an unbounded connected component of $M \setminus K$. Let $R > 0$ large enough so that $K \subset B(o, R)$ (recall that we have assumed here that the Riemannian manifold (M, g) is complete). For $k \in \mathbb{N}$ we choose $x \in \partial B(o, R + (2k+1)\varepsilon) \cap U$ and we consider $\gamma : [0, R + (2k+1)\varepsilon] \to M$ a minimizing geodesic from o to x (parametrize by arc- length). Necessary for all $t \in]R, R + (2k+1)\varepsilon]$ we have $\gamma(t) \in U$; moreover for $l = 0, 1, ..., k$, the open geodesic ball $B(\gamma(R + (2l+1)\varepsilon), \varepsilon)$ are in U and disjoint hence

$$\operatorname{vol} U \geq \sum_{l=0}^{k} \operatorname{vol} B\big(\gamma(R + (2l+1)\varepsilon), \varepsilon\big) \geq (k+1)v.$$



□

For instance, when the injectivity radius of (M, g) is positive, C. Croke has shown in ([20]) that for all $x \in M$ and all $r \leq \mathrm{inj}(M)$ then

$$\mathrm{vol}\, B(x, r) \geq C_n r^n.$$

Hence a complete Riemannian manifold with positive injectivity radius has all its ends with infinite volume.
About the second condition The condition (2.1) is clearly equivalent to

$$\lambda_0(M, g) := \inf_{f \in C_0^\infty(M)} \left\{ \frac{\int_M |df|^2 d\,\mathrm{vol}_g}{\int_M |f|^2 d\,\mathrm{vol}_g} \right\} > 0.$$

This condition is linked with the spectrum of the Laplacian on functions. In fact $\lambda_0(M, g)$ is the bottom of the spectrum of the operator $d^*d = \Delta$ acting on functions. Because the Riemannian manifold (M, g) is complete, the operator

$$d^*d = \Delta \,:\, C_0^\infty(M) \to L^2(M, d\,\mathrm{vol}_g)$$

has a unique self adjoint extension with domain :

$$\mathcal{D}(\Delta) = \{f \in L^2(M), \Delta f \in L^2(M)\} = \{f \in L^2(M), df \in L^2(M) \text{ and } d^*df \in L^2.\}$$

Hence $f \in \mathcal{D}(\Delta)$ if and only if $f \in \mathcal{D}^0(d)$ and there is $C \in \mathbb{R}$ such that

$$\forall \varphi \in C_0^\infty(M)\ ,\ |\langle df, d\varphi\rangle| \leq C\|\varphi\|_{L^2}.$$

The spectrum of Δ_g is a closed subspace of $[0, \infty[$ and the condition $\lambda_0(M, g) > 0$ is equivalent to zero not being in the spectrum of Δ. Also this condition depends only of the geometry at infinity, in fact we have that $\lambda_0(M, g) > 0$ if and only if there is $K \subset M$ a compact subset such that $\lambda_0(M \setminus K, g) > 0$ and $\mathrm{vol}(M) = \infty$.. From the proof of the proposition (2.4) it is clear that we have the following result :

Proposition 2.8. *Assume that (M, g) is a complete Riemannian manifold such that $\lambda_0(M, g) > 0$, then if we introduce*

$$H_v^1(M) = \frac{\{\alpha \in C_0^\infty(T^*M), \text{such that } d\alpha = 0\}}{\{df, f \in C^\infty(M), \text{such that } df \in C_0^\infty(T^*M) \text{ and } \mathrm{vol}(\mathrm{supp}\, f) < \infty\}}$$

then

$$\{0\} \to H^1_v(M) \to H^1_2(M).$$

Optimality of the results : The real line $(\mathbb{R}, (dt)^2)$ has clearly all his ends with infinite volume but $\lambda_0(\mathbb{R}, dt^2) = 0$; as a matter of fact if $u \in C_0^\infty(\mathbb{R})$ is not the zero function, then for $u_n(t) = u(t/n)$ we have

$$\int_{\mathbb{R}} |u_n'(t)|^2 dt = \frac{1}{n} \int_{\mathbb{R}} |u'(t)|^2 dt$$

and

$$\int_{\mathbb{R}} |u_n(t)|^2 dt = n \int_{\mathbb{R}} |u'(t)|^2 dt.$$

But the first group of cohomology with compact support of $\mathbb{R}$ has dimension 1 where as

$$\mathcal{H}^1(\mathbb{R}, dt^2) = \{f dt, f \in L^2 \text{ and } f' = 0\} = \{0\}.$$

We consider now the manifold $\Sigma = \mathbb{R} \times \mathbb{S}^1$ endowed with the warped product metric :

$$g = (dt)^2 + e^{2t}(d\theta)^2.$$

M have two ends one with finite volume and the other of infinite volume. Again the first group of cohomology with compact support of Σ has dimension 1. Moreover

$$\lambda_0(\Sigma) \geq \frac{1}{4} \ ;$$

as a matter of fact let $f \in C_0^\infty(\Sigma)$

$$\int_\Sigma |df|^2 d\operatorname{vol}_g \geq \int_\Sigma \left| \frac{\partial f}{\partial t}(t,\theta) \right|^2 e^t dt d\theta$$

Let $f = e^{-t/2} v$ then

$$\begin{aligned}
\int_\Sigma \left| \frac{\partial f}{\partial t}(t,\theta) \right|^2 e^t dt d\theta &= \int_\Sigma \left| \frac{\partial v}{\partial t} - \frac{1}{2} v \right|^2 dt d\theta \\
&= \int_{\mathbb{R}\times\mathbb{S}^1} \Big[\left| \frac{\partial v}{\partial t} \right|^2 + \frac{1}{4} |v|^2 + v \frac{\partial v}{\partial t} \Big] dt d\theta \\
&= \int_{\mathbb{R}\times\mathbb{S}^1} \Big[\left| \frac{\partial v}{\partial t} \right|^2 + \frac{1}{4} |v|^2 + \frac{1}{2} \frac{\partial v^2}{\partial t} \Big] dt d\theta \\
&= \int_{\mathbb{R}\times\mathbb{S}^1} \Big[\left| \frac{\partial v}{\partial t} \right|^2 + \frac{1}{4} |v|^2 \Big] dt d\theta \\
&\geq \frac{1}{4} \int_{\mathbb{R}\times\mathbb{S}^1} |v|^2 dt d\theta \\
&= \frac{1}{4} \int_\Sigma |f|^2 d\operatorname{vol}_g .
\end{aligned}$$

In fact we can show (using an appropriate test function $u_n(t,\theta) = \chi_n(t) e^{-t/2}$) that $\lambda_0(\Sigma) = \frac{1}{4}$.

In fact we can show that $\mathcal{H}^1(\Sigma)$ has infinite dimension (see the first part of theorem 2.16), however if the conclusion of the proposition 2.4) were true for (Σ, g) then by the alternative proof of (2.2) in (2.6), we would find a non constant harmonic function h with L^2 gradient. If we look at the construction of this function, it is not hard to check that h will only depends on the first variable that is

$$h(t,\theta)=f(t)$$

where f solves the O.D.E.

$$(e^t f')'=0\,;$$

hence there is constant A and B such that

$$h(t,\theta)=Ae^{-t}+B.$$

The fact that $dh\in L^2$ implies that

$$\int_{\mathbb{R}}(f')^2(t)e^t dt<\infty$$

that is $A=0$ and h is the constant function.

This simple proposition 2.4 or variant of it has been used frequently, I will mention only two results, the first one is the following very beautifully result of P. Li and J. Wang ([40]): (see also the other articles of P. Li and J. Wang for other related results [41, 43])

Theorem 2.9. *Assume that (M,g) is a complete Riemannian manifold of dimension $n>2$ with*

$$\lambda_0(M)\ge(n-2)$$

assume moreover that its Ricci curvature satisfies the lower bound

$$\mathrm{Ric}_g\ge-(n-1)g$$

then either (M,g) has only one end of infinite volume either (M,g) has two ends with infinite volume and is isometric to the warped product $\mathbb{R}\times N$ endowed with the metric

$$(dt)^2+\cosh^2(t)h$$

where (N,h) is a compact Riemannian manifold with $\mathrm{Ric}_g\ge-(n-2)g$

The second concerns the locally symmetric space ([16]) :

Theorem 2.10. *Let G/K be a symmetric space without any compact factor and without any factor isometric to a real or complex hyperbolic space. Assume that $\Gamma\subset G$ is a torsion-free, discrete subgroup of G such that $\Gamma\backslash G/K$ is non compact and that all ends of $\Gamma\backslash G/K$ have infinite volume. Then $\Gamma\backslash G/K$ has only one end, and*

$$H_{m-1}(\Gamma\backslash G/K,\mathbb{Z})=\{0\},$$

where $m=\dim(G/K)$.

