CONFORMALLY FLAT MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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ABSTRACT. We show that complete conformally flat manifolds of dimension $n \geqslant 3$ with nonnegative Ricci curvature enjoy nice rigidity properties: they are either flat, or locally isometric to a product of a sphere and a line, or are globally conformally equivalent to $\mathbb{R}^n$ or to a spherical spaceform $\mathbb{S}^n/\Gamma$. This extends previous results due to Q.-M. Cheng, M.-H. Noronha, B.-L. Chen and X.-P. Zhu, and S. Zhu.

In this note, we study complete conformally flat manifolds with nonnegative Ricci curvature. It is well known in dimension 2 that the sphere, the plane and their quotients are the only surfaces that can be endowed with a metric of nonnegative curvature. As Riemannian surfaces are always conformally flat, it seems natural to look at higher dimensional analogues of this fact. R. Schoen and S. T. Yau showed in [21] that conformal flatness together with nonnegative scalar curvature (or any variant of it involving the Yamabe constant) still allows much flexibility. On the contrary, if one concentrates on stronger curvature conditions such as Ricci curvature bounds, one might expect that they put some quite strong restrictions on the manifold.

For instance, in case the manifold is closed, a characterization of the spaceforms and the quotients of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ has been obtained by M.-H. Noronha [18]. If one adds extra assumptions like constant positive scalar curvature, further restrictions can moreover be obtained, see for instance Q.-M. Cheng [7] for results and references.

More generally, and without any compactness assumption, S. Zhu proved in [24] that the *universal covering* of a complete conformally flat manifold of nonnegative Ricci curvature either is conformally equivalent to $\mathbb{S}^n$ or $\mathbb{R}^n$, or is isometric to $\mathbb{R} \times \mathbb{S}^{n-1}$. In the same vein, B.-L. Chen and X.-P. Zhu proved in [5] that the only complete non-compact conformally flat manifolds with nonnegative Ricci curvature and fast curvature decay at infinity are the complete flat manifolds. One motivation for this result is that it stands as an analog on real manifolds of well-known rigidity and gap phenomena on Kähler manifolds with nonnegative holomorphic bisectional curvature [6, 13, 14, 15, 16, 17].

The goal of this paper is to complete the above results by exhibiting a full classification of the possible geometries of complete conformally flat manifolds with nonnegative Ricci curvature. As we shall see, quite a bit more can be said in this rather general setting on the topology of those manifolds, and also on their geometry. More precisely, we prove:

Theorem. *Let (M, g) be a complete conformally flat manifold of dimension $n \geqslant 3$ with nonnegative Ricci curvature. Then, exactly one of the following holds:*

1. M *is* globally conformally equivalent *to $\mathbb{R}^n$ with a conformal* non-flat *metric with nonnegative Ricci curvature;*

2. M *is* globally conformally equivalent *to a spaceform of positive curvature, endowed with a conformal metric with nonnegative Ricci curvature;*

3. M *is* locally isometric *to the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$;*

4. M *is* isometric *to a complete flat manifold.*

Obviously, the last three cases do appear. In particular, since the round metric on spherical spaceforms has positive curvature, any small conformal deformation will keep nonnegative Ricci curvature as well, hence the second class is rather large. We shall show below that the first case also occurs, by exhibiting explicit examples of non-flat globally conformally flat metrics with nonnegative Ricci curvature on $\mathbb{R}^n$.

The most interesting part of our theorem lies of course in the dichotomy between the first two cases, where deformations are permitted, and the last two cases, where some strong rigidity is obtained. The philosophy of our result is then : a complete conformally flat manifold with nonnegative Ricci curvature is either (globally) *topologically simple* (diffeomorphic to a vector space or a quotient of a sphere), or it is (locally) *metrically rigid*. Hence, only very general assumptions on the conformally flat manifold are enough to yield strong constraints either on its topology, or on its geometry. If one compares with S. Zhu's classification [24] of the possible universal coverings, one sees that in the non-compact case (*i.e.* when quotients of the sphere are excluded), the presence of a nontrivial fundamental group implies metric rigidity.

The vector space $\mathbb{R}^n$ is then the only space that admits non-flat complete conformally flat metrics with nonnegative Ricci curvature. As already alluded to, B.-L. Chen and X.-P. Zhu [5] showed that, in case the metric satisfies fast curvature decay assumptions at infinity, the metric is flat. It is well known under very general circumstances that fast curvature decay at infinity implies strong constraints on the topology at infinity, but our work implies further that, in the conformally flat case, either the metric is already flat or the topology is precisely that of $\mathbb{R}^n$. The examples we exhibit below of non-flat conformally flat metrics with nonnegative Ricci curvature on $\mathbb{R}^n$ support the idea that the decay rate chosen by B.-L. Chen and X.-P. Zhu [5] is indeed the optimal one. It remains an interesting question to understand if further constraints can be deduced on non-flat conformally flat metrics with nonnegative Ricci curvature on $\mathbb{R}^n$

1. NON TRIVIAL CONFORMALLY FLAT METRICS ON $\mathbb{R}^n$ WITH NONNEGATIVE RICCI CURVATURE

In this short section, we show that the Theorem above is indeed the best one can hope for. As a matter of fact, we exhibit examples of (rotationnally symmetric) metrics on $\mathbb{R}^n$ that are *non flat*, but *complete, conformally flat and with nonnegative Ricci curvature*. Hence this may exist, whereas existence of any analogous metric on quotients is forbidden by our result. A by-product is the optimality of the curvature decay rate imposed by B.-L. Chen and X.-P. Zhu in [5] to get a flat metric: our example lies exactly on the threshold where the result in [5] does not hold anymore.

