Dislocations and Kirchhoff ellipses

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Dislocations

We consider a nonlocal energy

$$I_\alpha(\mu) = \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} W_\alpha(x-y)\,d\mu(x)\,d\mu(y) + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2\,d\mu(x)$$

defined on probability measures $\mu \in \mathcal{P}(\mathbb{R}^2)$, where the interaction potential W_α is given by

$$W_\alpha(x_1,x_2) = -\frac{1}{2}\log(x_1^2+x_2^2)+\alpha\frac{x_1^2}{x_1^2+x_2^2}, \qquad x=(x_1,x_2)\in\mathbb{R}^2 .$$

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The kernel is obtained by adding to the Coulomb potential an anisotropic term weighted by a parameter $\alpha \in \mathbb{R}$.

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Aim: Characterise the minimiser (equilibrium measure)

- Does the minimiser exist? Is it compactly supported?
- Is the minimiser unique?
- Does the minimiser possess any symmetries?
- What is the dimension of its support?
- Can we find the minimiser explicitly?

Some available results

In general one can consider the above problem for a variety of interaction potentials and confinements

- Huge literature on existence, confinement, regularity of minimisers for a variety of potentials and for a variety of applications.

  (e.g. Cañizo, Carrillo, Castorina, Chipot, Choksi, Delgadino, Fetecau, Figalli, Hittmeir, Hoffmann, Huang, Kolokolnikov, Mainini, Mellet, Patacchini, Simione, Slepčev, Sugiyama, Topaloglu, Volzone, Yao, etc...)

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Some available results

Qualitative study of minimisers of nonlocal interaction energies

- typical assumption is radial symmetry of the potential
- dimensionality of the support for potential with subcritical singularity at 0 (Balagué-Carrillo-Laurent-Raoul)
- symmetrisation techniques (Carrillo-Castorina-Volzone)

Dislocations: The planar case

Coming back to our dislocation potential

$$W_\alpha(x) = -\frac{1}{2}\log(x_1^2 + x_2^2) + \alpha\frac{x_1^2}{x_1^2 + x_2^2}$$

When the anisotropy is *switched off*, namely for $\alpha = 0$, the minimiser is radial, and is given by the circle law $\mu_0 := \frac{1}{\pi}\chi_{B_1(0)}$, the normalised characteristic function of the unit disc.

(Coulomb gases, random matrices, Fekete sets, Ginibre ensemble; Sandier-Serfaty, Saff-Totik, Frostman, Wigner, Dyson)

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Aim: Characterise the minimiser (equilibrium measure)

- Does the minimiser exist? Is it compactly supported? YES!
- Is the minimiser unique? YES!
- Does the minimiser possess any symmetries? YES! It's radial.
- What is the dimension of its support? DIM = 2.
- Can we find the minimiser explicitly? YES! $\mu_0 = \frac{1}{\pi}\chi_{B_1(0)}$ (circle law).

Dislocations: The planar case

In the case $\alpha = 1$, The minimisers of I_1 were since long conjectured to be vertical walls of dislocations, and this has been confirmed by [Mora, Rondi, Scardia (2016)]. They proved that the only minimiser of I_1 is the semi-circle law.

$$\mu_1 := \frac{1}{\pi}\delta_0 \otimes \sqrt{2 - x_2^2}\,\mathcal{H}^1 \llcorner (-\sqrt{2}, \sqrt{2}) \tag{1}$$

on the vertical axis.

It is not difficult to prove that μ_1, that is the semicircle law, is the only minimiser of I_α for $\alpha \geq 1$, since
$I_\alpha(\mu_1) = I_1(\mu_1) < I_1(\mu) \leq I_\alpha(\mu) \ \forall \mu \neq \mu_1$.

The case $\alpha < 0$ can be recovered from the knowledge of the case $\alpha > 0$ by switching x_1 and x_2, so we can limit our analysis to $\alpha \in (0, 1)$.

Can we characterise the minimiser of I_α for every $0 < \alpha < 1$?

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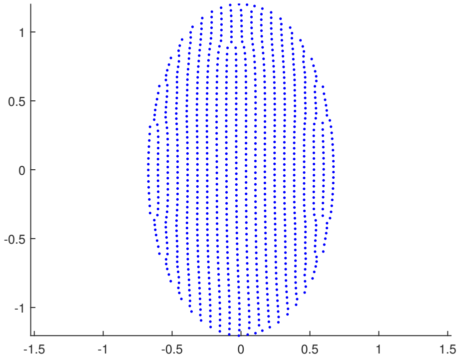

Dislocations: The planar case

Theorem (Carrillo, Mora, M. Rondi, Scardia, Verdera)

Let $0 \leq \alpha < 1$. The measure

$$\mu_\alpha := \frac{1}{\sqrt{1-\alpha^2}\pi} \chi_{\Omega(\sqrt{1-\alpha},\sqrt{1+\alpha})}, \tag{2}$$

where

$$\Omega(\sqrt{1-\alpha},\sqrt{1+\alpha}) := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{1-\alpha} + \frac{x_2^2}{1+\alpha} < 1 \right\},$$

is the unique minimiser of the functional I_α among probability measures $\mathcal{P}(\mathbb{R}^2)$.