2.2.3. *Condition involving a Sobolev inequality.*

Proposition 2.11. *Assume that (M,g) is a complete manifold that satisfies for a $\nu>2$ and $\mu>0$ the Sobolev inequality :*

$$\forall f\in C_0^\infty(M),\quad \mu\left(\int_M f^{\frac{2\nu}{\nu-2}}d\operatorname{vol}_g\right)^{1-\frac{2}{\nu}}\le\int_M|df|^2d\operatorname{vol}_g\,. \tag{2.3}$$

Then

$$\{0\}\to H_0^1(M)\to H_2^1(M).$$

The proof follows essentially the same path : if $[\alpha] \in H^1_0(M)$ is map to zero in $H^1_2(M)$, then we find $f \in L^{\frac{2\nu}{\nu-2}}$ such that

$$\alpha = df.$$

Hence f is locally constant outside the support of α. But according to (proposition 2.4 in [12]), we know that the Sobolev inequality (2.3) implies a uniform lower bound on the volume of geodesic balls :

$$\forall x \in M, \forall r \geq 0 \ : \ \operatorname{vol} B(x,r) \geq C(\nu) \left(\mu r^2\right)^{\nu/2}$$

for some explicit constant $C(\nu) > 0$ depending only on ν. Hence by (2.7), we know that such an estimate implies that all the unbounded connected components of $M \setminus \operatorname{supp} \alpha$ have infinite volume. Hence f has compact support and $[\alpha] = 0$ in $H^1_0(M)$.

Examples of manifolds satisfying the Sobolev inequality :

i) The Euclidean space $\mathbb{R}^n$ $(n \geq 3)$ satisfies the Sobolev inequality for $\nu = n$.
ii) If $M^n \subset \mathbb{R}^N$ is a minimal submanifold then M^n with the induced metric satisfies the Sobolev inequality for $\nu = n$ [51, 36].
iii) A Cartan-Hadamard manifold (a simply connected non positively curved complete Riemannian manifold) of dimension $n \geq 3$ satisfies the Sobolev inequality for $\nu = n$.
iv) The hyperbolic space of dimension n, satisfies the Sobolev inequality for any $\nu \in [n, \infty[\cap]2, \infty[$.

In fact the validity of the Sobolev inequality depends only of the geometry at infinity : according to [14], if $K \subset M$ is a compact subset of M then the Sobolev inequality (2.3) holds for M if and only if it holds for $M \setminus K$:

$$\forall f \in C^\infty_0(M \setminus K), \quad \mu \left(\int_{M \setminus K} f^{\frac{2\nu}{\nu-2}} d\operatorname{vol}_g \right)^{1-\frac{2}{\nu}} \leq \int_{M \setminus K} |df|^2 d\operatorname{vol}_g .$$

This proposition (2.11) has the following beautiful application due to Cao-Shen-Zhu [11] (see also [42]) however these authors haven't prove the vanishing of the first group of cohomology with compact support :

Corollary 2.12. *If $M^n \subset \mathbb{R}^{n+1}$ is a stable complete minimal hypersurface then M has only one end, moreover $H^1_0(M) = \{0\}$.*

Proof. We only need to show that $\mathcal{H}^1(M) = \{0\}$. Let $\alpha \in \mathcal{H}^1(M)$ then it satisfies the Bochner identity

$$\int_M \left[\, |\nabla \alpha|^2 + \operatorname{Ric}(\alpha, \alpha)\right] d\operatorname{vol}_g = 0.$$

However the Gauss equations imply that

$$|\operatorname{Ric}(\alpha,\alpha)| \leq |A|^2 \, |\alpha|^2 ,$$

where A is the second fundamental form of the hypersurface $M \subset \mathbb{R}^{n+1}$. The stability condition says that the second variation of the area is non negative that is

$$\forall f \in C^\infty_0(M), \ J(f) := \int_M \left[|df|^2 - |A|^2 \, |f|^2\right] d\operatorname{vol}_g \geq 0.$$

But the refined Kato inequality (due in this case to S.T. Yau) shows that

$$|\nabla\alpha|^2 \leq \frac{n}{n-1}\left|d|\alpha|\right|^2.$$

Hence we get

$$\begin{aligned}
0 &= \int_M \left[\, |\nabla\alpha|^2 + \mathrm{Ric}(\alpha,\alpha)\right] d\,\mathrm{vol}_g \\
&\geq \int_M \left[\frac{n}{n-1}\,|d|\alpha||^2 + \mathrm{Ric}(\alpha,\alpha)\right] d\,\mathrm{vol}_g \\
&\geq \int_M \left[\frac{1}{n-1}\,|d|\alpha||^2 + |d|\alpha||^2 - |A|^2\,|\alpha|^2\right] d\,\mathrm{vol}_g \\
&\geq J(|\alpha|) + \frac{1}{n-1}\int_M |d|\alpha||^2\, d\,\mathrm{vol}_g\,. \\
&\geq \frac{1}{n-1}\int_M |d|\alpha||^2\, d\,\mathrm{vol}_g\,.
\end{aligned}$$

Hence α has constant length, but because of the Sobolev inequality, the volume of (M,g) is infinite and $\alpha = 0$. □

2.2.4. *What is behind the injectivity of the map $H^1_0(M) \to H^1_2(M)$.* In fact there is a general notion from potential theory which is related to the injectivity of this map (see the survey of A. Ancona [1] or [39]).

Definition 2.13. Let $E \subset M$ be an open connected set with smooth compact boundary, the following properties are equivalent:

i) there is a positive super-harmonic function : $s : \overline{E} \to \mathbb{R}^*_+$ with

$$\liminf_{x\to\infty} s(x) = \inf_{x\in E} s(x) = 0$$

and

$$\inf_{x\in\partial E} s(x) \geq 1.$$

ii) There is a positive harmonic function : $h : \overline{E} \to \mathbb{R}^*_+$ with

$$\liminf_{x\to\infty} h(x) = \inf_{x\in E} h(x) = 0$$

and $h = 1$ on ∂E. Moreover $dh \in L^2$.

iii) The capacity of E is positive :

$$\mathrm{cap}(E) = \inf\left\{\int_E |dv|^2,\ v \in C^\infty_0(\overline{E}) \text{ and } v \geq 1 \text{ on } \partial E\right\} > 0.$$

iv) For any $U \subset E$ bounded open subset of E there is a constant $C_U > 0$ such that

$$\forall f \in C^\infty_0(\overline{E}), C_U \int_U f^2 \leq \int_E |df|^2.$$

v) For some $U \subset E$ bounded open subset of E there is a constant $C_U > 0$ such that

$$\forall f \in C^\infty_0(\overline{E}), C_U \int_U f^2 \leq \int_E |df|^2.$$

When one of these properties holds we say that E is non parabolic and if one of these properties fails we say that E is parabolic.

Proof. We clearly have $ii) \Rightarrow i)$ and $iv) \Rightarrow v)$. We first prove that $i) \Rightarrow iv)$. Let $\vec{\nu} : \partial E \to TE$ be the unit inward normal vector field along ∂E. When $v \in C_0^\infty(\overline{E})$ we set $\varphi = v/\sqrt{s}$ so that

$$\int_E |dv|^2 = \int_E s|d\varphi|^2 + \int_E \varphi\langle ds, d\varphi\rangle + \int_E \frac{|ds|^2}{4s}\varphi^2.$$

But

$$\begin{aligned} 2\int_E \varphi\langle ds, d\varphi\rangle &= \int_E \langle ds, d\varphi^2\rangle \\ &= \int_E (\Delta s)\varphi^2 - \int_{\partial E} \varphi^2 ds(\vec{\nu}). \end{aligned}$$

But s is assumed to be super-harmonic hence $\Delta s \geq 0$ and $ds(\vec{\nu}) \leq 0$ along ∂E, hence we finally obtain the lower bound

$$\int_E |dv|^2 \geq \int_E \frac{|ds|^2}{4s}\varphi^2 = \frac{1}{4}\int_E |d\log s|^2 v^2.$$

By assumption, s is not constant hence we can find an open bounded set $U \subset E$ and $\varepsilon > 0$ such that $|d\log s| > \varepsilon$ on U and we obtain that for all $v \in C_0^\infty(\overline{E})$

$$\int_E |dv|^2 \geq \frac{\varepsilon^2}{4}\int_U v^2.$$

We prove now that $v) \Rightarrow iv)$. Let $U \subset E$ such as in iv). For $o \in M$ a fixed point and R such that $U \cup \partial E \subset B(o,R)$, we will prove that there is a constant $C_R > 0$ such that

$$\forall f \in C_0^\infty(\overline{E}),\ C_R \int_{B(o,R)\cap E} f^2 \leq \int_E |df|^2.$$

Let $V \subset U$ be an non empty open set with $\bar{V} \subset U$ and let $\rho \in C^\infty(\overline{E})$ such that $\operatorname{supp}\rho \subset U$ and $0 \leq \rho \leq 1$ and $\rho = 1$ in V. Then for $f \in C_0^\infty(\overline{E})$ we have :

$$\begin{aligned} \int_{B(o,R)\cap E} f^2 &\leq 2\int_{B(o,R)\cap E} (\rho f)^2 + 2\int_{B(o,R)\cap E} ((1-\rho)f)^2 \\ &\leq 2\int_U f^2 + 2\int_{B(o,R)\cap E} ((1-\rho)f)^2 \\ &\leq \frac{2}{C_U}\int_E |df|^2 + \frac{2}{\lambda}\int_{B(o,R)\cap E} |d((1-\rho)f)|^2 \end{aligned}$$

Where $\lambda > 0$ is the first eigenvalue of the Laplacian on functions on $(B(o,R)\cap E)\setminus V$ for the Dirichlet boundary condition on ∂V. But

$$\begin{aligned} \int_{B(o,R)\cap E} |d((1-\rho)f)|^2 &\leq 2\int_{B(o,R)\cap E} |df|^2 + 2\|d\rho\|_{L^\infty}^2 \int_U |f|^2 \\ &\leq \left(2 + \frac{2\|d\rho\|_{L^\infty}^2}{C_U}\right)\int_E |df|^2. \end{aligned}$$

