Local smoothing estimates for wave equations

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Atelier d'Analyse Harmonique 2018

Aussois, March 26 - 30, 2018

joint work with
Jonathan Hickman (U. Chicago) and Chris Sogge (Johns Hopkins U.)

Wave equation on $\mathbb{R}^n$

Given $f_0, f_1 : \mathbb{R}^n \to \mathbb{C}$ consider the Cauchy problem for the wave equation

$$\begin{cases} (\partial_{tt}^2 - \Delta)u = 0 \\ u(\,\cdot\,,0) = f_0, \qquad \partial_t u(\,\cdot\,,0) = f_1. \end{cases}$$

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By Fourier transform, the solution u is given by

$$u(x,t) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos(t|\xi|) \widehat{f_0}(\xi)\, \mathrm{d}\xi + \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sin(t|\xi|) \frac{\widehat{f_1}(\xi)}{|\xi|}\, \mathrm{d}\xi.$$

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It can be re-written in terms of the half-wave propagator

$$e^{it\sqrt{-\Delta}}f(x) := \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|)}\widehat{f}(\xi)\,\mathrm{d}\xi.$$

(Fourier extension operator for the cone)

Fixed time estimates

For any fixed time t,

$$e^{it\sqrt{-\Delta}}f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|)}\widehat{f}(\xi)\,\mathrm{d}\xi$$

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For any fixed time t and any $1 < p < \infty$, Peral (1980, also Miyachi) proved that

$$\|u(\cdot,t)\|_{L^p_{-s_p}(\mathbb{R}^n)} \leqslant C_{t,p}(\|f_0\|_{L^p(\mathbb{R}^n)} + \|f_1\|_{L^p_{-1}(\mathbb{R}^n)})$$

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This is sharp: $L^p_{-s_p}$ cannot be replaced by L^p_α with $\alpha > -s_p$.

Integrating locally in time

One can integrate locally in time for $t \sim 1$:

$$\Big(\int_1^2 \|u(\cdot,t)\|^p_{L^p_{-s_p}(\mathbb{R}^n)} \mathrm{d}t \Big)^{1/p} \leqslant \Big(\int_1^2 C^p_{t,p} \mathrm{d}t \Big)^{1/p} \big(\|f_0\|_{L^p(\mathbb{R}^n)} + \|f_1\|_{L^p_{-1}(\mathbb{R}^n)} \big)$$
$$\lesssim \|f_0\|_{L^p(\mathbb{R}^n)} + \|f_1\|_{L^p_{-1}(\mathbb{R}^n)}$$

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Question: can one do better and replace $L^p_{-s_p}$ by L^p_{α} with $\alpha > -s_p$?

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YES: Sogge (1991) showed that the above estimate holds for $L^p_{-s_p+\varepsilon(p)}$ for some $\varepsilon(p) > 0$ if $2 < p < \infty$.

Local smoothing estimates

Local smoothing conjecture (Sogge)

The inequality

$$\Big(\int_1^2 \|u(\cdot,t)\|^p_{L^p_{-s_p+\sigma}(\mathbb{R}^n)} \mathrm{d}t \Big)^{1/p} \lesssim \|f_0\|_{L^p(\mathbb{R}^n)} + \|f_1\|_{L^p_{-1}(\mathbb{R}^n)}$$

holds for all $\sigma < 1/p$ *if* $\frac{2n}{n-1} \leqslant p < \infty$ *and* $\sigma < s_p$ *if* $2 < p < \frac{2n}{n-1}$.

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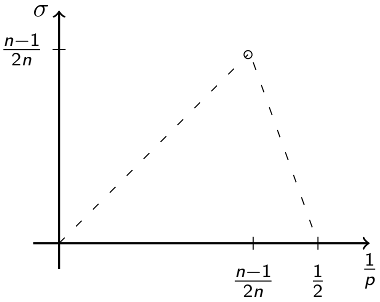

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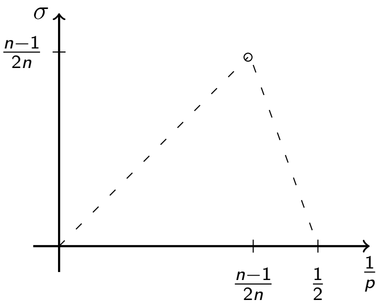

Interpolate the estimate

$$\|e^{it\sqrt{-\Delta}}f\|_{L^{\frac{2n}{n-1}}_{-\varepsilon}(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n)}$$

with the fixed time endpoints

$$\begin{cases} \|e^{it\sqrt{-\Delta}}f\|_{L^2(\mathbb{R}^n\times[1,2])} = \|f\|_{L^2(\mathbb{R}^n)} \\ \|e^{it\sqrt{-\Delta}}f\|_{L^\infty_{-\frac{(n-1)}{2}-\varepsilon}(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} \end{cases}$$

Conjectures

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$\Downarrow$

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Fourier Restriction conjecture for paraboloids

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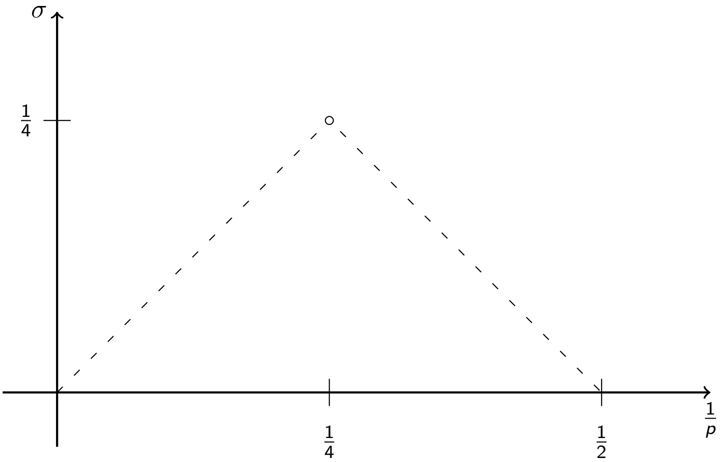
Kakeya conjecture

State of the art for the local smoothing conjecture ($n = 2$)



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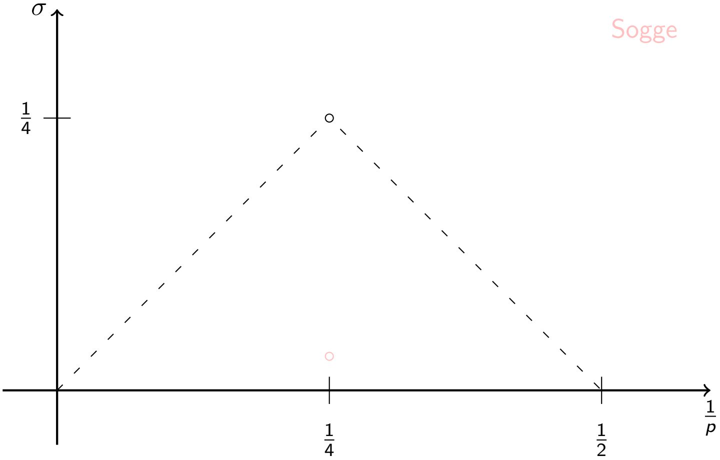