Construction of the example. Let f be a real function. From Besse's book [2, formula 1.159, page 59] we get the expression for the Ricci curvature of the metric $\mathrm{e}^{2f}g_0$ (g_0 the euclidean metric) is

$$(1.1) \qquad \mathrm{Ric} = -(n-2)(Ddf - df \otimes df) + (\Delta f - (n-2)|df|^2)g_0$$

which can be rewritten, in case $f = f(r)$ is a radial function and h is the unit round metric on $\mathbb{S}^{n-1}$, as

$$
\begin{aligned}
(1.2) \qquad \mathrm{Ric} = & -(n-2)\left(f''dr^2 - (f')^2dr^2 + rf'h\right) \\
& + \left(-f'' - (n-1)r^{-1}f' - (n-2)(f')^2\right)(dr^2 + r^2h) \\
= & -(n-1)\left(f'' + r^{-1}f'\right)dr^2 \\
& - \left(f'' + (2n-3)r^{-1}f' + (n-2)(f')^2\right)r^2h.
\end{aligned}
$$

The Ricci tensor is then nonnegative iff.

$$(1.3) \qquad f'' + r^{-1}f' \leqslant 0$$

and

$$(1.4) \qquad f'' + (2n-3)r^{-1}f' + (n-2)(f')^2 \leqslant 0.$$

It will be convenient to work with $f'(r) = r^{-1}a(r)$. With this choice, the conditions for Ricci to be nonnegative become:

$$(1.5) \qquad a' \leqslant 0$$

and

$$(1.6) \qquad a' + (n-2)r^{-1}(2a + a^2) \leqslant 0.$$

A function satisfying (1.5–1.6) is easily found: for instance, one can choose any smooth function a such that

$$(1.7) \qquad a(r) = 0 \ \ \forall r \in [0,1] \ , \ a(r) = -\frac{1}{2} \ \ \forall r \geqslant 2$$

so that $a'(r) \leqslant 0$ and $-2 \leqslant a(r) \leqslant 0$ everywhere. We then let

$$(1.8) \qquad f(r) = \int_0^r \frac{a(s)}{s}\,ds$$

Hence the metric $e^{2f} g_0$ is conformally flat and has nonnegative Ricci curvature. This metric behaves as $r^{-1}(dr^2 + r^2 h)$ at infinity, it is then complete since

$$(1.9) \qquad \int_1^\infty \frac{dr}{\sqrt{r}} = +\infty.$$

Optimality of the decay condition of B.-L. Chen and X.-P. Zhu [5]. Let us analyse a little further the examples above. It is clear that one may take:

$$\lim_{r \to \infty} a(r) = -(1 - \varepsilon) \quad \text{for any} \ \ \varepsilon > 0,$$

and the metric one obtains is still complete. A short computation leads to an estimate of the Ricci curvature:

$$\mathrm{Ric} \simeq (1 - \varepsilon^2) r^{-2\epsilon}.$$

But geodesic distance from the origin in this metric is $s = r^{-\epsilon}$ around infinity, hence

$$|\mathrm{Ric}| = O(s^{-2}).$$

In [5], B.-L. Chen and X.-P. Zhu proved that a complete non-compact conformally flat manifold with nonnegative Ricci curvature is necessarily flat if the scalar curvature is bounded, and if, for geodesic balls centered at some fixed origin,

$$\frac{s^2}{\mathrm{vol}\,B(s)} \int_{B(s)} |\mathrm{Ric}| \longrightarrow 0 \quad \text{as } s \text{ goes to infinity.}$$

In particular, this is satisfied if the following simplest conditions hold true:

$$\mathrm{vol}\,B(s) \geqslant Cs^n, \ \ s^{2+\delta}|\mathrm{Ric}| \longrightarrow 0 \quad \text{as } s \text{ goes to infinity, for some } \delta > 0.$$

As a result, we see that the assumptions in B.-L. Chen and X.-P. Zhu [5] is close to be optimal on manifolds conformally equivalent to $\mathbb{R}^n$.

2. The proof

The proof of the Theorem is divided into two parts: the first one is a classification of the possible holonomy coverings of (M, g) be a complete (compact or not) conformally flat manifold with nonnegative Ricci curvature. This is basically S. Zhu's result [24], but for sake of completeness, and also because this is quite a short proof, we have provided below the arguments leading to the results we will precisely need. Our arguments are roughly similar to those of M.-H. Noronha [18] and S. Zhu [24], although they differ at some places. It then turns out that the classification of the holonomy coverings is a classification of the universal coverings, and it then remains to show that, in the presence of a fundamental group, one gets metric rigidity (unless we are in the spherical spaceform case).

The very first tool one needs to achieve the holonomy covering classification is R. Schoen and S. T. Yau's analysis [21] of conformally flat manifolds with nonnegative scalar curvature. This applies in dimensions $n \geqslant 7$ without any further restrictions, and to manifolds

of dimensions $3 \leqslant n \leqslant 6$ provided that a very general positive mass holds true for manifolds with one asymptotically flat end and a huge (possibly infinite) number of other complete ends.