Hence the result for

$$C_R = \left(\frac{2}{C_U} + \frac{4}{\lambda}\left(1 + \frac{\|d\rho\|_{L^\infty}^2}{C_U}\right)\right)^{-1}.$$

Now we consider the implication $iii) \Rightarrow ii)$. We introduce

$$C(R) = \inf \int_{E \cap B(o,R)} |dv|^2$$

where the infimum runs over all functions $v \in C^\infty(\overline{E} \cap B(o,R))$ such that $v \geq 1$ on ∂E and $v = 0$ on $\partial B(o,R) \cap E$; where $o \in M$ is a fixed point and where $R > 0$ is chosen large enough so that $\partial E \subset B(o,R)$. We have $C(R) > 0$ and we have assumed that $C(\infty) = \inf_R C(R) > 0$. Each $C(R)$ is realized by the harmonic function h_R such that $h_R = 1$ on ∂E and $h_R = 0$ on $\partial B(o,R) \cap E$. We extend h_R by zero on $E \setminus B(o,R)$, an application of the maximum principle implies that when $R \geq R'$ then $h_{R'} \leq h_R$. We let $h(x) = \sup_R h_R(x)$. On compact subset of $\overline{E}$, h_R converge to h in the smooth topology moreover h is a harmonic function with $0 \leq h \leq 1$. The Green formula shows that

$$C(R) = -\int_{\partial E} dh_R(\vec{\nu}),$$

hence when $R \to \infty$ we obtain

$$C(\infty) = -\int_{\partial E} dh(\vec{\nu}) > 0.$$

In particular h is not the constant function and $h > 0$ on E by the maximum principle. We also have

$$\int_E |dh|^2 \leq \liminf_{R \to \infty} \int_E |dh_R|^2 = \liminf_{R \to \infty} C(R) = C(\infty) < \infty.$$

We must show that $\inf_E h = 0$. Let $v \in C_0^\infty(\overline{E})$ such that $v \geq 1$ on ∂E, we let $v = h\varphi$ then using the Green formula we get :

$$\begin{aligned}
\int_E |dv|^2 &= \int_E h^2 |d\varphi|^2 + 2\int_E h\varphi \langle dh, d\varphi \rangle + \int_E |dh|^2 \varphi^2 \\
&= \int_E h^2 |d\varphi|^2 + \frac{1}{2}\int_E \langle dh^2, d\varphi^2 \rangle + \int_E |dh|^2 \varphi^2 \\
&= \int_E h^2 |d\varphi|^2 + \frac{1}{2}\int_E \Delta(h^2)\varphi^2 - \int_{\partial E} \varphi^2 dh(\vec{\nu}) + \int_E |dh|^2 \varphi^2 \\
&\geq \int_E h^2 |d\varphi|^2 - \int_{\partial E} dh(\vec{\nu}) \\
&\geq (\inf_E h)^2 C(\infty) + C(\infty)
\end{aligned}$$

Taking the infimum over such v we obtain that

$$0 \geq (\inf_E h)^2 C(\infty)$$

hence $\inf_E h = 0$

It remains to show that $iv) \Rightarrow iii)$. We re consider the preceding notation. Under the hypothesis $iv)$, we must show that $C(\infty) > 0$. Or by contraposition that if $C(\infty) = 0$ then iv) can't be true. So we assume that $C(\infty) = 0$, in that case we know that h is the constant function 1 and we get for the functions h_R :

$$\lim_{R \to \infty} \int_U h_R^2 = \operatorname{vol} U$$

where as

$$\lim_{R\to\infty}\int_E |dh_R|^2 = 0.$$

That is iv) is not true

□

Proposition 2.14. *If all ends of M are non parabolic then*

$$\{0\} \to H^1_0(M) \to H^1_2(M).$$

In fact if M has more than k non parabolic ends then

$$\dim\left\{h \in C^\infty(M), \Delta h = 0 \text{ and } dh \in L^2\right\} \geq k.$$

The above inequality is due to P. Li and L-F Tam ([39]).

Proof. We remark that because M contains at most one non parabolic end, the above proof of (2.13 *v)⇒ iv)*) shows that for any $U \subset M$ bounded open subset there is $C_U > 0$ such that

$$\forall f \in C^\infty_0(M), C_U \int_U f^2 \leq \int_M |df|^2.$$

We consider a closed form $\alpha \in C^\infty_0(T^*M)$ which is map to zero in $H^1_2(M)$. Hence there is a sequence $v_k \in C^\infty_0(M)$ such that

$$\alpha = L^2 - \lim_{k\to\infty} dv_k.$$

By the above remark, we know that there is $v \in C^\infty(M)$ such that $dv = \alpha$ and

$$v = L^2_{loc} - \lim_{k\to\infty} v_k.$$

Let E be a unbounded connected component of $M \setminus \operatorname{supp}\alpha$, v is then constant on E

$$v = v(E) \text{ on } E.$$

We want to show that this constant is zero. We can enlarge $M \setminus E$ and assume that E has smooth boundary. On E we have

$$\lim_{k\to\infty}\int_E |dv_k|^2 = \int_E |\alpha|^2 = 0.$$

Choose U a non empty bounded open subset of E, we know that for a certain constant $C_U > 0$ we have the estimate

$$\int_E |dv_k|^2 \geq C_U \int_U v_k^2,$$

When k tends to ∞ we obtain

$$0 \geq C_U \operatorname{vol} U\, v(E)^2.$$

Hence v is zero on E and v has necessary compact support. □

2.3. **The two dimensional case.** In dimension 2, a remarkable property of the space of L^2 harmonic 1 forms is that it is an invariant of conformal structure (or of the complex structure).

2.3.1. *Conformal invariance.*

Proposition 2.15. *Let $\bar{g} = e^{2u}g$ be two conformally equivalent Riemannian metric* [5] *on a smooth manifold M^{2m} , then then we have the equality :*

$$\mathcal{H}^m(M, \bar{g}) = \mathcal{H}^m(M, g).$$

Proof. When $\alpha \in \Lambda^k T_x^* M$, we have

$$|\alpha|^2_{\bar{g}} = e^{-2ku(x)}|\alpha|^2_g.$$

and

$$d\,\mathrm{vol}_{\bar{g}} = e^{2mu} d\,\mathrm{vol}_g\,.$$

As a consequence, the two Hilbert spaces $L^2(\Lambda^m T^*M, \bar{g})$ and $L^2(\Lambda^m T^*M, g)$ are isometric, hence by definition the two space $Z_2^m(M, \bar{g}) = \{\alpha \in L^2(\Lambda^m T^*M, \bar{g}), d\alpha = 0\}$ and $Z_2^m(M, \bar{g}) = \{\alpha \in L^2(\Lambda^m T^*M, g), d\alpha = 0\}$ are the same. Moreover the orthogonal of $dC_0^\infty(\Lambda^{m-1}T^*M)$ is the same in $L^2(\Lambda^m T^*M, g)$ or $L^2(\Lambda^m T^*M, \bar{g})$. But by definition

$$\mathcal{H}^m(M, \bar{g}) = Z_2^m(M, \bar{g}) \cap \left(dC_0^\infty(\Lambda^{m-1}T^*M)\right)^\perp.$$

is also equal to $\mathcal{H}^m(M, g)$.

Another proof is to compute the codifferential $d^*_{\bar{g}}$: for $\alpha \in C^\infty(\Lambda^k T^*M)$ we obtain

$$d^*_{\bar{g}}\alpha = e^{-2u}\left(d^*_g\alpha - 2(m-k)\,\mathrm{int}_{\overrightarrow{grad\, u}}\,\alpha\right).$$

□

2.3.2. *Application.*

Theorem 2.16. *Let (S, g) be a complete connected Riemannian surface with finite topology (finite genus and finite number of ends) then either*

- $\dim \mathcal{H}^1(S, g) = \infty$
- *or* $\dim \mathcal{H}^1(S, g) < \infty$ *and M is conformally equivalent to a compact Riemannian surface $(\bar{S}, \bar{g})$ with a finite number of points removed :*

$$(S, g) \simeq (\bar{S} \setminus \{p_1, ..., p_k\}, \bar{g})$$

and

$$\mathcal{H}^1(S, g) \simeq \mathrm{Im}\left(H^1_0(S) \to H^1(S)\right) \simeq H^1(\bar{S}).$$

Proof. We know that a Riemannian surface with finite topology is necessary conformally equivalent to a compact Riemannian surface $(\bar{S}, \bar{g})$ with a finite number of points and disks removed :

$$(S, g) \simeq \left(\bar{S} \setminus \left(\cup_{l=1}^b D_l \cup \{p_1, ..., p_k\}\right), \bar{g}\right)$$

Hence from our previous result, we have

$$\mathcal{H}^1(S, g) = \mathcal{H}^1\left(\bar{S} \setminus \left(\cup_{l=1}^b D_l \cup \{p_1, ..., p_k\}\right), \bar{g}\right).$$

We first show that if $b \geq 1$ then $\dim \mathcal{H}^1(S) = \infty$. Let $f \in C^\infty(\partial D_1)$, then on $\bar{S} \setminus D_1$ we can solve the Dirichlet problem :

$$\begin{cases} \Delta^{\bar{g}} u = 0 & \text{on } \bar{S} \setminus D_1 \\ u = f & \text{on } \partial D_1 \end{cases}$$

[5] The metric $\bar{g}$ and g are not necessary complete.

