Aggregation equation and collapse to singular measure

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Collective behaviour and self-organization

1. **Ant Trails**

2. **Bird flocks**

3. **fish schools**

Questions

1. Why a large number of individuals want to move through a flock ?
2. How large-scale complexity emerge from microscopic local rules without centralized coordination ?
3. Identify the mechanisms governing the interactions between individuals and their environment.

Models

- Ideas from statistical physics(particle description),Kinetic theory (mean field limit).
- Hydrodynamic approach.

- Consider N particles moving through classical mechanics law :

$$\begin{cases} \dot{X}_i(t) = -\frac{1}{N}\sum_{j\neq i} \nabla K(X_i(t) - X_j(t)) \\ X_i(0) = X_i^0, i \in \{1, 2, .., N\} \end{cases}$$

We define the empirical measure

$$\mu_N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i(t)}$$

- If $(\mu_N(0))_N$ converges weakly to ρ_0, then "formally" $(\mu_N(t))_N$ converges weakly to the density ρ satisfying the continuity equation

$$\begin{cases} \partial_t \rho + \operatorname{div}(v\rho) = 0, \qquad\qquad x \in \mathbb{R}^d,\ t \geq 0 \\ v = -\nabla K \star \rho, \\ \rho_{|t=0} = \rho_0. \end{cases}$$

Aggregation equation with Newtonian potential

$$\begin{cases} \partial_t \rho + \text{div}(v\rho) = 0, & x \in \mathbb{R}^2,\ t \geq 0 \\ v = -\nabla K \star \rho, \quad K(x) = \frac{1}{2\pi} \log |x| \\ \rho_{|t=0} = \rho_0. \end{cases}$$

- The vector-field v is compressible :

$$\text{div } v = -\rho$$

- This equation is the compressible version of Euler equations where

$$v = \nabla^\perp K \star \rho = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \rho(y) dy$$

and ρ is the vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$, which satisfies the transport equation

$$\partial_t \omega + v \cdot \nabla \omega = 0$$

Classical solutions

- Wölibner (1933) : Euler equation is globally well-posed for $\omega_0 \in C^\alpha, \alpha > 0$. This follows from

$$\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}$$

- Nieto-Poupaud-Soler ; Bertozzi-Laurent-Léger : The aggregation equation is locally well-posed for $\rho_0 \in C^\alpha, \alpha > 0$: $\rho \in L^\infty([0, T^\star), C^\alpha)$.
- What about $T^\star$?

- The aggregation equation takes the form

$$\partial_t \rho + v \cdot \nabla \rho = \rho^2$$

- Using the characteristics we get

$$\rho(t, \psi(t,x)) = \frac{\rho_0(x)}{1 - t\rho_0(x)}$$

where ψ denotes the trajectories map :

$$\psi(t,x) = x + \int_0^t v(\tau, \psi(\tau, x)) d\tau$$

Consequences : Let $\rho_0 \in C_c^\alpha, \alpha > 0$, then

1. Global existence is satisfied when $\sup_{\mathbb{R}} \rho_0(x) \leq 0$.
2. Blow up in finite time happens if and only $\sup_{\mathbb{R}} \rho_0(x) > 0$,

$$T_\star = \frac{1}{\sup_{\mathbb{R}} \rho_0(x)} \quad \text{and} \quad \|\rho(t)\|_{L^\infty} = \frac{1}{T_\star - t}$$

Yudovich solutions type

- Yudovich : Euler admits global unique solution $\omega \in L^\infty(\mathbb{R}_+; L^1 \cap L^\infty)$
- N-P-S, B-L-L : Existence of solutions when $\rho_0 \in L^1 \cap L^\infty$, with

$$\rho \in L^\infty([0, T], L^\infty \cap L^1)$$

- For $\rho_0 \in L^1 \cap C_c$

$$\|\rho(t)\|_{L^1} = \|\rho_0\|_{L^1}$$

- The blow up does not occur for L^1 !

Concentration phenomenon

- Question : The L^1 norm is conserved but what about concentration phenomenon when $t \to T^\star$?