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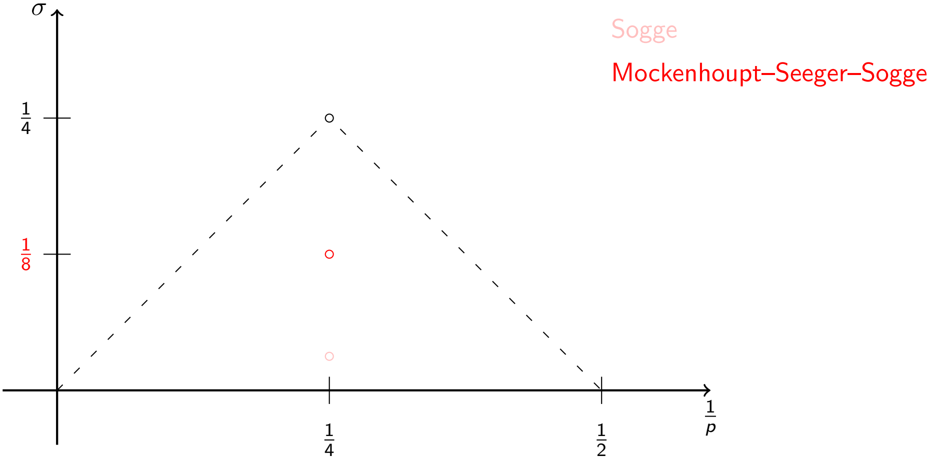

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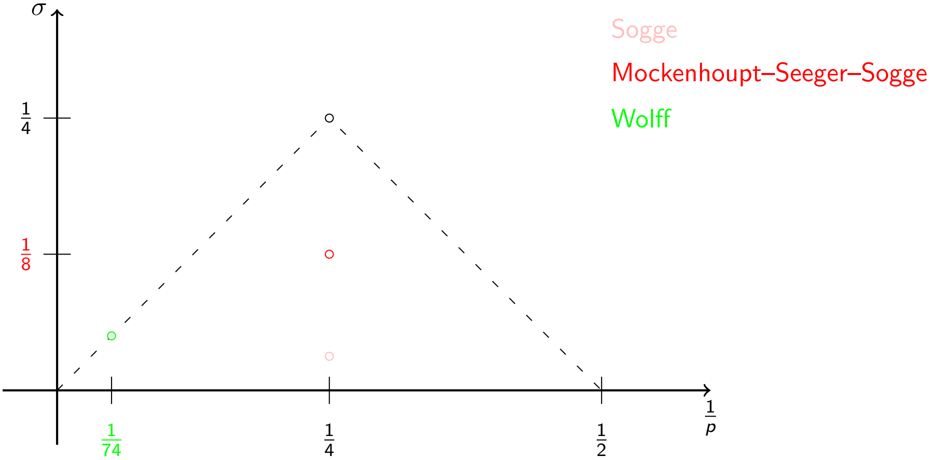

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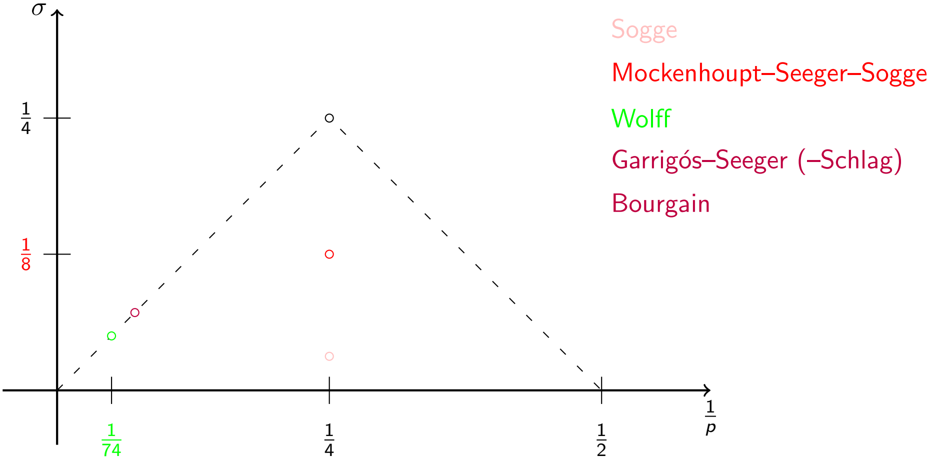

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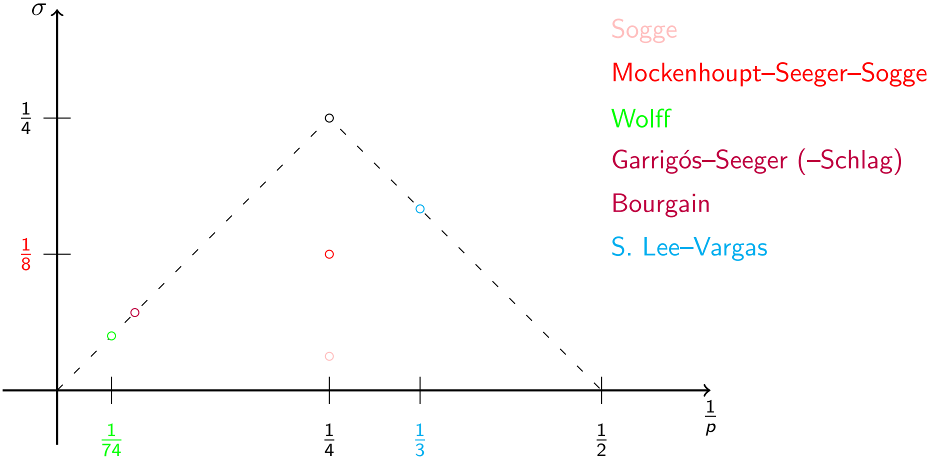

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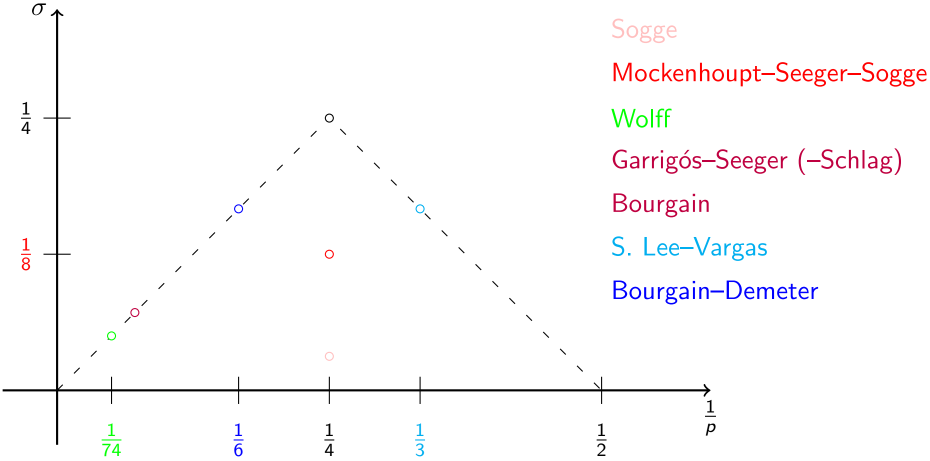