Then $u \in C^\infty(\bar S \setminus D_1)$ hence $du \in L^2(T^*\left(\bar S \setminus D_1\right), \bar g)$ is closed and coclosed hence $du \in \mathcal{H}^1(\bar S \setminus D_1, \bar g)$ its restriction to $S = \bar S \setminus \left(\cup_{l=1}^b D_l \cup \{p_1, ..., p_k\}\right)$ is also L^2 closed and co closed, hence we have build a linear map

$$f \in C^\infty(\partial D_1) \mapsto du\big|_S \in \mathcal{H}^1(S, g) = \mathcal{H}^1(S, \bar g)\ ,$$

the kernel of this map is the set of constant functions hence we have proved that

$$\dim \mathcal{H}^1(S, g) = \infty.$$

Now we assume that $b = 0$ that is

$$(S, g) \simeq (\bar S \setminus \{p_1, ..., p_k\}, \bar g)\ ,$$

we have

$$\mathcal{H}^1(S, g) = \mathcal{H}^1(\bar S \setminus \{p_1, ..., p_k\}, \bar g).$$

The main point is that when $\alpha \in \mathcal{H}^1(\bar S \setminus \{p_1, ..., p_k\}, \bar g)$ then α extends across the point $\{p_1, ..., p_k\}$. This is a direct consequence of the following lemma

Lemma 2.17. *Let $\mathbb{D}$ be the unit disk, and let $\alpha \in L^2\big(T^*(\mathbb{D} \setminus \{0\})\big)$ satisfying the equation $(d + d^*)\alpha = 0$, then α extends smoothly across 0 that is*

$$\mathcal{H}^1(\mathbb{D} \setminus \{0\}) = \mathcal{H}^1(\mathbb{D}).$$

□

In this lemma the metric on the disk can be considered as the Euclidean one, as a matter of fact any other Riemannian metric on $\mathbb{D}$ is conformally equivalent to the flat metric. We offer two proof of this result.

First proof of the lemma. Let $\alpha \in \mathcal{H}^1(\mathbb{D} \setminus \{0\})$, we are going to prove that $(d + d^*)\alpha = 0$ holds weakly on $\mathbb{D}$, then because the operator $(d + d^*)$ is elliptic it will hold strongly by elliptic regularity. Consider $\varphi \in C_0^\infty(\mathbb{D}) \oplus C_0^\infty(\Lambda^2 T^*\mathbb{D})$ we must show that

$$\langle \alpha, (d + d^*)\varphi\rangle = 0.$$

We consider the following sequence of cutoff functions

$$\chi_n(r, \theta) = \begin{cases} 0 & \text{when } r \le 1/n^2 \\ -\frac{\log(r/n^2)}{\log n} & \text{when } 1/n^2 \le r \le 1/n \\ 1 & \text{when } r \ge 1/n \end{cases}$$

By hypothesis

$$\langle \alpha, (d + d^*)(\chi_n \varphi)\rangle = 0.$$

But $(d + d^*)(\chi_n \varphi) = \chi_n (d + d^*)\varphi + d\chi_n \wedge \varphi + \mathrm{int}_{\overrightarrow{grad}\,\chi_n} \varphi$. We have then

$$0 = \langle \alpha, (d + d^*)(\chi_n \varphi)\rangle = \langle \alpha, \chi_n (d + d^*)\varphi)\rangle + \langle \alpha, d\chi_n \wedge \varphi + \mathrm{int}_{\overrightarrow{grad}\,\chi_n} \varphi\rangle$$

But when $n \to \infty$, the first term in the right hand side goes to $\langle \alpha, (d + d^*)\varphi\rangle$:

$$\lim_{n\to\infty} \langle \alpha, \chi_n (d + d^*)\varphi)\rangle = \langle \alpha, (d + d^*)\varphi)\rangle$$

and the second term is estimate as follow :

$$\left|\langle \alpha, d\chi_n \wedge \varphi + \mathrm{int}_{\overrightarrow{grad}\,\chi_n} \varphi\rangle\right| \le \|\alpha\|_{L^2} \|\varphi\|_{L^\infty} \|d\chi_n\|_{L^2}$$

But a direct computation shows that

$$\|d\chi_n\|_{L^2}^2 = \frac{2\pi}{\log n}.$$

Hence the result. □

Second proof of the lemma. Let $\alpha \in \mathcal{H}^1(\mathbb{D} \setminus \{0\})$. We first show that α is exact. That is we'll prove that

$$\int_{\mathbb{S}^1} \alpha = 0$$

Let $c = \int_{\mathbb{S}^1} \alpha$, this integral doesn't depend on the radius ; hence we get by Cauchy-Schwarz inequality we get :

$$c^2 \leq 2\pi r^2 \int_0^{2\pi} |\alpha|^2(r,\theta)d\theta$$

Dividing by r and integrating over $[\varepsilon, 1]$ we get :

$$-c^2 \log \varepsilon \leq 2\pi \int_{\mathbb{D}} |\alpha|^2.$$

Hence letting $\varepsilon \to 0+$ we obtain $c = 0$. So there is an harmonic function f on $\mathbb{D} \setminus \{0\}$ such that

$$\alpha = df.$$

We have

$$f(r,\theta) = \sum_{k\in\mathbb{Z}} u_k(r) e^{ik\theta}$$

where $u_k(r) = A_k r^{|k|} + B_k r^{|k|}$ when $k \neq 0$ and $u_0(r) = A_0 + B_0 \log(r)$, moreover

$$\int_{\mathbb{D}} |df|^2 = \sum_{k\in\mathbb{Z}} \int_0^1 \left[|u_k'|^2 + k^2|u_k|^2\right] rdr < \infty.$$

This implies that $B_k = 0$ for all $k \in \mathbb{Z}$ hence f is smooth at zero. □

2.4. **Bibliographical hints.** For the second lecture, we warmly recommend the reading of the paper of A. Ancona [1] about non parabolicity, we also recommend the beautiful survey by A. Grigor'yan [32] on non parabolicity, stochastic completeness and harmonic functions. The paper of P. Li-F.Tam [39] about the space of harmonic functions with L^2 differential has also to be read.

3. L^2 COHOMOLOGY OF CONFORMALLY COMPACT MANIFOLD

3.1. **The geometric setting.** Let $\overline{M}$ be a compact smooth manifold with boundary $N = \partial\overline{M}$. On $M = \mathrm{int}(\overline{M}) = \overline{M} \setminus N$, we say that a Riemannian metric g is *conformally compact* if

$$g = \frac{\bar{g}}{y^2}$$

where $\bar{g}$ is a smooth Riemannian metric on $\bar{M}$ (hence non complete) and $y : \overline{M} \to \mathbb{R}_+$ is smooth defining function for N that is

$$\begin{cases} y^{-1}\{0\} = N \\ dy \neq 0 \qquad\qquad \text{along } N \end{cases}$$

Such a metric is complete. The most famous example is the hyperbolic metric in the ball model :

$$g_{\text{hyp}} = \frac{4\|dx\|^2}{(1 - \|x\|^2)^2}$$

where $\|dx\|^2$ is the Euclidean metric on the unit ball of $\mathbb{R}^n$ and

$$y = \frac{1 - \|x\|^2}{2}$$

is a smooth defining function for $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$.

Conformally compact Riemannian metrics have a great interest in theoretical physics with regards to the AdS/CFT correspondence [7]. In fact these class of metric have been first study by C. Fefferman and R. Graham ([26]).

3.2. **The case where $k = \dim M/2$.**

Lemma 3.1. *Assume that (M, g) is a conformally compact Riemannian manifold of dimension $2k$ then*

$$\dim H_2^k(M) = \infty.$$

Proof. Because g and $\bar{g}$ are conformally equivalent we know that

$$H_2^k(M) = \mathcal{H}^k(M, g) = \mathcal{H}^k(M, \bar{g}).$$

Hence this lemma will a consequence of the following result

Lemma 3.2. *Assume that $(\overline{M}, \bar{g})$ is compact Riemannian manifold with smooth boundary, then for every $k \neq 0, \dim M$, we have*

$$\dim\{\alpha \in C^\infty(\Lambda^k T^*\overline{M}), d\alpha = d^*\alpha = 0\}] = \infty.$$

When $k = 0$, we know that $\mathcal{H}^0(\overline{M})$ consist of locally constant function hence is finite dimensional. Moreover, this equality for $k = 1$ can be proved by the same argument of the proof of the first assertion of (2.16), when $\overline{M}$ is connected, we have a linear map from the space of smooth function on the boundary of $\overline{M}$ to $\mathcal{H}^1(\overline{M})$ which associated to each $f \in C^\infty(\partial\overline{M})$ the differential of its harmonic extension to $\overline{M}$. The kernel of this map have dimension 1 hence when $n \geq 2$ we obtain that $\dim \mathcal{H}^1(\overline{M}) = \infty$.