This last result is considered as highly probable but has unfortunately remained unpublished so far (see [21, §4] and the appendix for details on this subtle point). Luckily enough, we only need Schoen-Yau's analysis to hold true in our setting where extra conditions on the Ricci curvature are available, and it turns out that a direct proof can be obtained using this extra assumption. For sake of completeness, we have included a description of the elements of that proof in an appendix to this paper. Note also that U. Christ and J. Lohkamp recently announced a proof of a general positive mass theorem with new techniques that might lead to the desired result [8].

We now come back to our main proof: Schoen-Yau's analysis then implies that the holonomy covering of M is a domain Ω included in the complement $\Omega(\Gamma)$ of the limit set of the fundamental group $\Gamma = \pi_1(M)$ of M in the sphere $\mathbb{S}^n$.

We now look at Ω: it is endowed with a complete metric conformal to the round metric and, due to nonnegativity of Ricci curvature, it has sub-euclidean volume growth:

$$\text{vol } B(r) \leqslant C r^n.$$

One can now apply [3, Prop. 2.2], which shows that the complement of Ω in the sphere is of n-capacity zero, hence of zero Hausdorff dimension. Moreover, manifolds with nonnegative Ricci curvature have a finite number of ends [12], and, arguing as in [3, Proof of Theorem 2.1], we can conclude that Ω must be the complement of a finite number of points in the sphere $\mathbb{S}^n$. Three cases can now occur:

(1) First of all, Ω might be the whole sphere. Then M is compact, with a metric g conformal to the round metric and with nonnegative Ricci curvature. As already noticed, there are plenty of such metrics, such as, $e.g.$, C^2-small conformal deformations of the round metric.

(2) Assume now that Ω has at least two ends. Since it has nonnegative Ricci curvature, Cheeger-Gromoll theorem [4] implies that Ω splits as a Riemannian product $\mathbb{R} \times Y$. Remarks due to Lafontaine [11] assert that a conformally flat manifold that is a Riemannian product with a line is necessarily a Riemannian product of a constant curvature manifold with the line. Hence, Y must be of constant curvature, and as Ω is simply connected, $Y = \mathbb{R}^{n-1}$ or $\mathbb{S}^{n-1}$ (hyperbolic space would lead to negative Ricci curvature). Since Ω has at least two ends, we end up with $Y = \mathbb{S}^{n-1}$ as the sole possibility in this case.

(3) If Ω has only one end, it forces it to be diffeomorphic to $\mathbb{R}^n$, endowed with a metric conformal to the euclidean metric and with nonnegative Ricci curvature. Either M itself is diffeomorphic to $\mathbb{R}^n$ (and we have seen in the previous section that the metric is not necessarily flat) or M is different from its universal covering Ω. From now on, we will assume that we are in that last case, and we will show that g is flat.

If $\Omega(\Gamma) = \mathbb{S}^n$, it implies that Γ is a group of conformal diffeomorphisms of the sphere acting properly discontinuously without fixed points on the whole sphere. It is then a

finite group, and it must fix the missing point $\Omega(\Gamma) - \Omega$. Any conformal map of finite order has at least two fixed points (if not they would be parabolic, and this would imply both $\Omega(\Gamma) = \Omega$ and infinite order) and Γ cannot act without fixed points on Ω, hence this case is excluded.

We are left with the case where $\Omega(\Gamma) = \Omega$ is diffeomorphic to $\mathbb{R}^n$ and Γ acts properly discontinuously without fixed points as conformal diffeomorphisms of the flat space. But the only possibility is then that Γ is a subgroup of the group of Euclidean isometries of flat space.

Let us note now that if M itself is compact, then it is (isometrically) a compact flat manifold. Indeed, M is diffeomorphic to a flat manifold, and is endowed with a conformal metric g with nonnegative Ricci curvature. By Bieberbach theorem [23] it is covered by a torus, and the Bochner technique [9, 4.37] (or, alternatively, the Gromov-Lawson scalar curvature obstruction [10]) applies to show that g itself is flat.

We now consider the case where M is non-compact. By Bieberbach theorem again, there exists a finite cover $\widehat{M}$ of M, which is conformally equivalent to a product of a flat torus and a flat space :

$$\widehat{M} = \mathbb{T}^k \times \mathbb{R}^{n-k}.$$

Moreover, M is obtained from $\widehat{M}$ by a finite group acting isometrically on the flat metric. Top conclude at this point, one could try to look at an extension of Gromov-Lawson's obstruction to nonnegative scalar curvature for general non-compact Riemannian manifolds of type $\mathbb{T}^k \times \mathbb{R}^{n-k}$. However, a careful reading of [10, pp. 95–96] shows that it is useless to hope for such an extension unless $k = n - 1$. We shall then need an *ad hoc* proof in the conformally flat case.

Let us now study the different possible values for k (with $1 \leqslant k \leqslant n - 1$ as the case $k = n$ has already been treated above and the case $k = 0$ bears no restriction, as we know from section 1).

(1) If $k = n - 1$, then $\widehat{M}$ has two ends, and Cheeger-Gromoll's theorem [4] applied again together with Lafontaine results [11] shows that M is isometric to a flat spaceform.

