Wave packet decompositions adapted to (non-self-adjoint) operators

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Outline

Wave packet decompositions

Our goal is to recall some techniques in "wave packet decompositions"

$$Wf(\rho) = \langle f(x), \varphi_\rho(x) \rangle$$

such as the Bargmann transform and to describe how one can adapt the decomposition to the operator investigated, in particular for quadratic non-self-adjoint operators.

Wave packet decompositions

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such as the Bargmann transform and to describe how one can adapt the decomposition to the operator investigated, in particular for quadratic non-self-adjoint operators.
Example: for the Bargmann transform, the family $\{\varphi_\rho\}_{\rho \in \mathbb{C}}$ is made up of phase-space translations of the Gaussian $\varphi_0(x) = \mathrm{e}^{-\pi x^2}$.

Application to non-self-adjoint operators

If $\theta \in (-\pi/2, \pi/2)$ and

$$Q_\theta = \pi \left(e^{i\theta} x^2 - e^{-i\theta} \frac{1}{4\pi^2} \frac{d^2}{dx^2} \right),$$

identify the eigenfunctions $u_{k,\theta}$ and the L^2-operator norm of the spectral projections

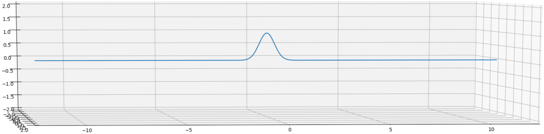
$$\frac{1}{k} \log \|\Pi_{k,\theta}\| \sim \left(\frac{1 + |\sin\theta|}{1 - |\sin\theta|} \right)^{1/2}, \quad \theta \neq 0.$$

[Davies, Kuijlaars 2004; Bagarello, 2010; V. 2013]

Phase-space translations

We know how to translate a function by x_0 in physical space:

$$\mathcal{T}_{(x_0,0)}f(x) = f(x - x_0).$$



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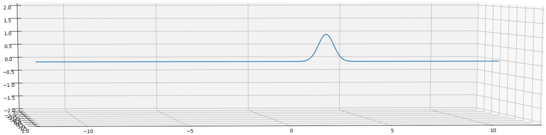

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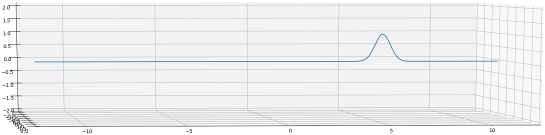

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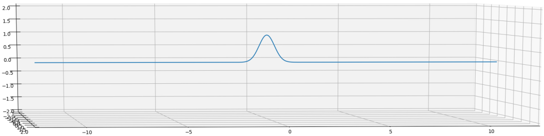

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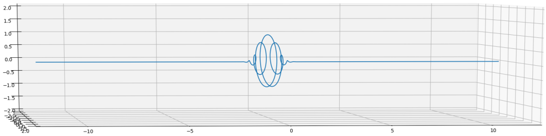

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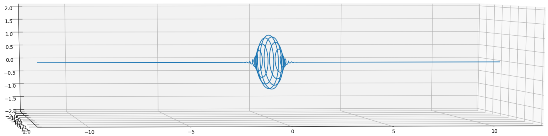

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To do both simultaneously, there's a natural correction factor.

$$\mathcal{T}_{(x_0,\xi_0)}f(x) = \mathrm{e}^{-\pi \mathrm{i} x_0\xi_0 + 2\pi \mathrm{i}\xi_0 x}f(x - x_0).$$

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$$D_t \mathrm{e}^{2\pi\mathrm{i}t\xi_0 x}f(x) = \xi_0 x \mathrm{e}^{2\pi\mathrm{i}t\xi_0 x}f(x).$$

Moreover, if $D_x = \frac{1}{2\pi\mathrm{i}}\partial_x$ as well,

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We'd actually like $\{\mathcal{T}_{t(x_0,\xi_0)}\}_{t\in\mathbb{R}}$ to be the group

$$\mathcal{T}_{t(x_0,\xi_0)} = \exp(2\pi\mathrm{i}t(\xi_0 x - x_0 D_x)).$$

Composition law

One can compute directly that, for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{2n}$ and

$$\sigma((v_x, v_\xi), (w_x, w_\xi)) = v_\xi w_x - w_\xi v_x,$$

the shifts obey

$$\mathcal{T}_{\mathbf{v}} \mathcal{T}_{\mathbf{w}} = \mathrm{e}^{\pi \mathrm{i} \sigma(\mathbf{v}, \mathbf{w})} \mathcal{T}_{\mathbf{v}+\mathbf{w}}.$$