A general proof using pseudo differential calculus shows that the solution space of an elliptic operator on a compact manifold with boundary has infinite dimension. The proof given below is more elementary, we will only used the *unique continuation principle*:

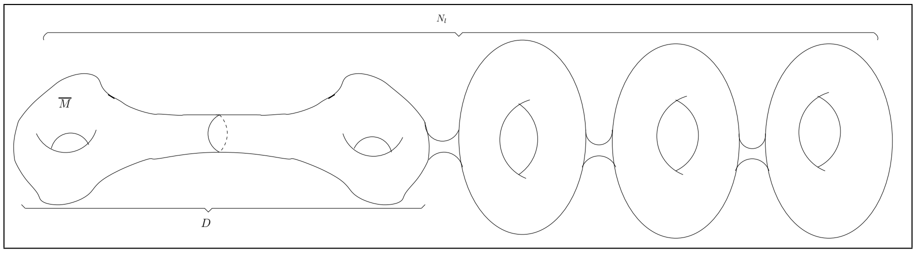
Proposition 3.3. *If N is a connected Riemannian manifold and $U \subset N$ is a non empty open subset of N. Then if $\alpha, \beta \in \mathcal{H}^k(N)$ satisfy $\alpha = \beta$ on U then*

$$\alpha = \beta \text{ on } N.$$

We consider $D = \overline{M}\#_{\partial\overline{M}}\overline{M}$ the double of $\overline{M}$, and we consider

$$N_l = D\#\Big(\#_l \mathbb{T}^n\Big)$$

the manifold obtained by making the connected sum of D with l copy of a n-torus $\mathbb{T}^n$ on the second copy of $\overline{M} \subset D$. We endowed N_l with a smooth Riemannian metric which coincide with $\bar{g}$ on the first copy.



Then by the unique continuation principle we know that the restriction map $\mathcal{H}^k(N_l) \to \mathcal{H}^k(\bar{M})$ on the first copy is injective hence

$$\dim \mathcal{H}^k(\bar{M}, \bar{g}) \geq b_k(N_l)\,.$$

However when $k \neq 0, \dim \bar{M}$ we have $b_k(N_l) = b_k(D) + l b_k(\mathbb{T}^n)$, hence letting $l \to \infty$ we obtain the desired result. □

3.3. **A reduction to exact metric.** The spaces of reduced L^2 cohomology depends only on the L^2 structures, hence if g_1 and g_2 are two Riemannian metric on a manifold M (g_1, g_2 need not to be complete) which are quasi isometric that is for a certain constant C

$$\frac{g_1}{C} \leq g_2 \leq C g_1\ ,$$

then clearly the Hilbert spaces $L^2(\Lambda^k T^* M, g_1)$ and $L^2(\Lambda^k T^* M, g_2)$ are the same with equivalent norms. Hence the quotient space defining reduced L^2 cohomology are the same[6] , that is

$$H_2^k(M, g_1) = H_2^k(M, g_2).$$

In particular we have the following :

Proposition 3.4. *If (M, g) is a conformally compact Riemannian metric, then $H_2^k(M, g)$ doesn't depend on the conformally compact metric on M.*

Proof. As a matter of fact, if $g = \frac{\bar{g}}{y^2}$ and $g_1 = \frac{\bar{g}_1}{y_1^2}$ are two conformally compact Riemannian metric on M, then because $\bar{g}$ and $\bar{g}_1$ are smooth Riemannian metric on a compact manifold, we know that there is a constant C_1 such that

$$\frac{\bar{g}_1}{C_1} \leq \bar{g} \leq C_1 \bar{g}_1.$$

Moreover there is always a smooth function $u : \overline{M} \to \mathbb{R}$ such that

$$y = e^u y_1$$

hence there is a constant C_2 such that

$$\frac{y_1}{C_2} \leq y \leq C_2 y_1.$$

Eventually we obtain for $C = C_1 C_2^2$,

$$C^{-1} g_1 \leq g \leq C g_1.$$

□

[6]Indeed the space $Z_2^k(M)$ depends only of the topology of $L^2(\Lambda^k T^* M, g)$: on $L^2(\Lambda^k T^* M, g) \times C_0^\infty(\Lambda^{k+1} T^* M)$ the bilinear form $(\alpha, \beta) \mapsto \langle \alpha, d^* \beta \rangle$ does not depends on g.

An simpler example of conformally compact Riemannian metric are the so-called exact conformally compact metric : let h be a smooth Riemannian metric on $N = \partial\overline{M}$ and $y : \overline{M} \to \mathbb{R}_+$ a boundary defining function, if $\varepsilon > 0$ is sufficiently small then

$$\frac{|dy|^2 + h}{y^2} = dr^2 + e^{2r}h, r = -\log y$$

is a conformally compact metric on the collar neighborhood $\{y < \varepsilon\}$ of $N \subset \overline{M}$. Any conformally compact Riemannian metric having such an expression near N will be called exact.

3.4. **A Rellich type identity and its applications.**

3.4.1. *The formula.* Let Ω be a Riemannian manifold with smooth boundary $\partial\Omega$

Lemma 3.5. *Let X be a vector field on Ω and $\alpha \in C^\infty(\Lambda^k T^*\Omega)$ then*

$$\begin{aligned}\mathcal{L}_X \alpha = \left.\frac{d}{dt}\right|_{t=0} \left(\Phi_X^t\right)^* \alpha &= d(\mathrm{int}_X\, \alpha) + \mathrm{int}_X(d\alpha)\\ &= \nabla X.\alpha + \nabla_X \alpha\ ,\end{aligned}$$

where Φ_X^t is the flow associated to X and for $(E_1, ..., E_n)$ a local orthonormal frame and $(\theta^1, ..., \theta^n)$ its dual frame then

$$\nabla X.\alpha = \sum_{i,j} \theta^i \wedge \theta^j\, (\nabla_{E_i} X, E_j)\, \mathrm{int}_{E_i}\, \mathrm{int}_{E_j}\, \alpha = \sum_{i=1}^n \theta^i \wedge (\mathrm{int}_{\nabla_{E_i} X}\, \alpha).$$

Proof. We used the formula (1.1)

$$d = \sum_{i=1}^n \theta^i \wedge \nabla_{E_i}\ .$$

Hence

$$\begin{aligned}d(\mathrm{int}_X\, \alpha) &= \sum_{i=1}^n \theta^i \wedge \nabla_{E_i}(\mathrm{int}_X\, \alpha)\\ &= \sum_{i=1}^n \theta^i \wedge (\mathrm{int}_{\nabla_{E_i} X}\, \alpha) + \sum_{i=1}^n \theta^i \wedge (\mathrm{int}_X\, \nabla_{E_i}\alpha)\\ &= \nabla X.\alpha + \sum_{i=1}^n \theta^i \wedge (\mathrm{int}_X\, \nabla_{E_i}\alpha)\end{aligned}$$

and

$$\begin{aligned}\mathrm{int}_X(d\alpha) &= \sum_{i=1}^n \mathrm{int}_X(\theta^i \wedge \nabla_{E_i}\alpha)\\ &= \sum_{i=1}^n \mathrm{int}_X(\theta^i)\, \nabla_{E_i}\alpha - \sum_{i=1}^n \theta^i \wedge \mathrm{int}_X(\nabla_{E_i}\alpha)\\ &= \nabla_X\alpha - \sum_{i=1}^n \theta^i \wedge \mathrm{int}_X(\nabla_{E_i}\alpha)\end{aligned}$$

Hence the result. ☐

Corollary 3.6. *Let $\vec{\nu} : \partial\Omega \to T\Omega$ be the inward unit vector field, then for $\alpha \in C_0^\infty(\Lambda^k T^*\overline{\Omega})$:*

$$\int_\Omega (\nabla X.\alpha, \alpha) + \frac{1}{2} \operatorname{div} X\, |\alpha|^2 = \int_\Omega (\operatorname{int}_X \alpha, d^*\alpha) + (\operatorname{int}_X d\alpha, \alpha)$$
$$+ \int_{\partial\Omega} \left[\frac{1}{2} (X, \vec{\nu}) |\alpha|^2 - (\operatorname{int}_{\vec{\nu}} \alpha, \operatorname{int}_X \alpha) \right] d\sigma.$$

Proof. The lemma (3.5) implies that

$$\int_\Omega [(\nabla X.\alpha, \alpha) + (\nabla_X \alpha, \alpha)]\, d\operatorname{vol}_g = \int_\Omega [(d(\operatorname{int}_X \alpha), \alpha) + (\operatorname{int}_X (d\alpha), \alpha)]\, d\operatorname{vol}_g .$$

Then the equality follows directly from the following two Green's type formulas :

$$\int_\Omega (\nabla_X \alpha, \alpha) d\operatorname{vol}_g = \frac{1}{2} \int_\Omega X.(\alpha, \alpha) d\operatorname{vol}_g = \frac{1}{2} \int_\Omega \operatorname{div} X |\alpha|^2\, d\operatorname{vol}_g - \frac{1}{2} \int_{\partial\Omega} (X, \vec{\nu}) |\alpha|^2 d\sigma$$

and

$$\int_\Omega (d(\operatorname{int}_X \alpha), \alpha) d\operatorname{vol}_g = \int_\Omega (\operatorname{int}_X \alpha, d^*\alpha) d\operatorname{vol}_g - \int_{\partial\Omega} (\operatorname{int}_{\vec{\nu}} \alpha, \operatorname{int}_X \alpha) d\sigma.$$

□

This integration by part formula is due to Donnelly-Xavier in [24], this formula has been used and refined by many authors (see for instance [25],[38]).