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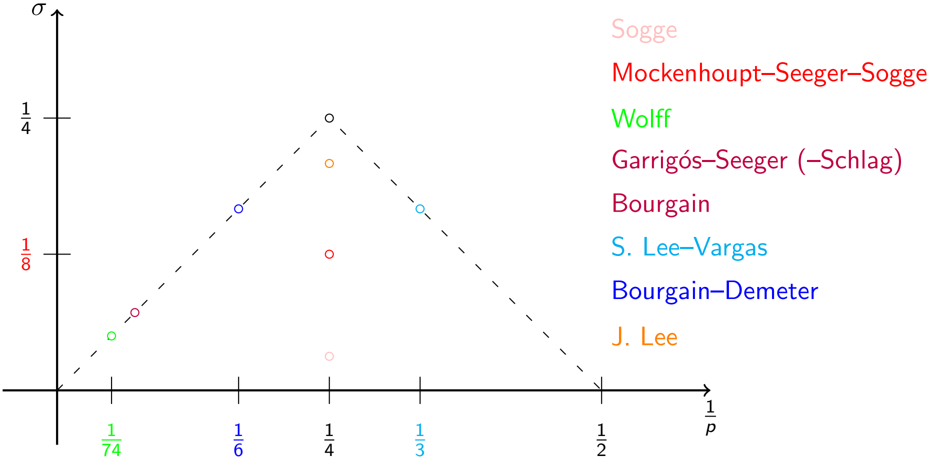

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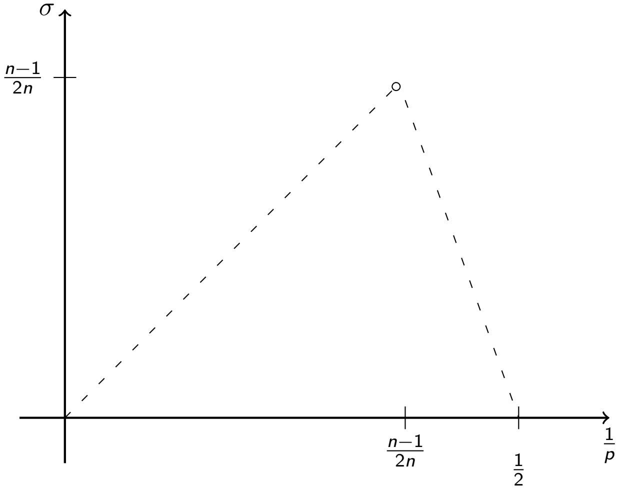

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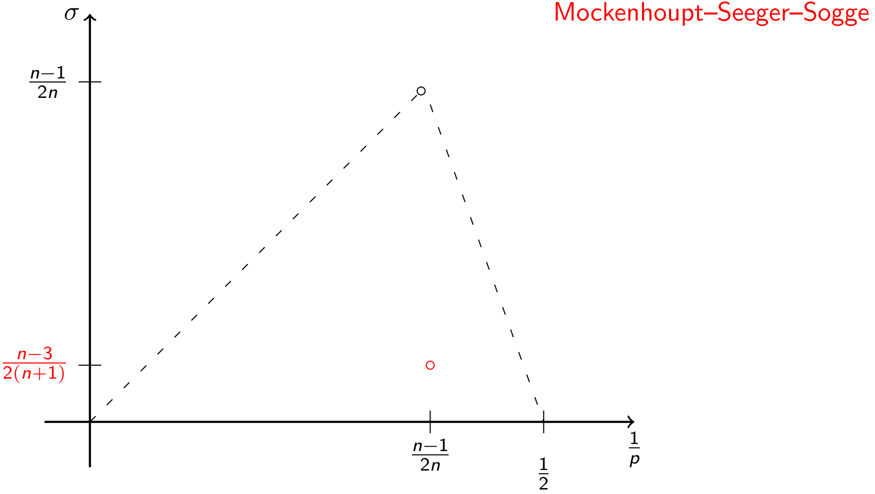

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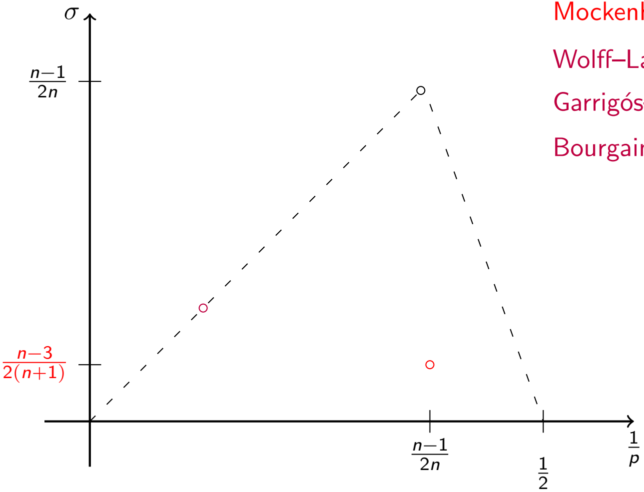

Mockenhoupt–Seeger–Sogge

Wolff–Laba

Garrigós–Seeger (–Schlag)

Bourgain

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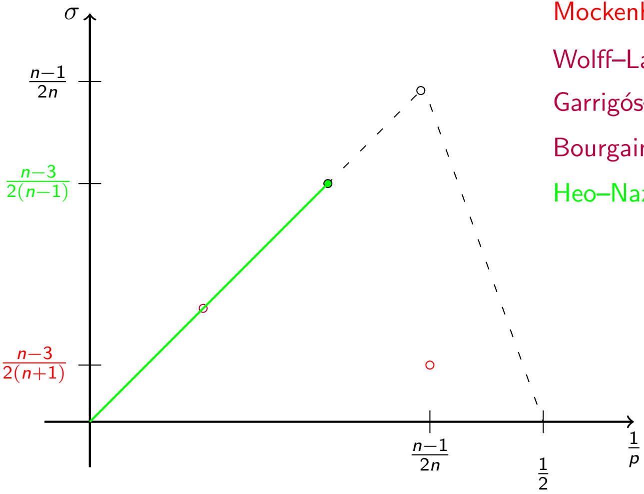

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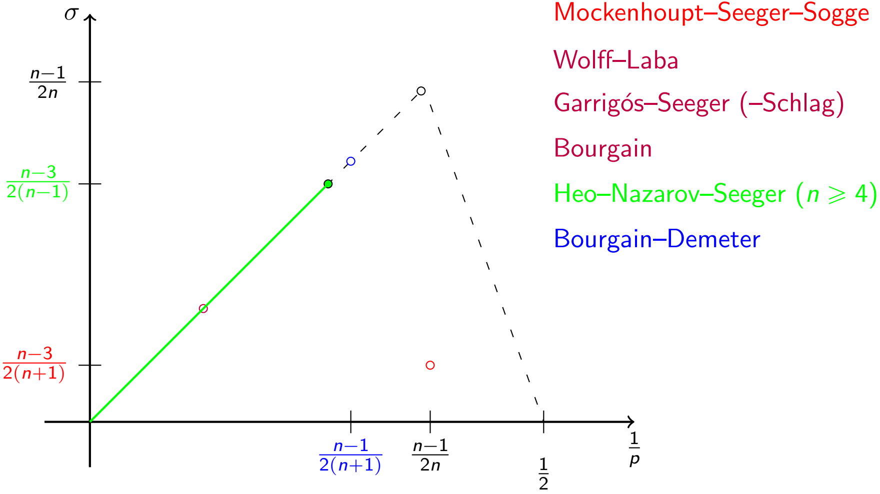