Two examples (from B-L-L)

1. Delta mass concentration : Let $\rho_0 = \frac{1}{\pi}\mathbf{1}_{D(0,1)}$ then weakly

$$\lim_{t\to 2\pi} \rho(t)dx \to \pi\delta_0$$

2. Vortex sheet concentration : Let $\rho_0 = (\pi ab)^{-1}\mathbf{1}_{\mathcal{E}}$ with $\mathcal{E}$ the ellipse centered at zero and with semi axes a and b. Then $\rho(t) = (\pi a(t)b(t))^{-1}\mathbf{1}_{\mathcal{E}_t}$, with $\mathcal{E}_t$ an ellipse and it converges weakly to the semi-cricle law

$$\lim_{t\to\pi ab} \rho(t)dx = \frac{2\sqrt{x_0^2 - x_1^2}}{\pi x_0^2}\mathbf{1}_{[-x_0,x_0]}dx_1, \; x_0 = a - b$$

Time rescaling and vortex patch problem

- Let $\rho_0 = \mathbf{1}_{D_0}$ with D_0 a bounded domain, then

$$\rho(t,x) = \frac{\mathbf{1}_{\widehat{D}_t}}{1-t}, 0 \le t < 1 = T_\star, \quad \widehat{D}_t = \psi(t, D_0)$$

Notice that

$$|\widehat{D}_t| = (1-t)|D_0|$$

- Let $\tau = -\ln(1-t)$ and $D_\tau = \widehat{D}_{1-e^{-\tau}}$ then $\mathbf{1}_{D_\tau}$ is a solution of

$$\partial_\tau \rho + v \cdot \nabla \rho = 0, \quad \rho_0 = \mathbf{1}_{D_0}, \quad v(\tau) = \nabla K \star \mathbf{1}_{\mathbf{D}_\tau}, \quad \tau \in [0, +\infty),$$

One also has

$$\forall \tau \ge 0, \quad |D_\tau| = e^{-\tau}|D_0|$$

- Chemin (Euler equation) : If $\partial D_0 \in C^{1+\varepsilon}$ then $\forall t \ge 0, \quad \partial D_t \in C^{1+\varepsilon}$,
- Bertozzi-Garnett-Laurent-Verdera (Aggregation) : similar result.

Asymptotic behavior

- Recall that $|D_t| = e^{-t}|D_0|$.
- This implies that

$$\int_0^{+\infty} \|v(t)\|_{L^\infty} dt < \infty$$

- This allows to define the limit shape

$$D_\infty := \left\{ \lim_{t\to+\infty} \psi(t,X), X \in D_0 \right\}$$

- What about the geometric structure of D_∞ ?
- What about the weak limit of the probability measure :

$$dP_t = e^t \frac{\mathbf{1}_{D_t}}{|D_0|} dX, \quad X = (x,y) \in \mathbb{R}^2$$

- Numerical simulations (B-L-L) : Collapse to skeleton shapes.

Proposition

Let D_0 be a simply connected domain symmetric with respect to an axis Δ. Denote by $L = Length(D_0 \cap \Delta)$. There exists an absoute constant C such that if

$$L > C|D_0|^{\frac{1}{2}}$$

then the shape D_∞ contains a segment of size $L - C|D_0|^{\frac{1}{2}}$.
Thin initial domains along their axis of symmetry generate concentration to segments.

- Applications to polygonal shapes.

One fold symmetric patches

Main goal : Find initial patches for which we can analyze the concentration to a collection of segments lying in the same straight line.

- Let D_0 be a planar set symmetric with respect to the real axis with

$$D_0 = \Big\{(x, y), x \in \mathbb{R}, -f(x) \leq y \leq f(x)\Big\}, \quad f \in C_c^1(\mathbb{R}, \mathbb{R}_+)$$

- D_0 is defined by a graph and what about D_t ?

- Note that D_t is symmetric with respect to the real axis for any $t \geq 0$.

Strategy

- Write the graph equation.
- Study local well-posedness : Hölder and continuous Dini spaces.
- Global well-posedness with small initial data.
- Collapse to singular measure.