(2) Suppose now that $1 \leqslant k \leqslant n - 2$. Using the computations done above, the Ricci curvature of the metric $g = \phi^{-2} g_0$ (g_0 the euclidean metric on $\mathbb{T}^k \times \mathbb{R}^{n-k}$) is

$$\mathrm{Ric}^g = (n-2)\frac{Dd\phi}{\phi} - \left(\frac{\Delta\phi}{\phi} + (n-1)\frac{|d\phi|^2}{\phi^2}\right) g_0.$$

Suppose now that $\widehat{M} = \mathbb{T}^k \times \mathbb{R}^{n-k}$ for $0 < k < n - 1$, and use polar coordinates on the $\mathbb{R}^{n-k}$-factor to get

$$\begin{aligned}
\phi\, \mathrm{Ric}^g(\partial_r, \partial_r) \;=\; &(n-2)(\partial_r^2 \phi) \\
&- \left(-(\partial_r^2 \phi) - \frac{n-k-1}{r}\partial_r \phi + (n-1)\frac{|d\phi|^2}{\phi} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-k-1}}\phi + \Delta_{\mathbb{T}^k}\phi\right)
\end{aligned}$$

so that

$$(2.1) \qquad 0 \leqslant (n-1)(\partial_r^2 \phi) + \frac{n-k-1}{r} \partial_r \phi - (n-1) \frac{(\partial_r \phi)^2}{\phi} - \frac{1}{r^2} \Delta_{\mathbb{S}^{n-k-1}} \phi - \Delta_{\mathbb{T}^k} \phi.$$

Similarly,

$$\phi \operatorname{tr}_{\mathbb{T}^k} \operatorname{Ric}^g = -(n-2) \Delta_{\mathbb{T}^k} \phi$$
$$- k \left(-(\partial_r^2 \phi) - \frac{n-k-1}{r} \partial_r \phi + (n-1) \frac{|d\phi|^2}{\phi} + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-k-1}} \phi + \Delta_{\mathbb{T}^k} \phi \right),$$

so that

$$(2.2) \qquad 0 \leqslant (\partial_r^2 \phi) + \frac{n-k-1}{r} \partial_r \phi - (n-1) \frac{(\partial_r \phi)^2}{\phi} - \frac{1}{r^2} \Delta_{\mathbb{S}^{n-k-1}} \phi - (1 + \frac{n-2}{k}) \Delta_{\mathbb{T}^k} \phi.$$

We now integrate equations (2.1) and (2.2) on $\mathbb{S}^{n-k-1} \times \mathbb{T}^k$, and we let

$$\bar{\phi} = \int_{\mathbb{S}^{n-k-1} \times \mathbb{T}^k} \phi.$$

This yields

$$(2.3) \qquad \begin{aligned} 0 &\leqslant (n-1) \partial_r^2 \bar{\phi} + \frac{n-k-1}{r} \partial_r \bar{\phi} - (n-1) \int_{\mathbb{S}^{n-k-1} \times \mathbb{T}^k} \frac{(\partial_r \phi)^2}{\phi} \; ; \\ 0 &\leqslant \partial_r^2 \bar{\phi} + \frac{n-k-1}{r} \partial_r \bar{\phi} - (n-1) \int_{\mathbb{S}^{n-k-1} \times \mathbb{T}^k} \frac{(\partial_r \phi)^2}{\phi} \; . \end{aligned}$$

By Hölder inequality,

$$\left(\partial_r \bar{\phi} \right)^2 = \left(\int_{\mathbb{S}^{n-k-1} \times \mathbb{T}^k} \partial_r \phi \right)^2 = \left(\int_{\mathbb{S}^{n-k-1} \times \mathbb{T}^k} \phi^{1/2} \phi^{-1/2} \partial_r \phi \right)^2$$
$$\leqslant \left(\int_{\mathbb{S}^{n-k-1} \times \mathbb{T}^k} \phi \right) \left(\int_{\mathbb{S}^{n-k-1} \times \mathbb{T}^k} \frac{(\partial_r \phi)^2}{\phi} \right) = \bar{\phi} \left(\int_{\mathbb{S}^{n-k-1} \times \mathbb{T}^k} \frac{(\partial_r \phi)^2}{\phi} \right).$$

Hence our pair of equations are transformed into the following:

$$(2.4) \qquad \begin{aligned} 0 &\leqslant (n-1) \partial_r^2 \bar{\phi} + \frac{n-k-1}{r} \partial_r \bar{\phi} - (n-1) \frac{(\partial_r \bar{\phi})^2}{\bar{\phi}} \; ; \\ 0 &\leqslant \partial_r^2 \bar{\phi} + \frac{n-k-1}{r} \partial_r \bar{\phi} - (n-1) \frac{(\partial_r \bar{\phi})^2}{\bar{\phi}} \; . \end{aligned}$$

Using this and the change of variables

$$r = e^t, \quad i.e. \; r \partial_r = \partial_t, \; r^2 \partial_r^2 = \partial_t^2 - \partial_t,$$

and denoting $\psi(t) = \bar{\phi}(e^t)$ for $t \in \mathbb{R}$, one gets:

$$(2.5) \qquad (n-1) \partial_t^2 \psi - k \, \partial_t \psi - (n-1) \frac{(\partial_t \psi)^2}{\psi} \geqslant 0 \; ,$$

and

$$(2.6) \qquad \partial_t^2 \psi + (n - k - 2)\, \partial_t \psi - (n-1) \frac{(\partial_t \psi)^2}{\psi} \; \geqslant \; 0 \; .$$

