On weak Mukenhoupt-Wheeden conjecture

Sheldy Ombrosi
Universidad Nacional del Sur

based in joint works with A. Lerner, F. Nazarov; and C. Pérez

March 26, 2018

The Hardy-Littlewood operator

The Hardy-Littlewood maximal operator on $\mathbb{R}$ is defined by

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy,$$

A weight w is a non-negative locally integrable function on $\mathbb{R}$, $w(E) = \int_E w$ for a measurable set $E \subset \mathbb{R}$.

C. Fefferman and E. Stein (1971) proved that there exists an absolute constant $C > 0$ such that for every weight w,

$$\sup_{\lambda > 0} \lambda w\{x \in \mathbb{R} : Mf(x) > \lambda\} \le C \int_{\mathbb{R}} |f(x)| Mw(x) dx$$

In particular, if we define the $[w]_{A_1}$ constant of the weight w by $[w]_{A_1} = \|Mw/w\|_{L^\infty}$, then (since $Mw(x) \le [w]_{A_1} w(x)$ a.e.)

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Calderón-Zygmund operators

The Hilbert transform:

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)dy.$$

It's bounded on L^p

$$\|Hf\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}$$

and of weak type (1,1)

$$|\{x \in \mathbb{R} : |Hf(x)| > t\}| \leq \frac{C}{t} \int_{\mathbb{R}} |f(x)|dx.$$

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Definition

A Calderón-Zygmund operator T (CZO) is an operator bounded on $L^2(\mathbb{R}^n)$ that admits the following representation

$$Tf(x) = \int K(x,y)f(y)dy$$

with $f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ and $x \notin \text{supp } f$ and where $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \longrightarrow \mathbb{R}$ has the following properties

Size condition: $|K(x,y)| \leq C_2 \frac{1}{|x-y|^n} \qquad x \neq 0.$

Smoothness condition (Hölder-Lipschitz):

$$|K(x,y) - K(x,z)| \leq C_1 \frac{|y-z|^\delta}{|x-y|^{n+\delta}} \qquad \frac{1}{2}|x-y| > |y-z|$$

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if and only if $w \in A_1$.
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The Conjectures

Muckenhoupt-Wheeden conjecture

Let w an arbitrary weight then

$$w(\{x \in \mathbb{R} : |Hf(x)| > \lambda\}) \lesssim \frac{1}{\lambda} \int_{\mathbb{R}} |f(x)| Mw(x) dx$$

where H stands for the Hilbert transform.

Weak Muckenhoupt-Wheeden conjecture

Let w an A_1 weight and H the Hilbert transform, then

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Orlicz averages

Definition

Let $\Phi : [0, \infty) \longrightarrow (0, \infty)$ be a Young function, i.e. a convex, increasing function such that $\Phi(0) = 0$,

$$\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 ; \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

If $\Phi(t) = t^r$

$$\|f\|_{\Phi,Q} = \left(\frac{1}{|Q|} \int_Q |f(x)|^r dx \right)^{\frac{1}{r}}.$$

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Orlicz maximal functions

Definition

We define the maximal operator associated to Φ as

$$M_\Phi f(x) = \sup_{x\in Q} \|f\|_{\Phi,Q}.$$

Some important particular cases

$$M_{L(\log L)^\alpha} f(x) = \sup_{x\in Q} \|f\|_{\Phi,Q}$$

where $\Phi(t) = t(1+\log^+ t)^\alpha \qquad \alpha > 0.$

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Let Φ and Ψ Young functions. If there exists $c > 0$ such that $\Phi(t) \leq \Psi(t) \qquad t > c$ then

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Open question

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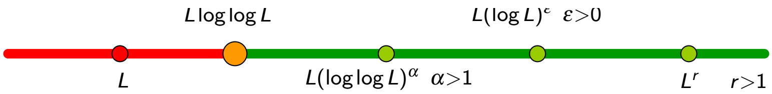

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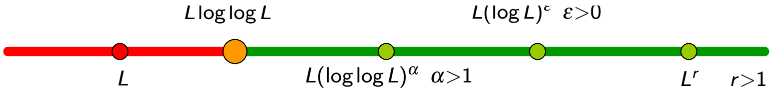

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- From well known results about weight theory (up to 2005) was simple to obtain for $\varepsilon > 0$

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Sharpness...

For $t \geq 1$, define

$$\varphi_H(t) = \sup_{[w]_{A_1} \leq t} \|H\|_{L^1(w) \to L^{1,\infty}(w)}.$$

We have that

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An overview of the proof

First step: show that the definition of φ_H along with the standard extrapolation and dualization arguments yields

$$\|H(w\chi_{[0,1)})\|_{L^2(\sigma)} \lesssim \varphi_H(\|M\|_{L^2(\sigma)\to L^2(\sigma)})\Big(\int_0^1 w\Big)^{1/2},$$

where $\sigma = w^{-1}$. $w \in A_2$, if

$$[w]_{A_2} = \sup_{I\subset\mathbb{R}} \frac{w(I)\sigma(I)}{|I|^2} < \infty$$

Second: for every $t \gg 1$, there exists an A_2 weight $w_t = w$ satisfying

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Rubio de Francia extrapolation trick

Given $g \geq 0$ with $\|g\|_{L^2(\sigma)} = 1$, define

$$\mathscr{R}g(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})^k}.$$

$$\begin{aligned} \alpha \int_{\{x:|Hf(x)|>\alpha\}} g &\leq \alpha \int_{\{x:|Hf(x)|>\alpha\}} \mathscr{R}g \leq \varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^1(\mathscr{R}g)} \\ &\leq \varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^2(w)}\|\mathscr{R}g\|_{L^2(\sigma)} \\ &\leq 2\varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^2(w)}. \end{aligned}$$

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Construction of the weight

Fix $t \gg 1$. Take $k \in \mathbb{N}$ such that $t \le 3^k \le 3t$. Let $\varepsilon = 3^{-k}$ and $p = \frac{1}{3\varepsilon}\left(\frac{1+\varepsilon}{2} + \frac{4\varepsilon^2}{1+\varepsilon}\right)$.
$p \sim \frac{1}{\varepsilon} \sim t \sim 3^k$.

