DEGENERATE POINCARÉ INEQUALITIES

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&

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CNRS-Paul Langevin Center

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Sheldy Ombrosi, Ezequiel Rela and Israel Rios-Rivera

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It goes back to Sobolev.

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- One of the points of this talk is to show how to **avoid** such representation formulae.
- We will use **Calderón-Zygmund** theory instead.

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• Chanillo-Wheeden, Franchi-Lu-Wheeden, Chua-Wheeden

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and there is gain for some δ:

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- Due to Fabes-Kenig-Serapioni.
- Method of proof is by fractional integrals.

Since there have been a lot of variants of these results:

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• First example of this property is due to L. Saloff-Coste.

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- Then we can define for a given a the optimal exponent

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- If $\alpha = m = 1, 2 \cdots$, $r = \frac{mp}{n-pm}$, the Sobolev exponent related to higher order PI.
- If $\nu \equiv 1$, a satisfies the D_r condition for every $r > 1$.

Variants of the D_r condition

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The (weighted) $D_r(w)$ condition for some $0 < r < \infty$: for each cube Q and for any family $\{Q_i\}$ of pairwise **disjoint** cubes contained in Q,

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Let $a \in D_r(w)$ for some $r > 0$ and let $w \in A_\infty$. Let f such that

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- Can be extended to the context of a space of homogeneous type.

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Definition 1 Let $L > 1$ and let Q be a cube. We will say that a family of pairwise disjoint subcubes $\{Q_i\}$ of Q is L-small if

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Definition 1
Let w be any weight and let $s > 1$. We say that the functional a satisfies the weighted $SD_p^s(w)$ condition for $0 < p < \infty$ if there is a constant c such that for any cube Q and any family $\{Q_i\}$ of pairwise disjoint subcubes of Q such that $\{Q_i\} \in S(L)$, the following inequality holds:

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A first result

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$$\frac{1}{|Q|}\int_Q |f-f_Q| \leq a(Q) \qquad (H)$$

for every cube Q. Then, there exists a dimensional constant c_n such that for any cube Q
Then

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^p\, w dx\right)^{\frac{1}{p}} \leq s\, c_n\, a(Q)$$

Corollary
Let $(u,v)\in A_p$. The the following Poincaré (p,p) inequality holds

$$\left(\frac{1}{u(Q)}\int_Q |f-f_Q|^p\, u\, dx\right)^{1/p} \leq c_n [u,v]_{A_p}^{\frac{1}{p}} \ell(Q)\left(\frac{1}{u(Q)}\int_Q |\nabla f|^p\, v\, dx\right)^{1/p},$$

where c_n is a dimensional constant.

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Corollary **(The Keith-Zhong phenomenon)**
Let $1 < p_0$ and let (f, g) be a couple of functions satisfying

$$\frac{1}{|Q|} \int_Q |f - f_Q| \, dx \leq C_{[w]_{A_{p_0}}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q g^{p_0} \, w dx \right)^{\frac{1}{p_0}} \qquad w \in A_{p_0}$$

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Then, for any $1 \leq p < p_0$, the following estimate holds for any $w \in A_p$

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|\, w dx\right)^{1/p} \leq c\, C_{[w]_{A_p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q g^p\, w dx\right)^{1/p}$$

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- Again, this is a very "clean" inequality.

Poincaré-Sobolev via Good-λ

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Let f be a locally integrable function and let w be a weight such that

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