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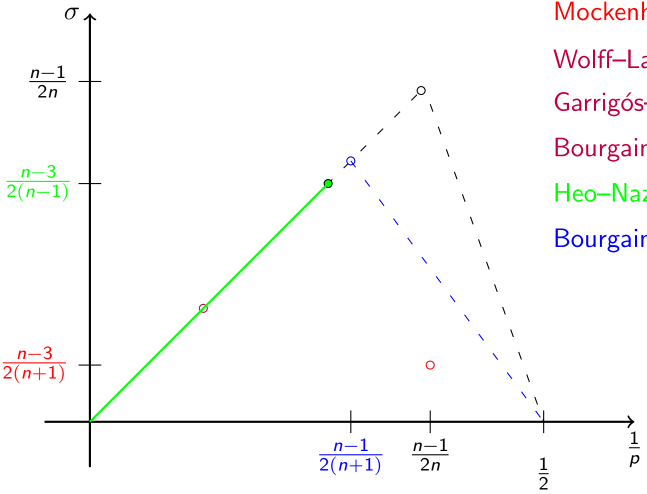

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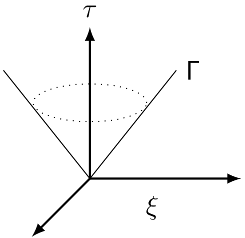
Bourgain–Demeter

Decoupling (or Wolff) inequalities

The space-time Fourier transform of $e^{it\sqrt{-\Delta}}f$ is

$$(e^{it\sqrt{-\Delta}}f)\widehat{\ }(\xi,\tau) = \widehat{f}(\xi)\delta(\tau - |\xi|)$$

so is supported in $\Gamma := \{(\xi,\tau) \in \mathbb{R}^{n+1} : \tau = |\xi|\}$.



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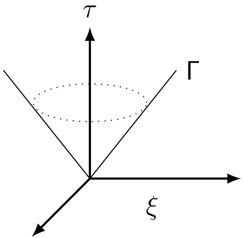

Decomposition into dyadic frequency scales in ξ:

$$\widehat{f} = \underbrace{\widehat{f}^{\lesssim 1}}_{|\xi|\lesssim 1} + \sum_{k\in\mathbb{N}} \underbrace{\widehat{f}^{k}}_{|\xi|\sim 2^k}$$

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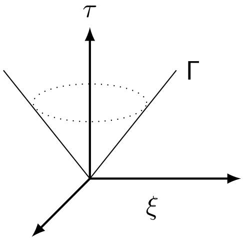

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Low frequency part is easy:

$$|e^{it\sqrt{-\Delta}}f^{\lesssim 1}| \lesssim K * f, \quad \text{for} \quad K \in L^1(\mathbb{R}^n).$$

If one is able to prove

$$\|e^{it\sqrt{-\Delta}}f^k\|_{L^p_\alpha(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

there's summability over $k \in \mathbb{N}$ to conclude

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p_{\alpha-\epsilon}(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for all $\epsilon > 0$.

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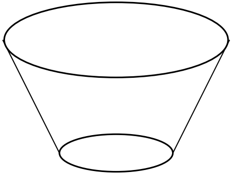

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Localising in $1 < t < 2$ has the effect of blurring out in $O(1)$ in frequency side.

Plates

Further decompose the frequency space so that we can better understand $e^{it\sqrt{-\Delta}}$.

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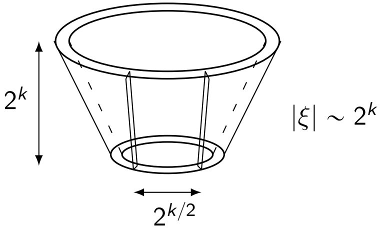

$$f^k := \sum_\theta f_\theta^k$$

θ : sectors of angular width $2^{-k/2}$

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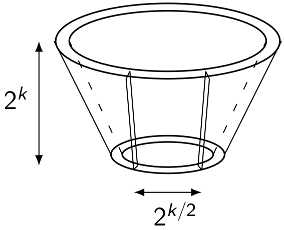

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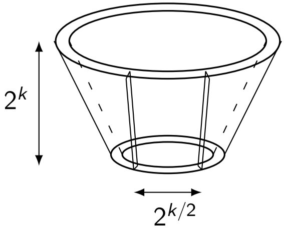

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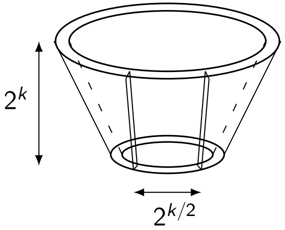

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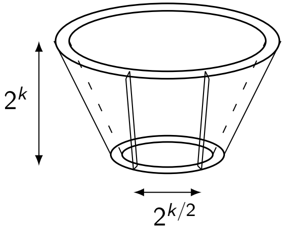

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$$\#\{\theta\} \sim 2^{(n-1)k/2}$$

$$\|\chi_{[1,2]}(t) e^{it\sqrt{-\Delta}} f^k\|_{L^p(\mathbb{R}^{n+1})} \overset{?}{\lesssim} 2^{k\gamma} \Big\| \Big(\sum_{\substack{\theta:\text{plates} \\ \angle(\theta)\sim 2^{-k/2}}} |\chi_{[1,2]}(t) e^{it\sqrt{-\Delta}} f^k_\theta|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^{n+1})} \qquad \text{(SF)}$$

$$\|\chi_{[1,2]}(t) e^{it\sqrt{-\Delta}} f^k\|_{L^p(\mathbb{R}^{n+1})} \overset{?}{\lesssim} 2^{k\gamma} \Big(\sum_{\substack{\theta:\text{plates} \\ \angle(\theta)\sim 2^{-k/2}}} \|\chi_{[1,2]}(t) e^{it\sqrt{-\Delta}} f^k_\theta\|^p_{L^p(\mathbb{R}^{n+1})} \Big)^{1/p} \qquad (\mathrm{D}_p)$$

Decoupling $\Rightarrow$ LS

The right hand-side in the decoupling inequality is "easy" to understand:

$$\Big(\sum_{\theta:\text{plates}} \|\chi_{[1,2]}(t) e^{it\sqrt{-\Delta}} f_\theta^k\|_{L^p(\mathbb{R}^{n+1})}\Big)^{1/p} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