Dislocations: The planar case

What is surprising is that techniques developed in the context of fluid mechanics turn out to be crucial for the characterisation of the minimisers of the anisotropic energy I_α.

In particular the minimality of the semi-circle law for the dislocation energy I_1 can be deduced from our result by a limiting argument based on Γ-convergence.

That is, we obtain again the result of [Mora, Rondi, Scardia], but with a different proof based on methods from fluid mechanics and complex analysis

Moreover the case $\alpha = 1$ is in the context of edge dislocations of metals.

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The Euler equation in the plane

In 1755 Euler propose the equation for $v(z,t)$, the velocity field of an incompressible and inviscid fluid

$$(E) \qquad \begin{cases} \partial_t v(z,t) + (v\cdot\nabla)v(z,t) = -\nabla p(z,t) \\ \nabla\cdot v = 0 \\ v(z,0) = v_0(z) \end{cases}$$

$$v\cdot\nabla = v_1\partial_1 + v_2\partial_2$$

The vorticity equation

Taking $curl$ in the 2-D Euler equation we get the vorticity equation also called Helmholtz equation (1858).

$$\begin{cases} \partial_t \omega + (v \cdot \nabla)\omega = 0 \\ v = \dfrac{i}{2\pi} \dfrac{1}{\bar{z}} * \omega \\ \omega(z,0) = \omega_0(z) \end{cases}$$

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Rotating Vortex patches

Definition

A V-state, also called rotating vortex patch, is a vortex patch that rotates with constant angular velocity. If the initial domain D_0 has the origin as center of mass, then $D_t = e^{it\Omega} D_0$ for a certain angular velocity Ω.

The two known explicit examples

Rankine vortex (1858)

If $\Omega = D(0,1)$ is the unit disc, then

$$\Omega_t = D(0,1), \quad 0 < t.$$

$\chi_{D(0,1)}(z)$ is a steady solution to the vorticity equation

Kirchhoff vortex (1876)

If $\Omega = \{(x,y) : x^2/a^2 + y^2/b^2 = 1\}$ is an ellipse then

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Proof of the Theorem

In relation with the existence and uniqueness we have

Proposition

Let $\alpha \in [0,1]$. Then the energy I_α is well defined on $\mathcal{P}(\mathbb{R}^2)$, is strictly convex on the class of measures with compact support and finite interaction energy, and has a unique minimiser in $\mathcal{P}(\mathbb{R}^2)$. Moreover, the minimiser has compact support and finite energy.

In fact the key point for the uniqueness is that the Fourier transform of our kernel never vanishes

$$\langle \hat{W}_\alpha, \varphi \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(1-\alpha)\xi_1^2 + (1+\alpha)\xi_2^2}{|\xi|^4} \varphi(\xi)\, d\xi$$

for every φ in the Schwarz class.

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To obtain the characterisation of the mimimisers we use that standard computations in potential theory shows that any minimiser μ of I_α must satisfy the following Euler-Lagrange conditions

$$(W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} = C_\alpha \quad \text{for every } x \in \Omega(\sqrt{1-\alpha}, \sqrt{1+\alpha}), \tag{3}$$

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with

$$C_\alpha = 2I_\alpha(\mu_\alpha) - \frac{1}{2}\int_{\mathbb{R}^2} |x|^2 \, d\mu_\alpha(x) =$$
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To prove that the ellipse law μ_α satisfies the Euler-Lagrange conditions, we evaluate explicitly the gradient of the convolution of the kernel W_α with the characteristic function of the domain enclosed by a general ellipse

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In fact, we compute *explicitly* the gradient of $W_\alpha * \chi_{\Omega(a,b)}$, both inside and outside $\Omega(a,b)$; and this is enough to check the Euler-Lagrange equations and to conclude the proof of Theorem.

$$\nabla(W_\alpha * \mu_\alpha)(x) + x = 0 \quad \text{for every } x \in \Omega(\sqrt{1-\alpha}, \sqrt{1+\alpha}),$$
$$x \cdot \nabla(W_\alpha * \mu_\alpha)(x) + |x|^2 \geq 0 \quad \text{for every } x \in \mathbb{R}^2.$$

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In order to evaluate the convolution $\nabla W_\alpha * \chi_{\Omega(a,b)}$, it is convenient to work in complex variables.

In complex variables the potential W_α reads as

$$W_\alpha(x) \equiv W_\alpha(z) = -\frac{1}{2}\log(z\bar{z}) + \frac{\alpha}{2}\left(1 + \frac{z}{2\bar{z}} + \frac{\bar{z}}{2z}\right),$$

and its gradient

$$\nabla W_\alpha(x) = -\frac{x}{|x|^2} + 2\alpha\frac{x_1x_2}{|x|^4}x^\perp \equiv 2\bar{\partial}W_\alpha(z) = -\frac{1}{\bar{z}} + \frac{\alpha}{2}\frac{1}{z} - \frac{\alpha}{2}\frac{z}{\bar{z}^2}, \tag{5}$$

where $x^\perp = (x_2, -x_1)$.