Graph reformulation

- Let $s \mapsto \gamma_t(s)$ be any parametrization of the boundary ∂D_t, then
$$\big(\partial_t\gamma_t(s) - v(t,\gamma_t(s))\big)\cdot \vec{n}(\gamma_t(s)) = 0$$
with $\vec{n}(\gamma_t)$ being a normal vector to the boundary at the point $\gamma_t(s)$
- By taking the parametrization $x \mapsto (x, f(t,x))$ we get
$$\begin{cases} \partial_t f(t,x) + u_1(t,x)\partial_x f(t,x) = u_2(t,x), \ t \geq 0, x \in \mathbb{R} \\ f(0,x) = f_0(x). \end{cases}$$
where $(u_1,u_2)(t,x)$ is the trace of (v_1,v_2) at the point $X = (x, f(t,x))$.
- One has the expressions
$$u_1(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}}\left\{\arctan\left(\frac{f_t(x+y)-f_t(x)}{y}\right) + \arctan\left(\frac{f_t(x+y)+f_t(x)}{y}\right)\right\}dy$$
$$u_2(t,x) = \frac{1}{4\pi}\int_{\mathbb{R}}\log\left[\frac{y^2+\big(f_t(x+y)-f_t(x)\big)^2}{y^2+\big(f_t(x+y)+f_t(x)\big)^2}\right]dy.$$

Qualitative properties

- Maximum principle :

$$\forall t \geq 0, \quad x \in \mathbb{R}, \quad 0 \leq f(t,x) \leq \|f_0\|_{L^\infty}.$$

  This can derived from the fact that

$$u_2(t,x) \leq 0.$$

- Confinement of the support : If $\mathrm{supp} f_0 \subset [a,b]$ then

$$t \geq 0, \quad \mathrm{supp} f(t) \subset [a,b]$$

Functional setting

- Hölder and Dini spaces : let $s \in (0,1)$ denote by C^s the usual Hölder space and

$$f \in C_D \Longleftrightarrow \|f\|_{L^\infty} + \int_0^1 \frac{\mu(r)}{r} dr < \infty$$

with $\mu(r) = \sup_{|x-y| \leq r} |f(x) - f(y)|$

- Dini space is an algebra, contrary to Besov space $B^0_{\infty,1}$.
- Beirao da Veiga (1984) : Euler equations are global when the vorticity belongs to Dini space. This is a consequence of the composition law :

$$\|f \circ \psi\|_{C_D} \leq C\|f\|_{C_D} \log(1 + \|\nabla\psi\|_{L^\infty})$$

- We have the embedding : for any $s \in (0,1)$

$$C^s \hookrightarrow C_D \hookrightarrow B^0_{\infty,1} \hookrightarrow C_b$$

Denote by $X = C^s, s \in (0,1), \quad C_D$.

Theorem (H.-Li (2018))

Let f_0 be a positive compactly supported function, s.t. $f_0' \in X$. Then

1. *The graph equation admits a unique local solution s.t. $f' \in L^\infty([0,T], X)$.*
2. *There exists $\varepsilon > 0$ such that if*

$$\|f_0'\|_{C^s} \leq \varepsilon,$$

then $T = +\infty$. Moreover

$$\|f(t)\|_{L^\infty} + \|f'(t)\|_{L^\infty} \leq C_0 e^{-t}$$

Theorem (H.-Li (2018))

Let f_0 be a positive compactly supported function, s.t. $f_0' \in C^s$ with small norm. Assume that supp f_0 is the union of $n-$disjoint segments.

1. *There exists a compact set $D_\infty \subset \mathbb{R}$ composed of exactly $n-$disjoints segments,*

$$\forall\, t \geq 0, \quad d_H(D_t, D_\infty) \leq C_0 e^{-t}$$

2. *The probability measure $dP_t = e^t \frac{\mathbf{1}_{D_t}}{|D_0|} dX$ converges weakly as $t \to +\infty$ to the probability measure*

$$dP_\infty(X) = \Phi(x)\mathbf{1}_{D_\infty} dx, \quad X = (x, y)$$

with $\Phi \in C^\alpha, \forall \alpha \in (0,1)$ and

$$\begin{aligned} \Phi(x) &= 2 \lim_{t \to +\infty} e^t f(t,x) \\ &= \frac{f_0(\psi_\infty^{-1}(x))}{\|f_0\|_{L^1}} e^{g(x)} \end{aligned}$$

where $\psi_\infty(x) = \lim_{t\to+\infty} \psi_1(t,x)$ and ψ_1 is the flow associated to u_1. The function g is recovered through the full dynamics of the graph.