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It turns out that for $\frac{2(n+1)}{n-1} \leqslant p < \infty$ the best possible value for γ in (D_p) is

$$\boxed{\gamma = s_p - 1/p}$$

so

$$\boxed{\text{sharp } p\text{-decoupling}} \Longrightarrow \boxed{\text{sharp LS estimates}}$$

Sharp decoupling theorem

Rescaling so that $1 \leqslant |\xi| \leqslant 2$ and in the language of Fourier extension operators

$$Ef(x,t) = \int_{1 \leqslant |\xi| \leqslant 2} e^{ix\cdot\xi + t|\xi|} f(\xi)\,\mathrm{d}\xi,$$

Theorem (Bourgain–Demeter, 2015)

For all $\epsilon > 0$ and $\lambda \geqslant 1$ there exists $C_{\epsilon,p}$ such that

$$\|Ef\|_{L^p(w_{B_\lambda})} \leqslant C_{\epsilon,p}\lambda^{\alpha(p)+\epsilon}\Big(\sum_{\theta:\lambda^{-1/2}-\text{plates}} \|Ef_\theta\|^p_{L^p(w_{B_\lambda})}\Big)^{1/p}$$

for $2 \leqslant p < \infty$, where

$$\alpha(p) := \begin{cases} s_p/2 & \textit{if} \quad 2 \leqslant p \leqslant \frac{2(n+1)}{n-1}, \\ s_p - 1/p & \textit{if} \quad \frac{2(n+1)}{n-1} \leqslant p < \infty. \end{cases}$$

They obtained the stronger ℓ^2-version, from which the ℓ^p follows from Hölder.

Wave equations on manifolds

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$$\|u(\,\cdot\,, t)\|_{L^p_{-s_p}(M)} \lesssim_{M,g} \|f_0\|_{L^p(M)} + \|f_1\|_{L^p_{-1}(M)}$$

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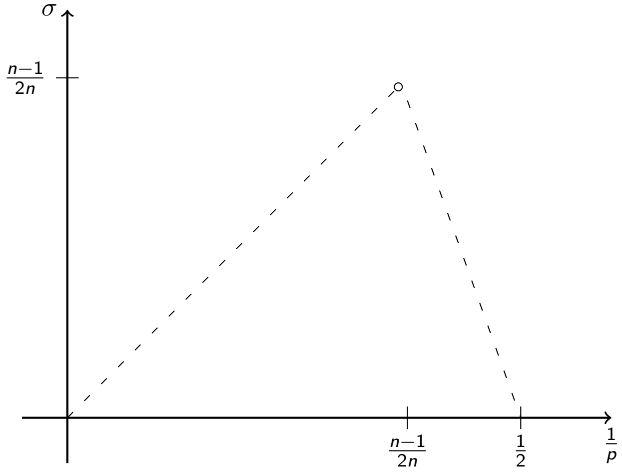
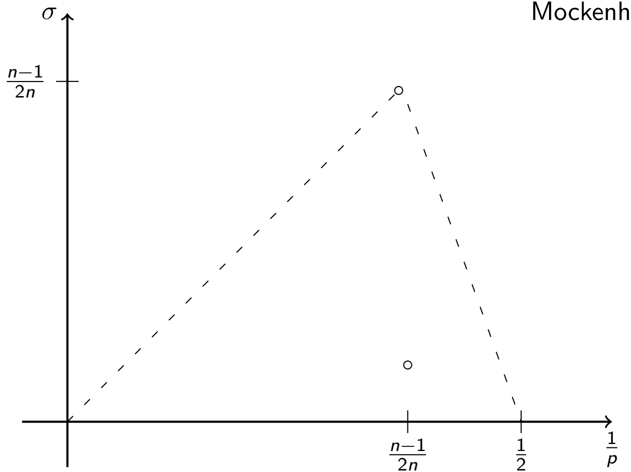
What about local smoothing estimates in this setting?

State of the art for the local smoothing conjecture



State of the art for the local smoothing conjecture

Mockenhoupt–Seeger–Sogge



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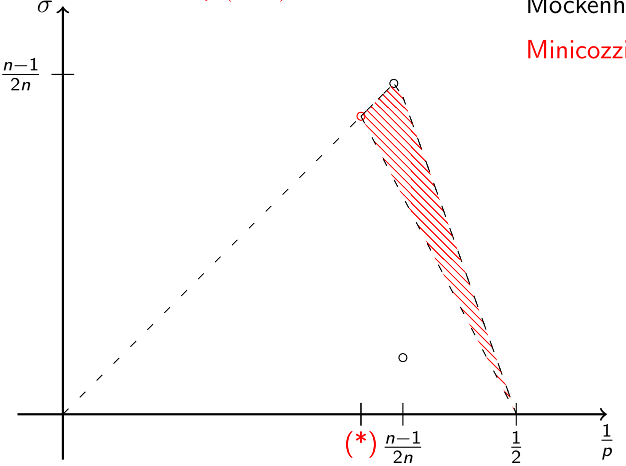
$$(*)\ \frac{1}{p} = \begin{cases} \frac{3n-3}{2(3n+1)} & \text{if } n \text{ odd} \\ \frac{3n-2}{2(3n+2)} & \text{if } n \text{ even} \end{cases}$$



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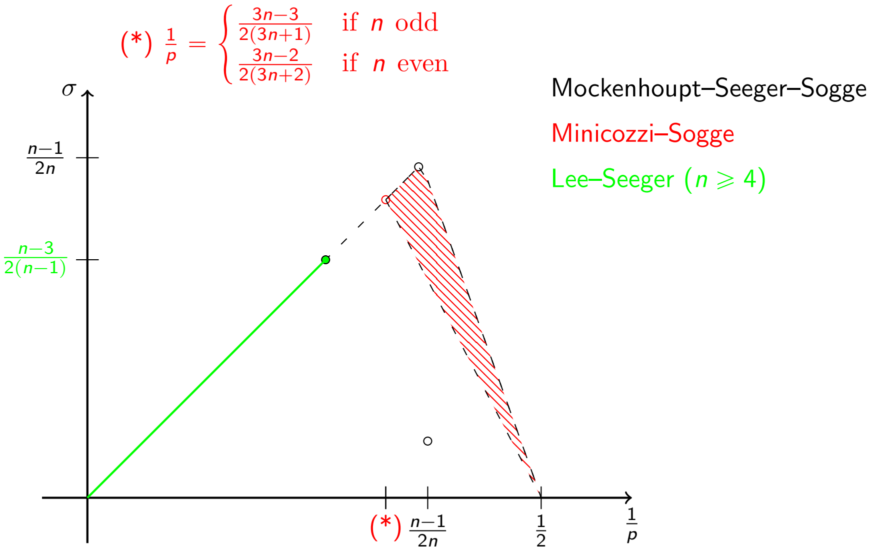