From equation (2.5) one gets, if $k > 0$:

$$(2.7) \qquad \partial_t \psi \; \leqslant \; \frac{n-1}{k} \partial_t^2 \psi - \frac{n-1}{k} \frac{(\partial_t \psi)^2}{\psi}.$$

Notice now that $k \leqslant n - 2$, so that injecting (2.7) into equation (2.6) leads to:

$$(2.8) \qquad \partial_t^2 \psi - \left(1 + \frac{k(n-2)}{k + (n-k-2)(n-1)} \right) \frac{(\partial_t \psi)^2}{\psi} \; \geqslant \; 0.$$

Define now

$$(2.9) \qquad \alpha = - \frac{k(n-2)}{k + (n-1)(n-k-2)} \; < \; 0,$$

so that

$$\begin{aligned}
\partial_t^2 (\psi^\alpha) &= \partial_t \left(\alpha\, \psi^{\alpha-1}\, \partial_t \psi \right) \\
&= \alpha \left(\psi^{\alpha-1} \partial_t^2 \psi + (\alpha - 1)\, \psi^{\alpha-2} (\partial_t \psi)^2 \right) \\
&= \alpha\, \psi^{\alpha-1} \left(\partial_t^2 \psi - (1 - \alpha) \frac{(\partial_t \psi)^2}{\psi} \right).
\end{aligned}$$

As $\alpha < 0$ and $\psi > 0$, this implies that ψ^α is a positive concave function on $\mathbb{R}$. But constants are the only positive concave functions on $\mathbb{R}$ that do not change signs, so that the function $\bar\phi$ is necessarily a constant in r.

We now define

$$\Psi : x \in \mathbb{R}^{n-k} \longmapsto \int_{\{x\} \times \mathbb{T}^k} \phi \; .$$

Using x in $\mathbb{R}^{n-k}$ as the origin, the work done above shows that, for each $r > 0$,

$$\begin{aligned}
\Psi(x) &= \lim_{s \to 0} \frac{1}{\mathrm{vol}\, \mathbb{S}^{n-k-1}} \bar\phi(s) \\
&= \frac{1}{\mathrm{vol}\, \mathbb{S}^{n-k-1}} \bar\phi(r) \\
&= \frac{1}{\mathrm{vol}\, \mathbb{S}^{n-k-1}} \int_{\{r\} \times \mathbb{S}^{n-k-1} \times \mathbb{T}^k} \phi \\
&= \frac{1}{\mathrm{vol}\, S_x(r)} \int_{S_x(r)} \Psi \; ,
\end{aligned}$$

or, equivalently: the function Ψ is harmonic on $\mathbb{R}^{n-k}$.

We are now in a position to end the proof. The condition for the scalar curvature to be nonnegative is :

$$(2.10) \qquad 0 \leqslant \mathrm{Scal}^g = -2(n-1)\Delta\phi - n(n-1)\frac{|d\phi|^2}{\phi},$$

so that

$$(2.11) \qquad \Delta\phi = \Delta_{\mathbb{R}^{n-k}}\phi + \Delta_{\mathbb{T}^k}\phi \leqslant 0 .$$

Integrating on $\mathbb{T}^k$, the first term in (2.11) vanishes since Ψ is harmonic, whereas the second vanishes by definition of the Laplacian on $\mathbb{T}^k$. As a result, ϕ itself must be harmonic and the sign of the scalar curvature equation (2.10) shows that $d\phi$ must vanish. Hence g must be flat. $\qquad\square$

Remark 2.1. Notice that the end of the proof given above works exactly when we expect the result to be true. Indeed, it fails if α in equation (2.9) vanishes, *i.e.* if $n = 2$ (surface case, where an alternative proof must be given) or if $k = 0$ (no torus factor, where the metric might be non-flat, in agreement with the example given in the first section).

Remark 2.2. For some values of k and n (namely $k \geqslant n-2$), a more geometric proof can be obtained along the following lines. One has

$$M = \left(\mathbb{T}^k \times \mathbb{R}^{n-k}\right)/\Delta$$

for some finite group Δ, so that $\Gamma = \pi_1(M)$ has polynomial growth of order k. Moreover, from Bishop-Gromov classical comparison theorem, volume growth is polynomial of order n at most. It was proved by Anderson [1] that if $\widehat{M} = \mathbb{T}^k \times \mathbb{R}^{n-k}$ has volume growth

$$\mathrm{vol}(B(r)) \geqslant Cr^p$$

then the group $\widehat{\Gamma} = \Gamma/\Delta$ has polynomial growth $n-p$ at most, hence Γ itself. This implies $p \leqslant n-k$. Applying Anderson's result once more, polynomial growth k of $\widehat{\Gamma}$ and volume growth

$$\mathrm{vol}(B(r)) \leqslant C'r^n$$

on Ω implies that $\widehat{M}$ has volume growth

$$\mathrm{vol}(B(r)) \leqslant C'r^{n-k}.$$

Now, if the curvature is bounded, and if $n-k \leqslant 2$, Anderson has proved in the same paper using minimal surfaces that $\widehat{M}$ must split off a factor $\mathbb{R}$ at least. Lafontaine's result [11] then ends this remark. Unfortunately, there seem to be no way to extend this conceptual argument to higher codimensions.