3.5. **Application to conformally compact Riemannian manifold.** We apply here this formula to $\Omega =]0, \infty[\times N$ endowed with the metric $dr^2 + e^{2r} h$ where h is a smooth Riemannian metric on the compact manifold N. We choose the vector field

$$X = -\frac{\partial}{\partial r}.$$

The curves $r \mapsto (r, \theta) \in \Omega$ are geodesic hence

$$\nabla_X \frac{\partial}{\partial r} = 0,$$

moreover it is not hard to verify that for $v \in TN$

$$\nabla_v X = -v \text{ [7]}.$$

A direct computation shows that for $\alpha \in \Lambda^k T^*\Omega$

$$(\nabla_X \alpha, \alpha) = -k|\alpha|^2 + |\operatorname{int}_X \alpha|^2 \text{ and } \operatorname{div} X = (n-1).$$

Hence

$$(\nabla_X \alpha, \alpha) + \frac{1}{2} \operatorname{div} X |\alpha|^2 = \left(\frac{n-1}{2} - k \right) |\alpha|^2 + |\operatorname{int}_X \alpha|^2$$

and we obtain that for all $\alpha \in C_0^\infty(\Lambda^k T^*\overline{\Omega})$:

(3.1)
$$\int_\Omega [(\operatorname{int}_X \alpha, d^*\alpha) + (\operatorname{int}_X d\alpha, \alpha)]\, d\operatorname{vol}_g \geq \left(\frac{n-1}{2} - k \right) \int_\Omega |\alpha|^2 d\operatorname{vol}_g$$
$$+ \int_{\partial\Omega} \left[\frac{1}{2} |\alpha|^2 - |\operatorname{int}_X \alpha|^2 \right] d\sigma.$$

[7] this comes from the fact that that level set of r are totally umbilical.

Corollary 3.7. *Assume that $k \le \frac{n-1}{2}$ and that $\alpha \in L^2(\Lambda^k T^*\Omega)$ satisfies*

$$d\alpha = d^*\alpha = 0$$

and

$$\operatorname{int}_{\vec{\nu}} \alpha = 0 \text{ along } \partial\Omega$$

then $\alpha = 0$. Hence by (1.6) we have that $H_2^k(\Omega) = \{0\}$ for $k < n/2$.

Proof. Let $\rho \in C_0^\infty(\mathbb{R}_+)$ having support in $[0,1]$ and such that $\rho = 1$ near 0 and let $\rho_N(t) = \rho(t/N)$. We apply the inequality (3.1) to $\alpha_N = \rho_N \alpha$:

$$\int_\Omega (\operatorname{int}_X d\alpha_N, \alpha_N) d\operatorname{vol}_g \ge \left(\frac{n-1}{2} - k\right) \int_\Omega |\alpha_N|^2 d\operatorname{vol}_g + \frac{1}{2}\int_{\partial\Omega} |\alpha|^2 d\sigma.$$

But

$$\int_\Omega (\operatorname{int}_X d\alpha_N, \alpha_N) d\operatorname{vol}_g = \int_\Omega \frac{1}{N}\rho'\left(\frac{r}{N}\right)|\alpha|^2 d\operatorname{vol}_g \le \frac{1}{N}\|\rho'\|_{L^\infty}\int_\Omega |\alpha|^2 d\operatorname{vol}_g .$$

Hence letting N going to infinity, we obtain that

$$0 = \left(\frac{n-1}{2} - k\right)\int_\Omega |\alpha|^2 d\operatorname{vol}_g + \frac{1}{2}\int_{\partial\Omega}|\alpha|^2 d\sigma.$$

Hence if $k < (n-1)/2$, we obtain $\alpha = 0$. When $k = (n-1)/2$, we obtain that $\alpha = 0$ along $\partial\Omega$. Then the unique continuation property for solution of elliptic operator of order 1 implies that $\alpha = 0$. □

Corollary 3.8. *Assume that $k < (n-1)/2$, then $\forall \alpha \in C_0^\infty(\Lambda^k T^*\Omega)$,*

$$\|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2 \ge \frac{1}{2}\left(\frac{n-1}{2} - k\right)^2 \|\alpha\|_{L^2}^2.$$

Proof. We apply the estimate (3.1) to $\alpha \in C_0^\infty(\Lambda^k T^*\Omega)$, then there is no boundary term and we get the inequality

$$\int_\Omega \left[(\operatorname{int}_X \alpha, d^*\alpha) + (\operatorname{int}_X d\alpha, \alpha)\right] d\operatorname{vol}_g \ge \left(\frac{n-1}{2} - k\right)\int_\Omega |\alpha|^2 d\operatorname{vol}_g .$$

But with the Cauchy-Schwarz inequality, we obtain :

$$\begin{aligned}\int_\Omega \left[(\operatorname{int}_X \alpha, d^*\alpha) + (\operatorname{int}_X d\alpha, \alpha)\right] d\operatorname{vol}_g &\le \|d^*\alpha\|_{L^2}\|\alpha\|_{L^2} + \|d\alpha\|_{L^2}\|\alpha\|_{L^2} \\ &\le \sqrt{2}\left[\|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2\right]^{1/2}\|\alpha\|_{L^2}.\end{aligned}$$

Hence the result. □

Remark 3.9. Using the vector field $-X$, the reader can check that the estimate of the corollary (3.8) is also true for $k > (n+1)/2$. Also with (1.7), the reader can also prove that $H_2^k(\Omega, \partial\Omega) = \{0\}$ for $k > n/2$.

3.6. **The spectrum of the Hodge-deRham Laplacian.**

3.6.1. *The essential spectrum.* Let $A : \mathcal{D}(A) \to H$ be a selfadjoint operator on a Hilbert space, then the spectrum of A is the subset $\operatorname{spec} A$ of $\mathbb{C}$ consisting of those $z \in \mathbb{C}$ such that $A - z\,\mathrm{Id}$ has not a bounded inverse. Because A is self adjoint, we have $\operatorname{spec} A \subset \mathbb{R}$. Moreover if $\mathcal{C} \subset \mathcal{D}(A)$ is a core for A then we have that $\lambda \in \mathbb{R}$ belongs to the spectrum of A, if and only if there is a sequence $(\varphi_n)_n$ of element of $\mathcal{C}$ such that

$$\begin{cases} \|\varphi_n\| = 1 \\ \lim_{n\to\infty} \|A\varphi_n - \lambda\varphi_n\| = 0. \end{cases}$$

We can separate the spectrum of A in two parts the discrete part and the essential part :

$$\operatorname{spec} A = \operatorname{spec}_d A \cup \operatorname{spec}_e A,$$

where $\operatorname{spec}_d A$ is the set of isolated point in $\operatorname{spec} A$ which are eigenvalue with finite multiplicity and $\operatorname{spec}_e A$ is the set of non-isolated point in $\operatorname{spec} A$ or of eigenvalue with infinite multiplicity. We have the following characterization of the essential spectrum : a real number λ belongs to the essential spectrum of A if and only if

there is a sequence $(\varphi_n)_n$ in $\mathcal{D}(A)$ with

$$\begin{cases} \|\varphi_n\| = 1 \\ \lim_{n\to\infty} \varphi_n = 0 \text{ weakly in } H \\ \lim_{n\to\infty} \|A\varphi_n - \lambda\varphi_n\| = 0. \end{cases}$$

Another useful characterisation of the complementary of the essential spectrum is the following :

Proposition 3.10. $\lambda \notin \operatorname{spec}_e A$ *if and only if there is a bounded operator* $G : H \to \mathcal{D}(A)$ *such that* $(A - \lambda\,\mathrm{Id})G - \mathrm{Id}$ *and* $G(A - \lambda\,\mathrm{Id}) - \mathrm{Id}$ *are compact operators. Moreover the operator* G *can be chosen so that* $(A-\lambda\,\mathrm{Id})G - \mathrm{Id} = G(A-\lambda\,\mathrm{Id}) - \mathrm{Id}$ *is the orthogonal projection on* $\ker(A - \lambda\,\mathrm{Id})$.

From these properties, it is not hard to verify that the essential spectrum is stable by compact perturbation :

Theorem 3.11. *Let* $A, B : \mathcal{D}(A) = \mathcal{D}(B) \to H$ *be two self-adjoint operator such that* $(A+i)^{-1} - (B+i)^{-1}$ *is a compact operator then* A *and* B *have the same essential spectrum :*

$$\operatorname{spec}_e(A) = \operatorname{spec}_e(B).$$

3.6.2. *Case of the Hodge-deRham Laplacian of complete Riemannian manifold.* Let (M, g) be a complete Riemannian manifold, then the operator $\Delta = dd^* + d^*d : C_0^\infty(\Lambda^k T^*M) \to L^2(\Lambda^k T^*M)$ has a unique self-adjoint extension to $L^2(\Lambda^k T^*M)$ which also denoted by Δ with domain

$$\mathcal{D}(\Delta) = \{\alpha \in L^2(\Lambda^k T^*M), \Delta\alpha \in L^2\}$$

That is $\alpha \in \mathcal{D}(\Delta)$ if and only if there is a constant $C > 0$ such that

$$\forall \varphi \in C_0^\infty(\Lambda^k T^*M),\ |\langle \alpha, \Delta\varphi\rangle| \le C\|\varphi\|_{L^2}.$$

In fact we can prove that

$$\mathcal{D}(\Delta) = \{\alpha \in L^2(\Lambda^k T^*M), d\alpha \in L^2, d^*\alpha \in L^2, dd^*\alpha \in L^2, d^*d\alpha \in L^2\}.$$