Slope equation

- Set $g(t,x) = \partial_x f(t,x) = f'(t,x)$ and define the operators

$$\Delta_y^{\pm} f(x) = f(x+y) \pm f(x)$$

then

$$\partial_t g + u_1 \partial_x g = F^-(t,x) - F^+(t,x)$$

with

$$F^{\pm}(t,x) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \frac{[\Delta_y^{\pm} f(x) \pm y f'(x)] \Delta_y^{\pm} f'(x)}{y^2 + [\Delta_y^{\pm} f(x)]^2} dy$$

- For local well-posedness we need to estimate

$$\|F^{\pm}(t)\|_X, \quad X = C^s, C_D$$

Estimate of F^-

- Recall that

$$F^-(t,x) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{R}} \frac{[\Delta_y^- f(x) - yf'(x)]\Delta_y^- f'(x)}{y^2 + [\Delta_y^- f(x)]^2} dy$$

- It is connected to Cauchy operator associated to the graph f :

$$\mathcal{C}_f g(x) = \int_{\mathbb{R}} \frac{g(x+y) - g(x)}{y + i(f(x+y) - f(x))} dy$$

- Coifman,McIntosh,Meyer : If $f \in W^{1,\infty}$, then for $p \in (1,\infty)$, $\mathcal{C}_f : L^p \to L^p$ is continuous.

- Wittmann : If K is a compact, $0 < s < 1$, $f \in C^{1+s}$ then $\mathcal{C}_f : \mathcal{C}_K^s \to C^s$ is continuous.

- This operator "could" fail to be continuous on Dini space !

Truncated bilinear Cauchy operators

- We shall be concerned with : for $M > 0, \theta \in [0,1]$

$$\mathcal{C}_f^\theta(g,h)(x) = \int_{-M}^{M} \frac{[g(x+\theta y) - g(x)][h(x+y) - h(x)]}{y + i(f(x+y) - f(x))} dy$$

Proposition

Let $0 < s < 1, K$ be a compact $X = C_K^s, C_{D,K}$ then we have the estimates

$$\|\mathcal{C}_f^\theta(g,h)\|_X \leq C\left[1 + \|f'\|_X^3\right]\left(\|g\|_D\|h\|_X + \|h\|_D\|g\|_X\right)$$

As an application :

$$\|F^-\|_X \leq C\|f'\|_D\left[\|f'\|_X + \|f'\|_X^3\right]$$

Estimate of F^+

- Recall that

$$F^+(t,x) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{R}} \frac{[\Delta_y^+ f(x) + yf'(x)]\Delta_y^+ f'(x)}{y^2 + [\Delta_y^+ f(x)]^2} dy$$

- It is connected to Cauchy operator of the second kind : let $\alpha, \beta \in [0,1]$

$$\mathcal{T}_f^{\alpha,\beta} g(x) = \text{p.v.}\int_{\mathbb{R}} \frac{y\, g(\alpha x + \beta y)}{y^2 + [f(x+y) + f(x)]^2} dy$$

Proposition

Let f be positive, compactly supported in K and $f' \in C_D$. Then

1. *The operator $\mathcal{T}_f^{\alpha,\beta} : C_{D,K} \to L^\infty$ is continuous indepen. on α, β*
2. *The modified operator $f'\mathcal{T}_f^{\alpha,\beta} : C_{D,K} \to C_{D,K}$ is continuous. The continuity constant depends only on $|\ln \beta|$ and finite for $\beta = 0$.*
3. *Similar estimates in C_K^s.*
4. *Application :*

$$\|F^+\|_X \le C\big(1 + \|f'\|_D^{\frac{1}{3}}\big)\big[\|f'\|_X + \|f'\|_X^{16}\big]$$

One key estimate

Lemma

Let K be a compact set of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous positive function supported in K such that $f' \in C_D$. Then we have,

$$\forall x \in \mathbb{R}, \quad |f'(x)| \leq C \frac{\|f'\|_{C_D}\left(1 + \ln_+(1/\|f'\|_D\right)}{1 + \ln_+\left(\frac{1}{f(x)}\right)},$$

If in addition $f' \in C^s(\mathbb{R})$ with $s \in (0,1)$, then

$$\forall x \in \mathbb{R}, \quad |f'(x)| \leq C\|f'\|_s^{\frac{1}{1+s}} [f(x)]^{\frac{s}{1+s}}$$

and the constant C depends only on s.