Appendix

In the last proof, we have used the following result of R. Schoen and S.T. Yau:

Theorem 2.3 (Schoen-Yau [21], Theorems 4.1–4.5). *Suppose (M, g) is complete, locally conformally flat with nonnegative scalar curvature, then the developing map :*

$$\Phi : \tilde{M} \to \mathbb{S}^n$$

is injective, the fundamental group $\pi_1(M)$ is isomorphic to a discrete subgroup of C_n (the conformal group of $\mathbb{S}^n$), and $\Phi(\tilde{M})$ is a domain Ω in the complement $\Omega(\Gamma)$ of the limit set of the action of $\pi_1(M)$ on the sphere $\mathbb{S}^n$.

The proof of this result is done in [21] in two steps: a first argument covers the case $n \geq 7$; another argument based on the positive mass theorem covers the case $3 \leq n \leq 6$. However, the authors remark that *"for this application it is necessary to extend the positive mass theorems to the case of complete manifolds; that is assuming that the manifold has an asymptotically flat end and other ends which are merely complete. This extension will be carried out in a future work"* [21, p. 65]. It is widely believed in the Riemannian Geometry community that the needed extension of the positive mass theorem is true, but, unfortunately, no such generalisation has been published so far. Of course, when the manifold is assumed to be spin, the Witten proof of the positive mass theorem [19, 20, 22] extends to this general framework but the spin assumption plays an essential role there. Note however that every orientable 3-manifold is spin. Hence, Schoen-Yau's analysis of conformally flat manifolds and Kleinian groups [21] for sure holds in dimension 3 as we only need to apply the positive mass theorem to the universal covering (mass increases as one takes quotients).

To avoid in our setting the problems caused by the absence of any written proof of the relevant positive mass theorem, it then remains to prove the adequate result for conformally flat manifolds with nonnegative Ricci curvature and dimensions $4 \leq n \leq 6$.

According to the Bishop-Gromov inequality, we know that on any covering of such a manifold the volume of geodesic balls is less than the volume of the euclidean balls of the same radius. We will show in this appendix that the argument given in [21] by Schoen and Yau to prove their theorem in dimension larger than 7 works in dimensions 4, 5, and 6 under just that very assumption on volume growth that can be obtained from nonnegativity of the Ricci curvature. More precisely, we prove below the following version of Theorem 3.1 in [21]:

Theorem 2.4 (Schoen-Yau's Theorem 3.1 revisited). *Assume that (M, g) is a complete Riemannian conformally flat manifold of dimension $n \in [4, 6]$ such that $\mathrm{Scal}_g \geq 0$, and, for some $x \in M$, there exist C such that $\mathrm{vol}\, B(x, r) \leq C r^n$ for every $r > 0$.*

Then if $\Phi : M \to \mathbb{S}^n$ is a conformal immersion then Φ is injective and yields a conformal diffeomorphism of M onto $\phi(M)$. Moreover the boundary of $\Phi(M)$ has zero Newtonian capacity.

Our proof closely follows [21], differing only at the points where dimension arguments enter the picture. We then refer to [21, p. 58–61] for the overall strategy, and focus below only on the steps where some modification is needed. In particular, we refer to [21] for the proof of the fact that Theorem 2.4 leads to the following final result:

Corollary 2.5. *Suppose (M, g) is complete, locally conformally flat with nonnegative Ricci curvature then the developing map :*

$$\Phi \ : \ \tilde{M} \to \mathbb{S}^n$$

is injective, $\pi_1(M)$ is isomorphic to a discrete subgroup of C_n (the conformal group of $\mathbb{S}^n$), and $\Phi(M)$ is a domain Ω included in the complement $\Omega(\Gamma)$ of the limit set of $\pi_1(M)$ in $\mathbb{S}^n$.

Proof of Theorem 2.4. – We will first show how the original proof of Schoen and Yau can be easily adapted for the case $n \geq 5$; then we examine the case $n = 4$ which needs a more elaborate argument.

The core of the proof of Theorem 3.1 in [21] lies in the following construction: let $o \in M$ be a point and G_o be the Green kernel of the conformal Laplacian of (M, g) with pole at o, and $\bar{G}_o$ be the pulled-back (by Φ) of the Green kernel of the conformal Laplacian on the sphere with pole at $\Phi(o)$, normalized so that $\bar{G}_o - G_o$ is smooth near o. We let $v = G_o/\bar{G}_o$. The metric $\bar{g} = G_o^{4/(n-2)} g$ is flat (and incomplete) and the function v is $\bar{g}$-harmonic and one has $0 < v \leq 1$. Then Schoen and Yau prove that their Theorem is obtained if one is able to show that $v = 1$ everywhere. It is that careful study of v, where the positive mass theorem plays a role in small dimensions, that we shall replace.

When $n \geq 5$ and $\alpha = 2(n-2)/n$, R. Schoen and S.T. Yau show in [21, equation (3.4), p. 59] that for every σ large enough :

$$\int_{B(o,\sigma/2)} |\bar{\nabla}v|^{\alpha-2}|\bar{\nabla}|\bar{\nabla}v||^2 dvol_{\bar{g}} \leq \frac{C}{\sigma^2} \int_{B(o,3\sigma)\setminus B(o,\sigma/3)} G_o^\alpha \, dvol_g \ .$$

Moreover, they have also shown that for every $p > n/(n-2)$ [21, Corollary 2.3]:

$$\int_{M\setminus B(o,1)} G_o^p \, dvol_g < \infty \ ;$$

in particular as $n > 4$ we have

$$\int_{M\setminus B(o,1)} G_o^2 \, dvol_g < \infty \ .$$

Using Hölder inequality and our assumption on volume growth, we then have

$$\int_{B(o,3\sigma)\setminus B(o,\sigma/3)} G_o^\alpha \, dvol_g \ \leq \ C\sigma^2 \left(\int_{B(o,3\sigma)\setminus B(o,\sigma/3)} G_o^2 \, dvol_g \right)^{1-2/n} .$$

Hence, letting $\sigma \to \infty$, we obtain that v is the constant function equal to 1. As already said, this is enough to continue the proof with the argument of [21].

