Why doing noncommutative Fourier analysis.

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Some definitions.

First Heisenberg group : $\mathbb{H}^1 \sim \mathbb{R}^3$ with the group law

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + 2(xy' - x'y)).$$

Heisenberg family $(\mathbb{H}^d)_{d \geq 1}$: $\mathbb{H}^d \sim \mathbb{R}^{2d+1}$ with

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- Lebesgue spaces : $L^p(\mathbb{H}^1) \sim L^p(\mathbb{R}^3)$.
- Convolution :

$$(f * g)(w) := \int_{\mathbb{H}^1} f(wv^{-1})g(v)dv.$$

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Some motivations.

What do the laws of phyics look like on $\mathbb{H}^1$? The geometry prevents dispersion ; consider the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^1} u = 0 \\ u(0) = u_0. \end{cases} \tag{1}$$

On $\mathbb{R}^3$: $\|u(t)\|_{L^\infty} \lesssim t^{-\frac{3}{2}} \|u_0\|_{L^1}$.
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Another example : NLS_3.

The defocusing nonlinear Schrödinger equation of order three (NLS_3)

$$\begin{cases} i\partial_t u + \Delta u = u|u|^2 \\ u(0) = u_0. \end{cases} \tag{2}$$

is :

- well-posed on $H^1(\mathbb{R}^3)$;
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Dispersive estimates are so weak, we are forced to live without them.
$\Rightarrow$ Akin to having strong nonlinearities and/or high (euclidean) dimensions.

Would like to have practical tools, like Fourier analysis... which already exists... somehow.

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A primer in abstract Fourier theory.

An irreducible unitary representation (IUR) of a group G is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$ is a group morphism.

Example (Euclidean case)

If $G = \mathbb{R}^n$, any IUR is of the type $(\pi_\xi, \mathbb{C})$ with $\xi \in (\mathbb{R}^n)^*$, where

$$\pi_\xi : \mathbb{R}^n \longrightarrow \mathcal{U}(\mathbb{C})$$
$$x \longmapsto M_{e^{i\langle \xi, x\rangle}} = \left(z \mapsto e^{i\langle \xi, x\rangle} z\right).$$

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The abstract Fourier transform.

For a 'reasonable' group G, the Fourier transform of $f \in L^1(G)$ is

$$\mathcal{F}(f)(\pi) := \int_G f(g)\overline{\pi(g)}dg \in \mathcal{L}(\mathcal{H}).$$

Example (Euclidean case)

If $G = \mathbb{R}^n$, we have

$$\mathcal{F}(f)(\pi_\xi) := \int_{\mathbb{R}^n} f(x) M_{e^{-i\langle \xi, x\rangle}} dx = M_{\hat{f}(\xi)} \in \mathcal{L}(\mathbb{C}).$$

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Basic abstract properties

Action on convolutions :

$$\mathcal{F}(f_1 * f_2)(\pi) = \mathcal{F}(f_1)(\pi) \circ \mathcal{F}(f_2)(\pi).$$

Abstract Parseval identity :

$$\|f\|^2_{L^2(G)} = \int_{\dots} \|\mathcal{F}(f)(\pi)\|^2 d\pi.$$

Inversion formula :

$$f(g) = \int_{\dots} \langle \mathcal{F}(f)(\pi), \pi(g) \rangle d\pi.$$

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From a matrix to its coefficients

$\mathcal{F}(f)(\pi_\lambda) \in \mathcal{L}(L^2(\mathbb{R})) \Rightarrow \mathcal{F}(f)(\pi_\lambda) \sim$ infinite matrix. Coefficients ?
Obtained by computing

$$\hat{f}(\lambda, n, m) := \langle \mathcal{F}(f)(\pi_\lambda) \cdot e_m, e_n \rangle_{L^2(\mathbb{R})}$$

for a suitable ONB $(e_n)_{n\in\mathbb{N}}$ of $L^2(\mathbb{R})$. Which $(e_n)_{n\in\mathbb{N}}$ do we choose ?
Hint : we are interested in the Schrödinger equation $\Rightarrow$ look at the laplacian.

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Pour qui sonne le glas-placien.

Motto : « the Fourier transform diagonalizes the laplacian ».