For every two positive numbers ω and σ such that $\omega\sigma = p$ and any interval $I \subset \mathbb{R}$, we define inductively the sequence of weights $w_n = w_n(\omega, \sigma, I)$ $(n = 0, 1, 2, \ldots)$ supported on I...

Let $u = \sqrt{p} + \sqrt{p-1}$ be the larger root of $u + \frac{1}{u} = 2\sqrt{p}$. Define

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Suppose that $w_{n-1}(\omega, \sigma, I)$ is already defined for all ω, σ with $\omega\sigma = p$ and all $I \subset \mathbb{R}$.

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$$w_0(\omega, \sigma, I) = \frac{\omega}{\sqrt{p}}\left(u\chi_{I_-} + \frac{1}{u}\chi_{I_+}\right),$$

where I_- and I_+ are the left and the right halves of I respectively.

Suppose that $w_{n-1}(\omega, \sigma, I)$ is already defined for all ω, σ with $\omega\sigma = p$ and all $I \subset \mathbb{R}$.

Construction of the weight

Fix $t \gg 1$. Take $k \in \mathbb{N}$ such that $t \leq 3^k \leq 3t$. Let $\varepsilon = 3^{-k}$ and $p = \frac{1}{3\varepsilon}\left(\frac{1+\varepsilon}{2} + \frac{4\varepsilon^2}{1+\varepsilon}\right)$.
$p \sim \frac{1}{\varepsilon} \sim t \sim 3^k$.

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$$I_{k-1}\subset I_{k-2}\subset\cdots\subset I_0=I$$

and $|I_{k-1}|=3\varepsilon|I|$.
Given an interval J, denote by $J^{(i)}, i=1,2,3$, the i-th from the left subinterval of J in the partition of J into 3 equal intervals.
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$$\begin{aligned} w_n(\omega,\sigma,I) &= \frac{\omega}{p}\left(\sum_{m=0}^{k-2}\chi_{I_m^{(1)}}+\chi_{I_{k-1}^{(1)}\cup I_{k-1}^{(2)}}+\frac{4\varepsilon}{1+\varepsilon}\chi_{I_{k-1}^{(3)}}\right) \\ &+ \sum_{m=0}^{k-2} w_{n-1}\left(2\omega,\frac{\sigma}{2},I_m^{(2)}\right). \end{aligned}$$

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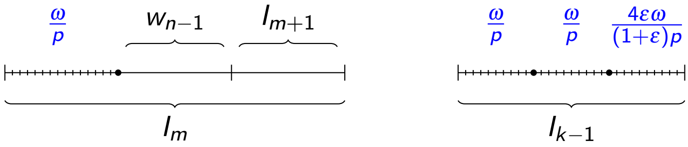
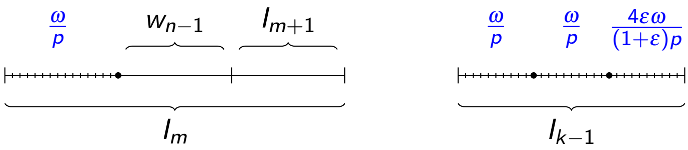


Figure: $w_n(\omega,\sigma,I)$ on intervals $I_m^{(i)}$ for $i=1,2$ and $0\le m\le k-2$ and on $I_{k-1}^{(i)}$ for $i=1,2,3$.

Proposition. For every $n\ge 0$,

$$\frac{1}{|I|}\int_I w_n(\omega,\sigma,I)dx=\omega \quad \text{and} \quad \frac{1}{|I|}\int_I w_n^{-1}(\omega,\sigma,I)dx=\sigma.$$



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Estimate of the maximal operator

Let $\mathscr{T}$ be the standard triadic lattice, that is,

$$\mathscr{T} = \{[3^j n, 3^j(n+1)), \quad j, n \in \mathbb{Z}\}.$$

Denote by $\mathscr{J}$ the family of all unions of two adjacent triadic intervals of equal length.
First:

$$\|M\|_{L^2(\sigma)\to L^2(\sigma)} \le 24 \sup_{J\in\mathscr{J}} \left(\frac{1}{w(J)}\int_J (M(w\chi_J))^2\sigma\right)^{1/2}.$$

Second: for every triadic interval J, we have

$$\left(\frac{1}{w(J)}\int_J (M(w\chi_J))^2\sigma\right)^{1/2} \lesssim p \sim t$$

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In other words, A_l^* is the union of all intervals $\frac{1}{2}J_{k-1}^{(3)}$ where $J \subset [0,1)$ carries w_{n-l}.
The sets A_l^* plays the central role in establishing the lower bound for $H(w\chi_{[0,1)})$, as the following proposition shows.

Proposition
There exists an absolute $C > 0$ such that for for all $l = 0, \ldots, n-1$ and for every $x \in A_l^*$

$$|H(w\chi_{[0,1)})(x)| \geq C\, k\, M(w\chi_{[0,1)})(x)$$

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