It remains to consider the case of the dimension 4 where, unfortunately, $\alpha = 1$. The first results we need are :

Lemma 2.6. *For $R > 0$ large enough and for all $\varepsilon > 0$,*

$$\int_{M \backslash B(o,R)} \frac{|\nabla G_o|^2}{G_o |\log G_o|^{1+\varepsilon}} d\mathrm{vol}_g \; + \; \int_{M \backslash B(o,R)} \frac{\mathrm{Scal}_g \, G_o}{|\log G_o|^\varepsilon} \, \mathrm{vol}_g < \infty$$

Proof. – This is inspired by [21, proof of Proposition 2.4(iii)].

Let $M = \cup_i U_i$ an exhaustion of M by a sequence of nested bounded open subsets with smooth boundaries, such that $o \in U_0$. Let $\varphi \in C^\infty(M)$ such that $\varphi = 1$ outside $B(o, R)$ and the support of φ does not meet $B(o, R/2)$ and let G_o^i be the Green kernel of the confomal Laplacian on U_i (with the Dirichlet boundary condition) of pole o.

We know that $G_o^i - G_o \to 0$ uniformly on compact subsets of M and that $G_o^i \leq G_o$. Moreover, nonnegativity of the scalar curvature implies that

$$(2.12) \qquad\qquad \Delta G_o^i + \mathrm{Scal}_g \, G_o^i / 6 \leq 0.$$

Hence we can integrate by parts the quantity

$$\int_{U_i} \varphi \langle \nabla G_o^i, \nabla(|\log G_o^i|^{-\varepsilon}) \rangle \, d\mathrm{vol}_g$$

with i large enough, so that $B(o, R) \subset U_i$. Note that it is shown in Lemma 3.2 in [21] that

$$\lim_{x \to \infty} G_o(x) = 0,$$

so that one may take i and R large enough to ensure that $G_o^i \leqslant 1/2$ on the support of φ Using inequality (2.12), and letting $i \to \infty$, this leads to :

$$\varepsilon \int_{U_i \backslash B(o,R)} \frac{|\nabla G_o^i|^2}{G_o^i |\log G_o^i|^{1+\varepsilon}} \, d\mathrm{vol}_g$$

$$+ \frac{1}{6} \int_{U_i \backslash B(o,R)} \frac{\mathrm{Scal}_g \, G_o}{|\log G_o|^\varepsilon} \, d\mathrm{vol}_g \leq C \int_{B(o,R) \backslash B(o,R/2)} |\nabla G_o^i| |\log G_o^i|^{-\varepsilon} d\mathrm{vol}_g \, .$$

The result follows from letting $i \to \infty$, because the right-hand side is an integral on a domain that does not depend on i and the convergence of G_o^i towards G_o is in C^∞ on compact subsets; moreover this is true for every $\varepsilon > 0$. □

Lemma 2.7. *For any $R > 0$,*

$$\int_{M \backslash B(o,R)} \frac{G_o^2}{|\log G_o|^4} d\mathrm{vol}_g < \infty.$$

Proof. – We notice that R. Schoen and S.T. Yau showed that the Yamabe invariant of M is positive (in fact it is exactly that of the standard sphere, see Proposition 2.2 in [21]), hence we have a constant $c > 0$ such that for any $f \in C_0^\infty(M)$,

$$c \left(\int_M f^4 \, d\mathrm{vol}_g \right)^{1/2} \leq \int_M [|df|^2 + \frac{1}{6} \mathrm{Scal}_g \, f^2] \, d\mathrm{vol}_g$$

In particular, we can apply this to $f = \varphi \sqrt{G_o^i} / \log G_o^i$ (with φ a cut-off function as above) and Lemma 2.6 with adequate choices of ε shows the desired result. $\qquad\square$

Now we can adapt the argument of [21] to the case $n = 4$: we start again from inequality (3.4) of that paper:

$$\forall \phi \in C_0^\infty(M), \quad \int_M \phi^2 |\bar{\nabla} v|^{-1} |\bar{\nabla}|\bar{\nabla} v||^2 \, dvol_{\bar{g}} \leq \int_M |d\phi|^2 G_o |dv| \, dvol_g \,.$$

Let k an integer large enough; we take the following as a test function :

$$\phi_k(x) = \begin{cases} 1 & \text{if} \quad x \in B(o, 2^k) \\ \frac{\log\left(2^{2k}/r(x)\right)}{\log(2^k)} \leqslant 1 & \text{if} \quad x \in B(o, 2^{2k}) \setminus B(o, 2^k) \\ 0 & \text{if} \quad x \notin B(o, 2^{2k}) \end{cases}$$

where $r(x) = \text{dist}(x, o)$. As we have

$$G_o |dv| \leq |dG_o| + G_o |d \log \bar{G}_o|,$$

we obtain :

$$(2.13) \quad \int_{B(o,2^k)} |\bar{\nabla} v|^{-1} |\bar{\nabla}|\bar{\nabla} v||^2 dvol_{\bar{g}} \leq \frac{c}{k^2} \int_{B(o,2^{2k})\setminus B(o,2^k)} [|dG_o| + G_o |d \log \bar{G}_o|] \frac{dvol_g}{r^2(x)}.$$