Moreover, $C_0^\infty(\Lambda^k T^*M)$ is dense in $\mathcal{D}(\Delta)$ for the graph norm : $\alpha \mapsto \|\alpha\|_{L^2} + \|\Delta\alpha\|_{L^2}$. We have the following very useful result of H. Donnelly and I.Glazman (see [3, 6, 23, 29])

Theorem 3.12. *Zero is not in the essential spectrum of* Δ $(0 \notin \mathrm{spec}_e(\Delta))$ *if and only if there is a compact set* $K \subset M$ *and a constant* $\varepsilon > 0$ *such that*

$$\forall \varphi \in C_0^\infty(\Lambda^k T^*(M \setminus K)), \|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2 = \langle \alpha, \Delta\alpha \rangle \geq \varepsilon \|\alpha\|_{L^2}^2.$$

Remark 3.13. When $\Omega \subset M$ is a open set with smooth compact boundary $\partial\Omega$, we can introduce two operators :

- The Laplacian with the relative boundary condition :
$$\mathcal{C}_{rel} = \{\alpha \in C_0^\infty(\Lambda^k T^*\overline{\Omega}), \text{ such that } \iota^*\alpha = \iota^*(d^*\alpha) = 0\}$$
then $\Delta : \mathcal{C}_{rel} \to L^2$ has a unique selfadjoint extension Δ_{rel}.
- The Laplacian with the absolute boundary condition :
$$\mathcal{C}_{abs} = \{\alpha \in C_0^\infty(\Lambda^k T^*\overline{\Omega}), \text{ such that } \mathrm{int}_{\vec{\nu}}\, \alpha = \mathrm{int}_{\vec{\nu}}\, d\alpha = 0\}$$
then $\Delta : \mathcal{C}_{abs} \to L^2$ has a unique selfadjoint extension Δ_{abs}.

Moreover when $M \setminus \Omega$ is a compact set we have

$$0 \notin \mathrm{spec}_e \Delta \Leftrightarrow 0 \notin \mathrm{spec}_e \Delta_{abs} \Leftrightarrow 0 \notin \mathrm{spec}_e \Delta_{rel}.$$

3.7. **Applications to conformally compact manifolds.**

3.7.1. *The essential spectrum.*

Proposition 3.14. *Let* (M, g) *be a conformally compact Riemannian manifold endowed with an exact metric, then for* $k \notin \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$, *zero is not in the essential spectrum of the Hodge deRham Laplacian acting on* $k-$*forms.*

Proof. This is a direct corollary of the result (3.12) of I.Glatzmann and H.Donnelly and of the corollary (3.8). □

Corollary 3.15. *Assume again that* (M, g) *be a conformally compact Riemannian manifold endowed with an exact metric. For* $k \leq (n-1)/2$, *we consider* $\alpha \in Z_2^k(M)$ *which is zero in* $H_2^k(M)$ *then there is* $\beta \in L^2(\Lambda^{k-1}T^*M)$ *such that*

$$\alpha = d\beta.$$

Proof. By hypothesis, we know that there is a sequence $\varphi_l \in C_0^\infty(\Lambda^{k-1}T^*M)$ such that

$$\alpha = L^2 - \lim_{l\to\infty} d\varphi_l.$$

Because $k - 1 < (n-1)/2$, from (3.10) we know that there is a bounded operator

$$G : L^2(\Lambda^{k-1}T^*M) \to L^2(\Lambda^{k-1}T^*M)$$

such that $\forall \varphi \in C_0^\infty(\Lambda^{k-1}T^*M)$

$$\Delta G\varphi = \varphi - h(\varphi),$$

where $h(\varphi)$ is the orthogonal projection on the kernel of Δ that is on $\mathcal{H}^k(M)$. Hence for $\psi_l = G\varphi_l$ we obtain

$$\varphi_l = h(\varphi_l) + dd^*\psi_l + d^*d\psi_l$$

and

$$\alpha = L^2 - \lim_{l\to\infty} dd^*d\psi_l.$$

We let $\eta_l = d^*d\psi_l$, we have $d^*\eta_l = 0$ and $d\eta_l = d\varphi_l$, hence, by elliptic regularity, η_l is smooth. In particular, $d\eta_l \in L^2$ and $d^*\eta_l \in L^2$ and $h(\eta_l) = 0$. We have

$$\begin{aligned}
\|\eta_l - \eta_k\|_{L^2}^2 &= \langle \Delta G(\eta_l - \eta_k), (\eta_l - \eta_k)\rangle \\
&= \langle dG(\eta_l - \eta_k), d(\eta_l - \eta_k)\rangle + \langle d^*G(\eta_l - \eta_k), d^*(\eta_l - \eta_k)\rangle \\
&\leq \|dG(\eta_l - \eta_k)\|_{L^2}\|d(\eta_l - \eta_k)\|_{L^2} \\
&\leq \left[\,\langle \Delta G(\eta_l - \eta_k), G(\eta_l - \eta_k)\rangle\,\right]^{1/2} \|d(\eta_l - \eta_k)\|_{L^2} \\
&\leq \left(\|\eta_l - \eta_k\|_{L^2}\|G(\eta_l - \eta_k)\|_{L^2}\right)^{1/2} \|d(\eta_l - \eta_k)\|_{L^2} \\
&\leq C\|(\eta_l - \eta_k)\|_{L^2}\|d(\eta_l - \eta_k)\|_{L^2}.
\end{aligned}$$

where C^2 is the operator norm of G. Hence

$$\|\eta_l - \eta_k\|_{L^2} \leq C\|d(\eta_l - \eta_k)\|_{L^2} = C\|d(\varphi_l - \varphi_k)\|_{L^2}.$$

The sequence $(d\varphi_l)_l$ is a Cauchy sequence in L^2 hence $(\eta_k)_k$ is also a Cauchy sequence in L^2 converging to some $\beta \in L^2$ and we have

$$\alpha = d\beta.$$

□

Remark 3.16. This proof also shows that the primitive β obtained satisfies the equation $d^*\beta = 0$. Hence if α is smooth then β will be also smooth.

3.7.2. *At infinity.* With the same method, and because we have the vanishing result for the L^2 cohomology (3.7), we have

Proposition 3.17. *Let $\Omega =]0,\infty[\times N$ endowed with the warped product metric $(dr)^2 + e^{2r}h$, then for $k \leq (n-1)/2$ and $\alpha \in Z_2^k(\Omega)$ there is $\beta \in L^2(\Lambda^{k-1}T^*\Omega)$ such that*

$$\alpha = d\beta.$$

Proof. We offer a proof which is more elementary than the one passing through the essential spectrum and the vanishing result, this proof has an independent interest; the argument comes from an article by P. Pansu ([54]) where such techniques were used in order to obtain clever negative pinching results. We introduce the vector field $T = -X = \frac{\partial}{\partial r}$ and we consider its flow

$$\Phi^t(r,\theta) = (r+t,\theta).$$

When $\alpha = dr \wedge \alpha_1 + \alpha_2 \in C^\infty(\Lambda^k T^*\Omega)$ where $\alpha_1 \in C^\infty(\mathbb{R}_+, C^\infty(\Lambda^{k-1}T^*N))$ and $\alpha_2 \in C^\infty(\mathbb{R}_+, C^\infty(\Lambda^k T^*N))$, we easily obtain the estimate :

$$\begin{aligned}
\left|\left(\Phi^t\right)^* \alpha\right|^2 (r,\theta) &= e^{2(k-1)t}\left|\alpha_1\right|^2 (r+t,\theta) + e^{2kt}\left|\alpha_2\right|^2 (r+t,\theta) \\
&\leq e^{2kt}\left|\alpha\right|^2 (r+t,\theta)
\end{aligned} \tag{3.2}$$

But when $\alpha \in Z_2^k(\Omega)$, the Cartan formula says that

$$\left(\Phi^t\right)^* \alpha - \alpha = d\beta_t. \tag{3.3}$$

where

$$\beta_t = \int_0^t \left(\Phi^s\right)^* (\mathrm{int}_T\, \alpha)\, ds.$$

From our estimate (3.2), when $k \leq (n-1)/2$ we easily obtain

$$
\begin{aligned}
\left\| \left(\Phi^t\right)^* \alpha \right\|_{L^2}^2 &\leq \int_{]0,\infty[\times N} e^{2kt} \left|\alpha\right|^2 (r+t,\theta) e^{(n-1)r} dr d\,\mathrm{vol}_h(\theta) \\
&\leq \int_{]0,\infty[\times N} \left|\alpha\right|^2 (r+t,\theta) e^{(n-1)(r+t)} dr d\,\mathrm{vol}_h(\theta) \\
&\leq \int_{]t,\infty[\times N} \left|\alpha\right|^2 (r,\theta) e^{(n-1)r} dr d\,\mathrm{vol}_h(\theta) = \|\alpha\|^2_{L^2(\Omega_t)} ,
\end{aligned}
\tag{3.4}
$$

where $\Omega_t =]t,\infty[\times N$. Hence

$$
L^2 - \lim_{t\to+\infty} \left(\Phi^t\right)^* \alpha = 0.
$$

Moreover for $t \geq 0$, we have

$$
\begin{aligned}
|\beta_t|(r,\theta) &\leq \int_0^t |\alpha|(r+s,\theta) e^{(k-1)s} ds \\
&\leq e^{-(k-1)r} \int_r^\infty |\alpha|(s,\theta) e^{(k-1)s} ds.
\end{aligned}
\tag{3.5}
$$

We need the following lemma

Lemma 3.18. *Let $v \in C_0^\infty(\mathbb{R}_+)$ and let*

$$
Mv(r) = e^{-(k-1)r} \int_r^\infty v(s) e^{(k-1)s} ds
$$

then for $k-1 < (n-1)/2$ we have

$$
\left(\frac{n-1}{2} - (k-1)\right)^2 \int_0^\infty |Mv|^2(t) e^{(n-1)t} dt \leq \int_0^\infty |v|^2(t) e^{(n-1)t} dt.
$$

That is the operator M extends as a bounded operator in $L^2(\mathbb{R}_+, e^{(n-1)t}dt)$.