On $\mathbb{R}^n$:

$$\mathcal{F}(\Delta f)(\pi_\xi) = \mathcal{F}(f)(\pi_\xi) \circ M_{-|\xi|^2}$$

$$\left(\Leftrightarrow \widehat{\Delta f}(\xi) = -|\xi|^2 \hat{f}(\xi)\right)$$

On $\mathbb{H}^1$:

$$\mathcal{F}(\Delta_{\mathbb{H}^1} f)(\pi_\lambda) = \mathcal{F}(f)(\pi_\lambda) \circ \Delta_{\mathrm{osc},\lambda}$$

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Eigenfunctions : (rescaled) Hermite functions $H_{n,\lambda} := |\lambda|^{\frac{1}{4}} H_n(|\lambda|^{\frac{1}{2}} \cdot)$.

Eigenvalues : $\Delta_{\text{osc},\lambda} H_{n,\lambda} = -4|\lambda|(2n+1)H_{n,\lambda}$.

The eigenvalues mix the IUR parameter and the Hermite index !

If we choose $e_n = H_{n,\lambda}$, we get

$$\begin{aligned}\widehat{\Delta f}(\lambda, n, m) &= \langle \mathcal{F}(f)(\pi_\lambda) \cdot (\Delta_{\text{osc},\lambda} H_{m,\lambda}), H_{n,\lambda} \rangle_{L^2(\mathbb{R})} \\ &= \langle \mathcal{F}(f)(\pi_\lambda) \cdot (-4|\lambda|(2m+1)H_{m,\lambda}), H_{n,\lambda} \rangle_{L^2(\mathbb{R})} \\ &= -4|\lambda|(2m+1)\hat{f}(\lambda, n, m).\end{aligned}$$

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$$\begin{aligned}\widehat{\Delta f}(\lambda, n, m) &= \langle \mathcal{F}(f)(\pi_\lambda) \cdot (\Delta_{osc,\lambda} H_{m,\lambda}), H_{n,\lambda} \rangle_{L^2(\mathbb{R})} \\ &= \langle \mathcal{F}(f)(\pi_\lambda) \cdot (-4|\lambda|(2m+1)H_{m,\lambda}), H_{n,\lambda} \rangle_{L^2(\mathbb{R})} \\ &= -4|\lambda|(2m+1)\hat{f}(\lambda, n, m).\end{aligned}$$

The Fourier kernel of $\mathbb{H}^1$

Expanding the definition of $\hat{f}(\lambda, n, m)$ yields ($\hat{w} = (\lambda, n, m)$)

$$\hat{f}(\hat{w}) = \int_{\mathbb{H}^1} f(w)\overline{\Theta(w, \hat{w})}dw$$

where Θ satisfies

- $|\Theta(w, \hat{w})| \leq 1$;
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Recasting the abstract properties

Action on convolutions :

$$\widehat{(f_1 * f_2)}(\lambda, n, m) = \sum_{\ell} \hat{f}_1(\lambda, n, \ell)\hat{f}_2(\lambda, \ell, m).$$

Parseval identity :

$$\int_{\mathbb{H}^1} |f(w)|^2 dw = \sum_{n,m} \int_{\mathbb{R}} |\hat{f}(\lambda, n, m)|^2 |\lambda| d\lambda.$$

Inversion formula :

$$f(w) = c \sum_{n,m} \int_{\mathbb{R}} \hat{f}(\hat{w})\Theta(w, \hat{w})|\lambda| d\lambda.$$

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Application : 'resolution' of the Schrödinger equation

Taking the Fourier transform in

$$\begin{cases} i\partial_t u + \Delta_{\mathbb{H}^1} u = 0 \\ u(0) = u_0 \end{cases}$$

gives

$$\begin{cases} i\partial_t \hat{u}(t,\lambda,n,m) - 4|\lambda|(2m+1)\hat{u}(t,\lambda,n,m) = 0 \\ \hat{u}(0,\lambda,n,m) = \hat{u}_0(\lambda,n,m) \end{cases}$$

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Where do the coefficients live ?