We first treat the first term in the right-hand side of the previous estimate. One has:

$$\int_{B(o,2^{2k})\setminus B(o,2^k)} |dG_o| \frac{dvol_g}{r^2(x)} \leq \left(\int_{B(o,2^{2k})\setminus B(o,2^k)} |dG_o|^2 dvol_g \right)^{1/2} \left(\int_{B(o,2^{2k})\setminus B(o,2^k)} \frac{dvol_g}{r^4(x)} \right)^{1/2}.$$

Now the same proof as that of Lemma 2.6 above (see also [21, Proposition 2.4(iii-a)]) shows that

$$\int_{M\setminus B(o,1)} |dG_o|^2 \, dvol_g < \infty \,,$$

and the upper bound on the volume growth yields

$$\int_{B(o,2^{2k})\setminus B(o,2^k)} \frac{dvol_g}{r^4(x)} \leq C \, k$$

for a constant C independent of k. Hence we get that the first term goes to zero when k goes to infinity.

Now for the second term in the right-hand side of (2.13):

$$\int_{B(o,2^{2k})\setminus B(o,2^k)} G_o |d \log \bar{G}_o| \frac{dvol_g}{r^2(x)}$$

$$\leqslant \left(\int_{B(o,2^{2k})\setminus B(o,2^k)} G_o |\log G_o|^2 \frac{dvol_g}{r^4(x)} \right)^{1/2} \left(\int_{B(o,2^{2k})\setminus B(o,2^k)} \frac{G_o}{|\log G_o|^2} |d \log \bar{G}_o|^2 \, dvol_g \right)^{1/2}$$

The function G_0 tends to 0 at infinity by Lemma 3.2 in [21], hence we can again choose $R > 0$ large enough so that $G_o \leq 1/2$ outside the ball $B(o, R/2)$. As a result, we have:

$$\int_{B(o,2^{2k}) \setminus B(o,2^k)} G_o |\log G_o|^2 \frac{d\text{vol}_g}{r^4(x)} \leq C k .$$

We now use another estimate due to R. Schoen and S.T. Yau which is valid in every dimension: they show at the end of page 60 in [21] that for any $\psi \in C_0^\infty(M)$ vanishing near o we have :

$$\int_M \psi^2 |\nabla \log \bar{G}_o|^2 \, d\text{vol}_g \leq c \int_M \psi^2 |\nabla \log G_o|^2 \, d\text{vol}_g + c \int_M |d\psi|^2 \, d\text{vol}_g$$

Let now ϕ a function with the following properties:

- it has compact support in $B(o, 2^{2k+1}) \setminus B(o, 2^{k-1})$;
- it is identically 1 on $B(o, 2^{2k}) \setminus B(o, 2^k)$;
- its gradient is bounded by $2^{-2k} c$ on $B(o, 2^{2k+1}) \setminus B(o, 2^{2k})$;
- and its gradient is bounded by $2^{-k} c$ on $B(o, 2^k) \setminus B(o, 2^{k-1})$.

This does exist and we can apply the last estimate to $\psi = (\phi \sqrt{G_o})/\log G_o$. We obtain :

$$\int_{B(o,2^{2k}) \setminus B(o,2^k)} G_o |\log G_o|^{-2} |d \log \bar{G}_o|^2 \, d\text{vol}_g$$

$$\leq C \int_{B(o,2^{2k+1}) \setminus B(o,2^{k-1})} |d\phi|^2 G_o \, |\log G_o|^{-2} \, d\text{vol}_g$$

$$+ C \int_{B(o,2^{2k+1}) \setminus B(o,2^{k-1})} G_o |\,(\log G_o|^{-2} + |\log G_o|^{-4}) \, |d \log G_o|^2 \, d\text{vol}_g$$

From Lemma 2.6 we know that the second term in the right-hand side of this last estimate goes to zero when k goes to infinity. For the first term, Cauchy-Schwartz inequality leads to :

$$\int_{B(o,2^{2k+1}) \setminus B(o,2^{k-1})} |d\phi|^2 \frac{G_o}{|\log G_o|^2} \, d\text{vol}_g$$

$$\leqslant \left(\int_{B(o,2^{2k+1}) \setminus B(o,2^{k-1})} |d\phi|^4 \, d\text{vol}_g \right)^{1/2} \left(\int_{B(o,2^{2k+1}) \setminus B(o,2^{k-1})} G_o^2 |\log G_o|^{-4} \, d\text{vol}_g \right)^{1/2}$$

But our volume growth assumptions imply that

$$\int_{B(o,2^{2k+1}) \setminus B(o,2^{k-1})} |d\phi|^4 \, d\text{vol}_g \leq C$$

where C is a constant independent of k. Using Lemma 2.7, we end with

$$\lim_{k \to \infty} \int_{B(o,2^{2k+1}) \setminus B(o,2^{k-1})} G_o^2 |\log G_o|^{-4} \, d\text{vol}_g = 0 .$$

Hence we obtain again that v is the constant function equal to 1 everywhere. And this is what is needed to resume the proof of [21]. $\qquad \square$

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