Proof of the lemma (3.18) . Let $w(r) = e^{(k-1)r} Mv(r)$, then w has compact support and $w'(r) = -e^{(k-1)r} v(r)$ and let

$$
w(r) = f(r) e^{-\left(\frac{n-1}{2} - (k-1)\right) r}
$$

and $\epsilon_k = n-1-2(k-1) > 0$. Then

$$
\begin{aligned}
\int_0^\infty |v|^2(r) e^{(n-1)r} dr &= \int_0^\infty |w'|^2(r) e^{\epsilon_k r} dr \\
&= \int_0^\infty \left[|f'|^2(r) - \epsilon_k f' f + \left(\frac{\epsilon_k}{2}\right)^2 |f|^2(r) \right] dr \\
&\geq -\frac{\epsilon_k}{2} \int_0^\infty \left(f^2\right)'(r) dr + \left(\frac{\epsilon_k}{2})\right)^2 \int_0^\infty |f|^2(r) dr \\
&\geq \frac{\epsilon_k}{2} |f|^2(0) + \left(\frac{\epsilon_k}{2}\right)^2 \int_0^\infty |f|^2(r) dr \\
&\geq \left(\frac{\epsilon_k}{2}\right)^2 \int_0^\infty |w|^2(r) e^{(n-1-2(k-1))r} dr \\
&= \left(\frac{\epsilon_k}{2}\right)^2 \int_0^\infty |Mv|^2(r) e^{(n-1)r} dr.
\end{aligned}
$$

□

From the lemma (3.18) and the estimate (3.5) we obtain :

$$\|\beta_t\|_{L^2} \le \frac{2}{\frac{n-1}{2}-(k-1)}\|\alpha\|_{L^2}. \tag{3.6}$$

Hence if we let $t\to\infty$ in the equation (3.3), the estimates (3.4,3.6) imply that that $\alpha = d\beta_\infty$ where

$$\beta_\infty = \int_0^\infty (\Phi^s)^* (\operatorname{int}_T \alpha)\, ds \in L^2.$$

□

Remark 3.19. When $k \ge (n+1)/2$, we can also show with a similar proof that $\mathcal{H}^k_{rel}(\Omega) = 0$, even one can obtain that every $\alpha \in Z^k(\Omega,\partial\Omega)$ has an L^2 primitive given by

$$\beta = \int_{-\infty}^0 (\Phi^s)^* (\operatorname{int}_T \tilde\alpha)\, ds$$

where $\tilde\alpha$ is the extension of α to $\mathbb{R}\times N$ by letting $\tilde\alpha = 0$ on $(\mathbb{R}\times N)\setminus\Omega =]-\infty,0[\times N$.

3.8. **Mazzeo's result.** We will now prove R.Mazzeo's result :

Theorem 3.20. *Let (M,g) be a conformally compact Riemannian manifold then*

i) when $k<\dim M/2$, then $H^k_2(M)\simeq H^k_0(M)\simeq H^k(\overline{M},\partial\overline{M})$.
ii) when $k>\dim M/2$, then $H^k_2(M)\simeq H^k(M)$.
iii) when $k=\dim M/2$, then $\dim H^k_2(M)=\infty$.

Proof. We have already prove iii), we will only prove i) and indicate how we can prove ii) with similar arguments. We can assume that the metric is exact. So let $\Omega =]0,\infty[\times N$ being a neighborhood of infinity endowed with the warped product metric $(dr)^2+e^{2r}h$. And we will denote $K = M\setminus\Omega$ and $j_\Omega : \Omega\to M$, $j_K : K\to M$ the inclusion maps.

We assume that $k<n/2$. We are going to prove that the natural map $H^k_0(M)\simeq H^k(K,\partial K)\to H^k_2(M)$ is an isomorphism.

fact 1: The map $H^k_0(M)\simeq H^k(K,\partial K)\to H^k_2(M)$ is injective. Let $\alpha\in C^\infty_0(\Lambda^kT^*M)$ a closed form with support in K which is mapped to zero in $H^k_2(M)$. According to the corollary (3.15) we know that α has a L^2 primitive : there is $\beta\in L^2(\Lambda^{k-1}T^*M)$ such that

$$\alpha = d\beta.$$

Let $j_\Omega : \Omega\to M$ be the inclusion map, then we have that

$$j_\Omega^*\beta \in Z^{k-1}_2(\Omega)$$

and by the proposition (3.17), we obtain $\eta\in L^2(\Lambda^{k-2}T^*\Omega)$ such that $j^*_\Omega\beta = d\eta$. Consider $\bar\eta$ an extension of η to M. Then $\beta - d\bar\eta$ has compact support and $d(\beta - d\bar\eta)=\alpha$.

fact 2: The map $H^k_0(M)\simeq H^k(K,\partial K)\to H^k_2(M)$ is surjective. Let $\alpha\in\mathcal{H}^k(M)$ then we know that $j^*_\Omega(\alpha)\in Z^k_2(\Omega)$ hence by the proposition (3.17), we obtain a $\eta\in L^2(\Lambda^{k-1}T^*\Omega)$ such that $d\eta=\alpha$. Now we choose $\bar\eta\in L^2((\Lambda^{k-1}T^*M)$ an extension of η which is in the domain of d and we obtain that $\alpha - d\bar\eta$ has compact support moreover α and $\alpha - d\bar\eta$ belong to the same reduced cohomology class

The proof of the case ii) is done similarly. So assume that $k>\dim M/2$. First we recall that according to (3.9), we have $H^k_2(\Omega,\partial\Omega)=\{0\}$ hence the natural map $[j^*_K] : H^k_2(M)\to H^k(K)\simeq H^k(M)$ is injective (see 1.15).

In order to show that this map is surjective, we proceed as follow: let $c \in H^k(K)$ and $\alpha \in c$ if $\bar{\alpha} \in C_0^\infty(\Lambda^k T^*M)$ is a smooth extension of α then clearly

$$j_\Omega^*(d\bar{\alpha}) \in Z^{k+1}(\Omega, \partial\Omega).$$

We are then able to find some $\eta \in L^2(\Lambda^k T^*\Omega)$ which is smooth such that $j_\Omega^*(d\bar{\alpha}) = d\eta$ and $\iota^*\eta = 0$ (where $\iota : \partial\Omega \to \Omega$ is the inclusion map). Then as in the proof of the lemma 1.14, we can show that the k form $\tilde{\alpha}$ defined by $\tilde{\alpha} = \alpha$ on K and $\tilde{\alpha} = \bar{\alpha} - \eta$ is closed and L^2 on M. Moreover it is clear that its L^2 cohomology class is map to c by $[j_K^*]$.

When $k > (n-1)/2$, the surjectivity of this map is more easy to prove : consider $\pi : M \to K$ the natural retraction which is the identity on K and which is $\pi(r,\theta) = (0,\theta)$ on Ω. Then a small verification shows that $\pi^*\alpha$ is a L^2 closed form on M and clearly $j_K^*(\pi^*\alpha) = \alpha$. □

Remark 3.21. In fact, similar arguments shows that when (M,g) is a complete Riemannian manifold such that 0 is not in the spectrum of the Hodge-deRham Laplacian on k forms then for $K \subset M$ a compact subset and $\Omega = M \subset K$, we have the two short exact sequences :

$$H_2^{k-1}(\Omega) \to H^k(K,\partial K) \to H_2^k(M) \to H_2^k(\Omega) \to H^{k+1}(K,\partial K) \to H_2^{k+1}(M) \to H_2^{k+1}(\Omega).$$

$$H_2^{k-1}(K) \to H^k(\Omega,\partial\Omega) \to H_2^k(M) \to H_2^k(K) \to H^{k+1}(\Omega,\partial\Omega) \to H_2^{k+1}(M) \to H_2^{k+1}(K).$$

In fact, these exact sequences always hold for the non reduced L^2 cohomology and when 0 is not in the spectrum of the Hodge-deRham Laplacian on k forms then the non reduced and reduced L^2 cohomology coincide in degree k and $k-1$ (this is exactly the assertion of the proposition (3.15).

3.9. **Bibliographical hints.** The last lectures is based on a new proof of R. Mazzeo's results [48] by N. Yeganefar [65], some part of our arguments can be found in the paper of V.M. Gol'dshteĭn, V. I. Kuz'minov, I. A. Shvedov [30]. A very good discussion on the essential spectrum can be find in the papers of H. Donnely [23] and C. Bär [6] . The integration by part formula is due to H.Donnelly and F.Xavier [24] and has been revisited by many authors for instance by J.Escobar-A.Freire [25] and A. Kasue [38] .

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