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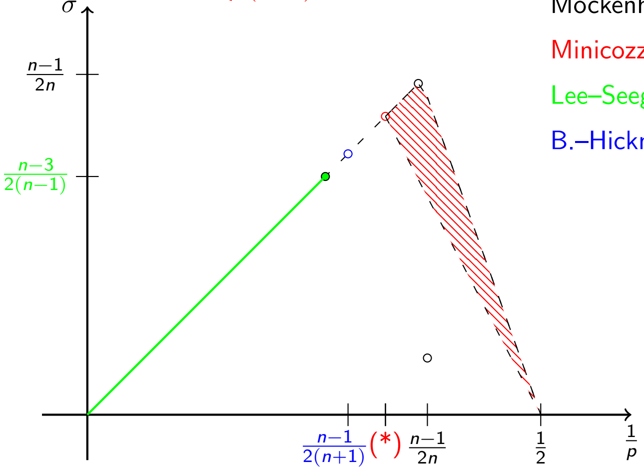

Mockenhoupt–Seeger–Sogge

Minicozzi–Sogge

Lee–Seeger ($n \geqslant 4$)

B.–Hickman–Sogge

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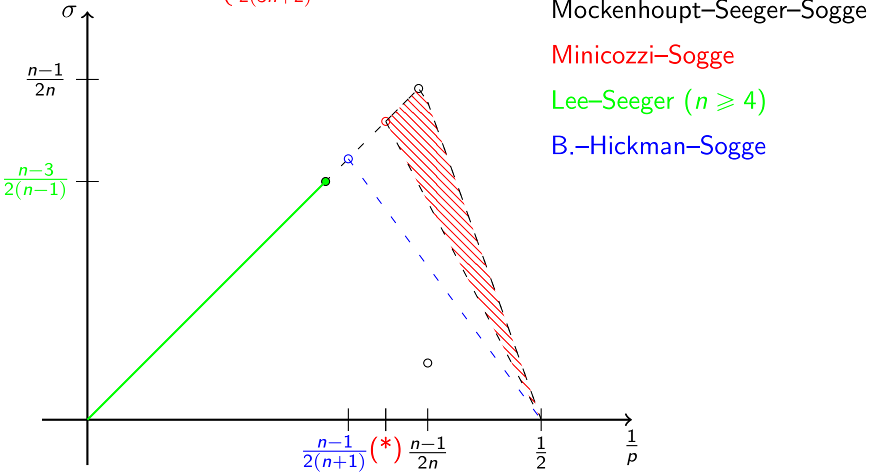

Local smoothing estimates

Theorem (B.–Hickman–Sogge)

With the previous setting, and $\frac{2(n+1)}{n-1} \leqslant p < \infty$*, the estimate*

$$\Big(\int_1^2 \|u(\,\cdot\,,t)\|^p_{L^p_{-s_p+\sigma}(M)} \,\mathrm{d}t \Big)^{1/p} \lesssim_{M,g} \|f_0\|_{L^p(M)} + \|f_1\|_{L^p_{-1}(M)}$$

holds for all $\sigma < 1/p$.

The solution u to the Cauchy problem is given by

$$u(x,t) = \mathcal{F}_0 f_0(x,t) + \mathcal{F}_1 f_1(x,t)$$

where each $\mathcal{F}_\mu$ can be written in local coordinates as a

$$\mathcal{F}_\mu f(x,t) := \int_{\hat{\mathbb{R}}^n} e^{i\phi(x,t;\xi)} \frac{b(x,t;\xi)}{(1+|\xi|^2)^{-\mu/2}} \hat{f}(\xi)\, \mathrm{d}\xi$$

where

- b is a symbol of order 0 (with compact support in the (x,t) variables)
- ϕ satisfies certain *non-degeneracy* and *curvature* hypothesis:

  For fixed (x_0, t_0),

  $$\xi \mapsto \partial_{xt}\phi(x_0, t_0; \xi)$$

  is "essentially a cone", i.e., a smooth hypersurface with $(n-1)$ non-vanishing principal curvatures.

  Remember, for $\phi(x,t;\xi) = x\cdot\xi + t|\xi|$, one has $\partial_{x,t}\phi(x,t;\xi) = (\xi, |\xi|)$.

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So enough to show

$$\|\mathcal{F}f^k\|_{L^p(\mathbb{R}^{n+1})} \lesssim 2^{k(s_p - 1/p + \epsilon)} \|f\|_{L^p(\mathbb{R}^n)}$$

for $\frac{2(n+1)}{n-1} \leqslant p < \infty$.

Phase conditions

Reduction to $1/2 \leqslant |\xi| \leqslant 2$, indeed suffices to consider a conic domain

$$\Gamma_1 := \left\{ \xi \in \hat{\mathbb{R}}^n : 1/2 \leqslant \xi_n \leqslant 2 \text{ and } |\xi_j| \leqslant |\xi_n| \text{ for } 1 \leqslant j \leqslant n-1 \right\}.$$

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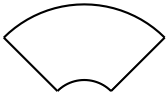

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Let $a = a_1 \otimes a_2 \in C_c^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ where

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H2) Defining the generalised Gauss map by $G(z;\xi) := \frac{G_0(z;\xi)}{|G_0(z;\xi)|}$ where

$$G_0(z;\xi) := \bigwedge_{j=1}^{n} \partial_{\xi_j}\partial_z \phi(z;\xi),$$

one has

$$\mathrm{rank}\, \partial^2_{\eta\eta}\langle \partial_z\phi(z;\eta), G(z;\xi)\rangle|_{\eta=\xi} = n-1$$

for all $(z;\xi) \in \mathrm{supp}\, a \backslash 0$.

The oscillatory integral operators

The local smoothing estimates for $\mathcal{F}$ will be deduced from a decoupling inequality for a closely related class of oscillatory integral operators.

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Given $\lambda \geqslant 1$, define the rescaled phase and amplitude

$$\phi^\lambda(x,t;\xi) := \lambda\phi(x/\lambda, t/\lambda;\xi) \qquad \text{and} \qquad a^\lambda(x,t;\xi) := a_1(x/\lambda, t/\lambda)a_2(\xi)$$

and, with this data, let

$$T^\lambda f(x,t) := \int_{\hat{\mathbb{R}}^n} e^{i\phi^\lambda(x,t;\xi)} a^\lambda(x,t;\xi) f(\xi)\,\mathrm{d}\xi.$$

Recall the Fourier extension operator

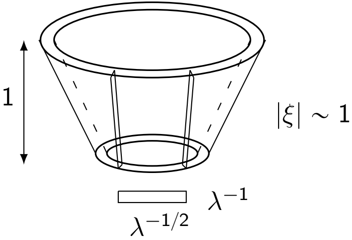
$$Ef(x,t) = \int_{1\leqslant|\xi|\leqslant 2} e^{i(x\cdot\xi + t|\xi|)} f(\xi)\,\mathrm{d}\xi$$

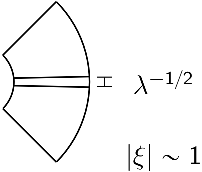
for which we studied bounds on B_λ.

$$\|Ef\|_{L^p(B_\lambda)} \quad \text{reads now} \quad \|T^\lambda f\|_{L^p(\mathbb{R}^{n+1})}.$$

The plates

Remember the constant coefficient case

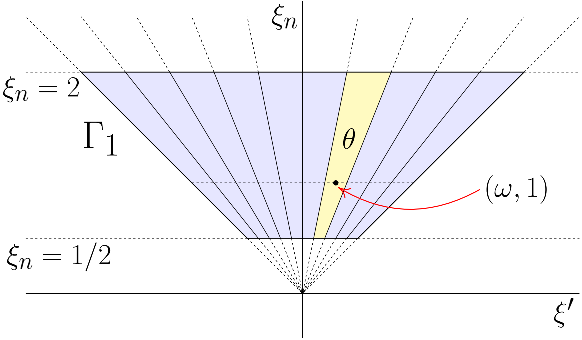




The plates

Fix a second spatial parameter $1 \leqslant R \leqslant \lambda$.
Fix a maximally $R^{-1/2}$-separated subset of $[-1,1]^{n-1} \times \{1\}$.
For each ω belonging to this subset define the $R^{-1/2}$-*plate*

$$\theta := \left\{(\xi', \xi_n) \in \hat{\mathbb{R}}^n : 1/2 \leqslant \xi_n \leqslant 2 \text{ and } |\xi'/\xi_n - \omega| \leqslant R^{-1/2}\right\}.$$



Define $f_\theta := \chi_\theta f$.