- Here $X = C^s_K, 0 < s < 1$. Recall that

$$\|F^-\|_{C^s} \leq C\|f'\|_D\big[\|f'\|_{C^s} + \|f'\|^3_{C^s}\big]$$

and

$$\|F^+\|_{C^s} \leq C\big(1 + \|f'\|_D^{\frac{1}{3}}\big)\big[\|f'\|_{C^s} + \|f'\|^{16}_{C^s}\big]$$

- By interpolation

$$\begin{aligned}\|f'(t)\|_D &\lesssim \|f(t)\|^{\theta}_{L^1}\|f'(t)\|^{1-\theta}_{C^s}\\ &\lesssim e^{-Ct}\|f'(t)\|^{1-\theta}_{C^s}\end{aligned}$$

- We establish the refined estimate :

$$F^+(t,x) = f'(x) + L(t,x) + N(t,x)$$

$$\|L(t)\|_{C^s} \leq \|f'\|_{C^s} + C\|f'\|^s_D\|f'\|_{C^s}$$

$$\|N(t)\|_{C^s} \leq C\|f'\|^{\frac{1}{3}}_D\big[\|f'\|_{C^s} + \|f'\|^{16}_{C^s}\big]$$

- $g = f'$ satisfies

$$\partial_t g + u_1\partial_x g + g = -L(t,x) - N(t,x) + F^-(t,x).$$

- Define the probability measure

$$dP_t = e^t \mathbf{1}_{D_t} dX, \quad X = (x, y), |D_0| = 1$$

Note that supp $dP_t \subset K_0$ a fixed compact independent of the time

- Let $\varphi \in C_c^\infty(\mathbb{R}^2)$, then by Fubini

$$\begin{aligned} I_t &:= \int_{\mathbb{R}^2} \varphi(x, y) dP_t(X) \\ &= e^t \int_{\mathbb{R}} \int_{-f(t,x)}^{f(t,x)} \varphi(x, y) dy dx \end{aligned}$$

- Taylor expansion on $y : \varphi(x, y) = \varphi(x, 0) + y\eta(x, y), \quad \eta \in L^\infty(\mathbb{R}^2)$ and

$$I_t = 2e^t \int_{\mathbb{R}} f(t, x)\varphi(x, 0)dx + O(e^{-t}) \quad \text{since} \quad \|f(t)\|_{L^\infty} \le Ce^{-t}$$

- Main result : there exists $\Phi \in C^\alpha(\mathbb{R}), \forall \alpha \in (0, 1)$ such that

$$\|e^t f(t) - \Phi\|_{L^\infty} = O(e^{-\epsilon t})$$

One has

$$\partial_t f(t,x) + u_1 \partial_x f(t,x) + f(t,x) = -f(t,x)R(t,x) \tag{1}$$

with

$$\begin{aligned} \|R(t)\|_{L^\infty} &\leq \|f'(t)\|_D \Big(1 + \|f'(t)\|_\infty^5\Big) \\ &\leq Ce^{-\eta t}, \quad \eta > 0. \end{aligned} \tag{2}$$

From the characteristic method we get the representation

$$e^t f\Big(t, \psi_1(t,x)\Big) = f_0(x) e^{\int_0^t R(\tau,\psi_1(\tau,x))d\tau}. \tag{3}$$

Thus

$$\lim_{t\to+\infty} \|e^t f\Big(t, \psi_1(t,\cdot)\Big) - R_2(\cdot)\|_{L^\infty} = 0,$$

with

$$R_2 : x \mapsto f_0(x) e^{\int_0^{+\infty} R(\tau,\psi_1(\tau,x))d\tau}. \tag{4}$$

Thank you ! !