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Proof of the Theorem

Let $b \geq a > 0$ and $\mu_{a,b} := \frac{1}{\pi ab}\,\chi_{\Omega(a,b)}$ be the (normalised) characteristic function of the ellipse of semi-axes a and b.

We compute

$\frac{1}{z} * \chi_{\Omega(a,b)} = z - \lambda\bar{z}$ inside Ω, being $\lambda = \frac{a-b}{a+b}$

By taking the conjugate we obtain directly

$\frac{1}{\bar{z}} * \chi_{\Omega(a,b)} = \bar{z} - \lambda z$ inside Ω.

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Using the above computations inside and the fact that all the potential are linear inside one obtains

$$\begin{cases} -1 - \alpha\lambda + ab = 0, \\ \lambda + \dfrac{\alpha}{2} + \lambda^2 \dfrac{\alpha}{2} = 0, \end{cases}$$

Now, it is easy to check that $a = \sqrt{1-\alpha}$ and $b = \sqrt{1+\alpha}$ are the unique solution of the system.

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Proof of the Theorem

V-states or rotating vortex patches can be viewed as stationary solutions in a reference system that rotates with the patch, and they can be described by means of an equation involving the stream function of the initial patch D_0

$$-\log|\cdot| * \chi_{D_0} + \Omega|z|^2 = C, \quad \text{on the boundary of } D_0,$$

where Ω is the angular velocity of the patch and C is a constant.

Proof of the Theorem

which is formally similar to the first Euler-Lagrange equation

$$(W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} = C_\alpha \quad \text{for every } x \in \Omega(\sqrt{1-\alpha}, \sqrt{1+\alpha}).$$

We consider a nonlocal energy

$$I_\alpha(\mu) = \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} W_\alpha(x-y)\, d\mu(x)\, d\mu(y) + \frac{1}{2} \int_{\mathbb{R}^3} |x|^2\, d\mu(x) \quad (6)$$

defined on probability measures $\mu \in \mathcal{P}(\mathbb{R}^3)$, where the interaction potential W_α is given by

$$W_\alpha(x_1, x_2, x_3) = \frac{1}{|x|} + \alpha \frac{x_1^2}{|x|^3}, \qquad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad (7)$$

Dislocations: The 3-D case

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Again the problem is to describe the minimisers of the above energy.

The kernel is obtained by adding to the 3 dimensional Coulomb potential an anisotropic term weighted by a parameter $\alpha \in \mathbb{R}$.

In the particular case where $\alpha = 0$, the minimiser is radial, and is given by $\mu_0 := \frac{3}{4\pi}\chi_{B_1(0)}$, the normalised characteristic function of the unit ball.

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Dislocations: The 3-D case

Theorem

Let $-1 < \alpha < 1$. *There exist constants* $a(\alpha)$ *and* $b(\alpha)$ *such that the measure*

$$\mu_\alpha := \frac{3}{ab^2 4\pi}\chi_{\Omega(a,b,b)}, \tag{8}$$

where

$$\Omega(a,b,b) := \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{b^2} < 1 \right\},$$

is the unique minimiser of the functional I_α *among probability measures* $\mathcal{P}(\mathbb{R}^3)$.

Dislocations: 3D

The minimizer is given by the probability measure of the characteristic functions of an oblate ellipsoid ($0 < \alpha < 1$), the characteristic function of a ball ($\alpha = 0$) or the characteristic function of a prolate ellipsoid ($-1 < \alpha < 0$).

The computations are much more involved and we can not use some of the advantages of the complex numbers.

The idea is again to check the Euler-Lagrage Equations:

$$(W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} = C_\alpha \quad \text{for every } x \in \Omega(a,b,b), \tag{9}$$

$$(W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} \geq C_\alpha \quad \text{for every } x \in \mathbb{R}^3, \tag{10}$$

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To do that, one has to compute explicitly the kernel associated to this energy inside and outside our ellipsoid.

Then it is better to use apropiate coordenates, that is the oblate espheroidal coordinates or the prolate spheroidal coordinates.

This solution, when α goes to 1 or to -1, converges to the characteristic function of an oblate or a prolate ellipsoid.

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Open problems

We would like to consider the problem of dislocations using the Coulomb potentials, the same confinement, but replacing the anisotropic term.

For instance, let's consider the nonlocal energy

$$I_\alpha(\mu) = \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} W_\alpha(x-y)\,d\mu(x)\,d\mu(y) + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2\,d\mu(x)$$

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Open problems

In general we can consider an anisotropic case being

$$\frac{x_1^{2n}}{(x_1^2 + x_2^2)^n}$$

For $n = 2$ the candidates are a family of ellipses.

This is due to the fact that some even Calderon-Zygmund integrals of the characteristic funtion of an ellipse are constants inside the ellipses.

The key point is to prove that the second Euler-Lagrange condition is satisfied.

It means how can we go from a condition inside to a condition outside ?

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Thank you for your attention