We would like to study the coefficient set

$$\widetilde{\mathbb{H}}^1 = (\mathbb{R} \setminus \{0\}) \times \mathbb{N} \times \mathbb{N}.$$

Topology ? Metric ? « Regularity implies decay » ?
Hint : from the identity

$$\widehat{\Delta f}(\lambda, n, m) = -4|\lambda|(2m+1)\hat{f}(\lambda, n, m),$$

smoothness on $\mathbb{H}^1$ implies decay $\sim (|\lambda| m + |\lambda|)^{-p}$.
Another identity :

$$(n-m)\Theta(w, \hat{w}) \sim w \cdot \nabla_w \Theta(w, \hat{w}).$$

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The natural distance on $\widetilde{\mathbb{H}}^1$

We endow $\widetilde{\mathbb{H}}^1$ with the distance

$$\hat{d}(\hat{w}, \hat{w}') := |\lambda - \lambda'| + \left|\lambda m - \lambda' m'\right| + |(n - m) - (n' - m')|.$$

Let's embed $\widetilde{\mathbb{H}}^1$ to euclideanize the distance : $\left(\widetilde{\mathbb{H}}^1, \hat{d}\right)$ is isometric to

$$\widetilde{\mathbb{H}}^1 := \{(\lambda, \lambda m, n - m)\} \subset (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \times \mathbb{Z}$$

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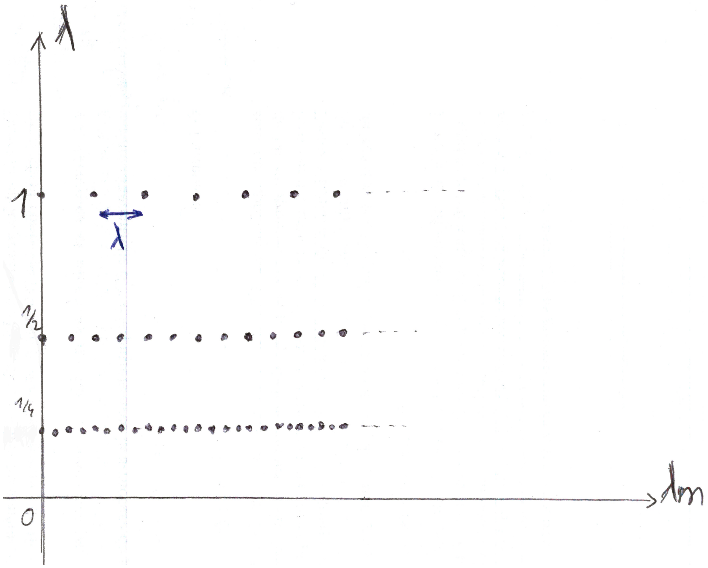
Drawing the Euclidean $\widetilde{\mathbb{H}}^1$



The missing points

The space $\widetilde{\mathbb{H}}^1$ is not complete : the sequence $(\frac{1}{m}, 1, 1)_{m\in\mathbb{N}}$ has its limit outside !

Completion of $\widetilde{\mathbb{H}}^1$:

$$\widehat{\mathbb{H}}^1 := \widetilde{\mathbb{H}}^1 \cup \widehat{\mathbb{H}}^1_0$$

Boundary :

$$\widehat{\mathbb{H}}^1_0 := \{0\} \times \mathbb{R} \times \mathbb{Z}.$$

Note : this incompleteness can *only* be seen through matrix coefficients.
What does it mean ? What do the Fourier transforms become near $\widehat{\mathbb{H}}^1_0$?

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The boundary kernel

On the boundary, the kernel becomes ($\sim$ semiclassical limit)

$$\Theta(w, \lambda, \lambda m, n-m) \xrightarrow[\lambda \to 0]{\lambda m \to z} \mathcal{K}(w, 0, z, n-m)$$

$$\mathcal{K}(w, 0, z, n-m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\left((n-m)\theta + 2|z|^{\frac{1}{2}}(x \sin\theta + y \cos\theta)\right)} d\theta.$$

Example :

$$\hat{f}\left(\frac{1}{m}, m+1, m\right) \xrightarrow[m \to \infty]{} \frac{1}{2\pi} \int_{\mathbb{H}^1} f(w) \left(\int_{-\pi}^{\pi} e^{-i(\theta + 2(x \sin\theta + y \cos\theta)} d\theta\right) dw.$$

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Familiar statements

- Riemann-Lebesgue lemma : the FT is continuous from $L^1(\mathbb{H}^1)$ to $\mathcal{C}_0(\hat{\mathbb{H}}^1)$.
- Schwartz duality : the FT is a bicontinuous isomorphism between $\mathcal{S}(\mathbb{H}^1)$ and $\mathcal{S}(\widehat{\mathbb{H}}^1)$.
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- Look at matrix coefficients $\longrightarrow$ things look much nicer/familiar.
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