The variable coefficient decoupling theorem

Let

$$\alpha(p) := \begin{cases} s_p/2 & \text{if } 2 \leqslant p \leqslant \frac{2(n+1)}{n-1}, \\ s_p - 1/p & \text{if } \frac{2(n+1)}{n-1} \leqslant p < \infty. \end{cases}$$

Theorem (B.–Hickman–Sogge, 2018)

Let T^λ be an operator of the form described above and $2 \leqslant p \leqslant \infty$. For all $\varepsilon > 0$ and $M \in \mathbb{N}$ one has

$$\|T^\lambda f\|_{L^p(\mathbb{R}^{n+1})} \lesssim_{\varepsilon,M,\phi,a} \lambda^{\alpha(p)+\varepsilon}\Big(\sum_{\theta:\lambda^{-1/2}-\text{plate}} \|T^\lambda f_\theta\|^p_{L^p(\mathbb{R}^{n+1})}\Big)^{1/p} + \lambda^{-M}\|f\|_{L^2(\hat{\mathbb{R}}^n)}.$$

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Decoupling inequalities are “stable” : instance in Pramanik–Seeger (2007).

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Scheme of the proof

We will prove that for $1 \leqslant R \leqslant \lambda$,

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- Trivial for small scales ($R \sim 1$).
- At sufficiently small scales ($< \lambda^{1/2}$), T^λ may be effectively approximated by extension operators.

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We will prove that for $1 \leqslant R \leqslant \lambda$,

$$\|T^\lambda f\|_{L^p(B_R)} \lesssim_{\varepsilon,M,\phi,a} R^{\alpha(p)+\varepsilon}\Big(\sum_{\theta: R^{-1/2}-\mathrm{plate}} \|T^\lambda f_\theta\|^p_{L^p(B_R)}\Big)^{1/p}$$

where $B_R \subseteq B(0,\lambda)$.

Induction on scales.

- Trivial for small scales ($R \sim 1$).
- At sufficiently small scales ($< \lambda^{1/2}$), T^λ may be effectively approximated by extension operators.
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- Parabolic rescaling.

A trivial decoupling inequality

As

$$T^\lambda f = \sum_{\theta: R^{-1/2}-\text{plate}} T^\lambda f_\theta,$$

one may trivially bound

$$\begin{aligned}\|T^\lambda f\|_{L^p(B_R)} &\leqslant \sum_{\theta: R^{-1/2}-\text{plate}} \|T^\lambda f_\theta\|_{L^p(B_R)} \\ &\leqslant \Big(\sum_{\theta: R^{-1/2}-\text{plate}} \|T^\lambda f_\theta\|_{L^p(B_R)}^p \Big)^{1/p} \Big(\sum_{\theta: R^{-1/2}-\text{plate}} 1 \Big)^{1/p'} \\ &\lesssim R^{(n-1)/2p'} \Big(\sum_{\theta: R^{-1/2}-\text{plate}} \|T^\lambda f_\theta\|_{L^p(B_R)}^p \Big)^{1/p}\end{aligned}$$

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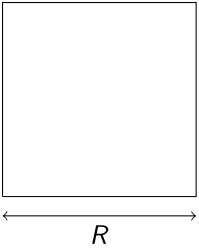
This settles the desired decoupling inequality for $R \sim 1$.

Approximation by extension operators



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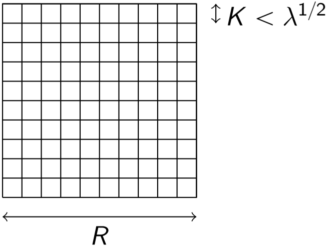

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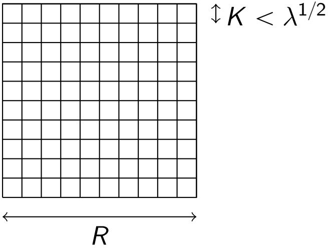

$$B_R = \bigcup_{B_K \subset B_R} B_K$$

On each B_K, one morally has

$$\|T^\lambda f\|_{L^p(B_K)} \sim \|E_K f\|_{L^p(B_K)}$$

for some Fourier extension operator E_K associated to a *conic* hypersurface.

Aproximation by extension operators, cont'd

Fix a $B_K = B(\bar{z}, K)$.

Apply a nonlinear change of variables $\xi = \Psi^\lambda_{\bar{z}}(\eta)$ and a Taylor expansion of ϕ^λ around the point $\bar{z}$,

$$\begin{aligned} T^\lambda f(z) &= \int_{\hat{\mathbb{R}}^n} e^{i\phi^\lambda(z;\xi)} a_1^\lambda(z) a_2(\xi) f(\xi)\,\mathrm{d}\xi \\ &= \int_{\hat{\mathbb{R}}^n} e^{i(z-\bar{z})\cdot(\eta, h_{\bar{z}}(\eta)) + i\mathcal{E}^\lambda_{\bar{z}}(z-\bar{z};\eta)} a_1^\lambda(z) a_{\bar{z}}(\eta) f_{\bar{z}}(\eta)\,\mathrm{d}\eta \end{aligned}$$

where

- $a_{\bar{z}}(\eta) := a_2 \circ \Psi^\lambda_{\bar{z}}(\eta) |\det \partial_\eta \Psi^\lambda_{\bar{z}}(\eta)|$
- $f_{\bar{z}}(\eta) := e^{i\phi^\lambda(\bar{z};\Psi^\lambda_{\bar{z}}(\eta))} f \circ \Psi^\lambda_{\bar{z}}(\eta)$
- $h_{\bar{z}}(\eta)$ is a smooth function homogeneous of degree 1.
- and, by Taylor's theorem,

$$\mathcal{E}^\lambda_{\bar{z}}(v;\eta) = \frac{1}{\lambda}\int_0^1 (1-r)\langle(\partial^2_{zz}\phi)((\bar{z}+rv)/\lambda; \Psi^\lambda_{\bar{z}}(\eta))v, v\rangle\,\mathrm{d}r.$$

Aproximation by extension operators, cont'd

Fix a $B_K = B(\bar{z}, K)$.

Apply a nonlinear change of variables $\xi = \Psi_{\bar{z}}^{\lambda}(\eta)$ and a Taylor expansion of ϕ^{λ} around the point $\bar{z}$,

$$T^{\lambda} f(z) = \int_{\hat{\mathbb{R}}^n} e^{i\phi^{\lambda}(z;\xi)} a_1^{\lambda}(z) a_2(\xi) f(\xi)\, \mathrm{d}\xi$$
$$= \int_{\hat{\mathbb{R}}^n} e^{i(z-\bar{z})\cdot(\eta, h_{\bar{z}}(\eta)) + i\mathcal{E}_{\bar{z}}^{\lambda}(z-\bar{z};\eta)} a_1^{\lambda}(z) a_{\bar{z}}(\eta) f_{\bar{z}}(\eta)\, \mathrm{d}\eta$$

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Since $|v| = |z-\bar{z}| \leqslant K \leqslant \lambda^{1/2}$, for all $\beta \in \mathbb{N}_0^n$ one has

$$\sup_{(v;\eta)\in B(0,K)\times \operatorname{supp} a_{\bar{z}}} |\partial_{\eta}^{\beta} \mathcal{E}_{\bar{z}}^{\lambda}(v;\eta)| \lesssim_{|\beta|} 1.$$

Use of constant coefficient decoupling

On each B_K, the approximation

$$\|T^\lambda f\|_{L^p(B_K)} \sim \|E_K f_{\bar{z}}\|_{L^p(B(0,K))}$$

allows to apply the Bourgain–Demeter theorem for such E_K:

$$\|T^\lambda f\|_{L^p(B_K)} \sim \|E_K f_{\bar{z}}\|_{L^p(B(0,K))} \lesssim K^{\alpha(p)+\varepsilon/2}\Big(\sum_{\theta: K^{-1/2}-\mathrm{plates}} \|E_K f_{\bar{z}}\|^p_{L^p(B(0,K))}\Big)^{1/p}$$

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and summing over $B_K \subset B_R$

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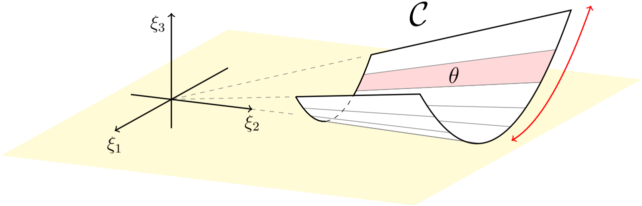
Parabolic rescaling

ξ space



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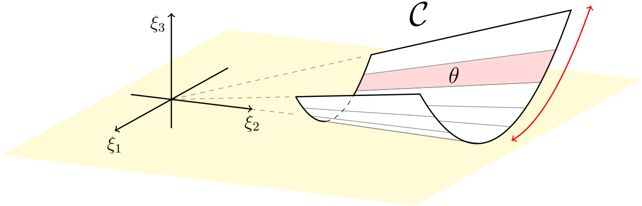
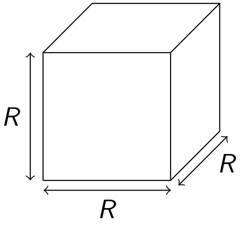
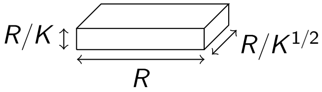


(x, t) space





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By induction hypothesis, one assumes the inequality

$$\| T^{\lambda} f \|_{L^p(B_\rho)} \lesssim \rho^{\alpha(p)+\varepsilon} \Big(\sum_{\theta:\rho^{-1/2}-\text{plates}} \| T^{\lambda} f_\theta \|^p_{L^p(B_\rho)} \Big)^{1/p}$$

to hold for $\rho \leqslant R/2$.

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If θ is a $K^{-1/2}$-plate, by rescaling and setting $\rho = R/K$,

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Summing over all $K^{-1/2}$-plates θ,

$$\Big(\sum_{\theta:K^{-1/2}-\text{plates}} \|T^\lambda f_\theta\|_{L^p(B_R)}^p\Big)^{1/p} \lesssim (R/K)^{\alpha(p)+\varepsilon}\Big(\sum_{\sigma:R^{-1/2}-\text{plates}} \|T^\lambda f_\sigma\|_{L^p(B_R)}^p\Big)^{1/p}$$

Closing induction

We saw:

Approximation + Bourgain–Demeter constant coefficient implies

$$\|T^\lambda f\|_{L^p(B_R)} \lesssim K^{\alpha(p)+\varepsilon/2}\Big(\sum_{\theta:K^{-1/2}-\text{plates}} \|T^\lambda f_\theta\|_{L^p(B_R)}^p\Big)^{1/p}.$$

Parabolic rescaling + induction on the radius implies

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So altogehter,

$$\|T^\lambda f\|_{L^p(B_R)} \lesssim K^{-\varepsilon/2} R^{\alpha(p)+\varepsilon}\Big(\sum_{\sigma: R^{-1/2}-\text{plates}} \|T^\lambda f_\sigma\|_{L^p(B_R)}^p\Big)^{1/p}.$$

Choose K large enough so that $C_\varepsilon K^{-\varepsilon/2} \leqslant 1$.

Bounds for Hörmander-type operators

Decouping theory for Hörmander type operators is the same as for extension operators.

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Case of non-homogeneous phase functions:

Restriction conjecture:

$$\|Ef\|_{L^{\frac{2n}{n-1}}(B_\lambda)} \lesssim \lambda^{\varepsilon}\|f\|_{L^\infty(B^{n-1})}.$$

Hörmander conjectured the same estimate to hold for T^λ.

True for $n = 2$: Carleson–Sjolin (1972).

Bounds for Hörmader type operators

Bourgain (1991): false for $n \geqslant 3$,

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fails to hold uniformly in $\lambda \geqslant 1$ whenever $p < \frac{2(n+1)}{n-1}$.

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What if principal curvatures are all assumed to be positive?

Positive definite phases

It is possible to go beyond the above exponents.

After contributions of Lee (2006) and Bourgain–Guth (2011), the sharp bounds were recently stablished by Guth–Hickman–Iliopoulou (2017):

$$\|T_+^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^\varepsilon \|f\|_{L^p(B^{n-1})}$$

holds for all $\lambda \geqslant 1$ whenever

$$p \geqslant \frac{2(3n+1)}{3n-3} \quad \text{if } n \geqslant 3 \text{ is odd,}$$
$$p \geqslant \frac{2(3n+2)}{3n-2} \quad \text{if } n \geqslant 4 \text{ is even.}$$

Sharp: for instance by the examples of Minicozzi–Sogge (1997).

Sharp local smoothing for Fourier integral operators

We showed in general, that

$$\|\mathcal{F}f\|_{L^p_{-s_p+1/p+\epsilon}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

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	n odd	n even
$n-1$ non-vanishing curvatures	$\frac{2(n+1)}{n-1}$	$\frac{2(n+2)}{n}$
$n-1$ positive curvatures	$\frac{2(3n+1)}{3n-3}$	$\frac{2(3n+2)}{3n-2}$

Merci!