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Remark: the sympletic product also appears in

$$\mathcal{T}_{\mathbf{v}} = \exp\left(2\pi \mathrm{i} \sigma(\mathbf{v}, (x, D_x))\right)$$

What the correction factor is good for

Let

$$\mathcal{F}f(x) = \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}} \int \mathrm{e}^{-2\pi \mathrm{i} xy} f(y)\, \mathrm{d}y$$

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We know that $\mathcal{F}f(\cdot - x_0)(x) = \mathrm{e}^{-2\pi \mathrm{i} x_0 x}\mathcal{F}f(x)$ and that $\mathcal{F}\mathrm{e}^{2\pi \mathrm{i}\xi_0 \cdot} f(\cdot)(x) = \mathcal{F}f(x - \xi_0)$. But the two together…

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We have the general rule

$$\mathcal{F}\mathcal{T}_{(x_0,\xi_0)} = \mathcal{T}_{(\xi_0,-x_0)}\mathcal{F},$$

with no constants.

A more complicated example

A more involved computation comes from tracking the quantum Schrödinger evolution

$$e^{-\mathrm{i}tQ_0}\varphi_0(x - x_0)$$

when

$$\begin{aligned} \varphi_0(x) &= e^{-\pi x^2}, \\ Q_0 &= \pi(D_x^2 + x^2). \end{aligned}$$

Ansatz and ODEs

We could guess that $\mathrm{e}^{-\mathrm{i}tQ_0}\varphi_0(x-x_0)$ should take the form

$$\mathrm{e}^{-\mathrm{i}tQ_0}\varphi_0(x-x_0) = \mathrm{e}^{-\pi(x^2+a(t)x+b(t))}$$

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for $a(0)=-2x_0$ and $b_0=x_0^2$.
We can obtain ODEs for a' and b' which give us

$$a(t)=-2\mathrm{e}^{-\mathrm{i}t}x_0,\quad b(t)=x_0^2\mathrm{e}^{-\mathrm{i}t}\cos t+\frac{\mathrm{i}t}{2\pi}.$$

Using Egorov

Instead, we can use that $\varphi_0(x)$ is chosen such that

$$Q_0\varphi_0(x) = \pi\left(-\frac{1}{(2\pi)^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2\right)\mathrm{e}^{-\pi x^2} = \frac{1}{2}\varphi_0(x)$$

and that $\mathrm{e}^{-\mathrm{i}tQ_0}$ follows a rule for shifts like that of $\mathcal{F}$:

$$\mathrm{e}^{-\mathrm{i}tQ_0}\mathcal{T}_{\mathbf{v}} = \mathcal{T}_{\mathbf{F}^t\mathbf{v}}\mathrm{e}^{-\mathrm{i}tQ_0}, \quad \mathbf{F}^t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

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Therefore

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}tQ_0}\mathcal{T}_{(x_0,0)}\varphi_0(x) &= \mathcal{T}_{x_0(\cos t, -\sin t)}\mathrm{e}^{-\mathrm{i}tQ_0}\varphi_0(x) \\ &= \mathrm{e}^{-\frac{\mathrm{i}t}{2}}\mathcal{T}_{x_0(\cos t, -\sin t)}\varphi_0(x).\end{aligned}$$

Metaplectic operators

There are many such operators $\mathcal{K}$, unitary on $L^2(\mathbb{R}^n)$, associated with linear transformations $\mathbf{K}$ such that

$$\mathcal{K}\mathcal{T}_{\mathbf{v}} = \mathcal{T}_{\mathbf{K}\mathbf{v}}\mathcal{K}.$$

Generators of this set are

- The Fourier transform $\mathcal{F}_1$ in x_1 associated with $\mathbf{F}_1(x_1, x', \xi_1, \xi') = (\xi_1, x', -x_1, \xi')$.
- A linear change of variables $\mathcal{V}_G f(x) = (\det G)^{1/2} f(Gx)$ is associated with $\mathbf{V}_G(x, \xi) = (G^{-1}x, G^\top \xi)$.
- The multiplication operator $\mathcal{W}_A f(x) = \mathrm{e}^{\pi \mathrm{i} x \cdot Ax} f(x)$, where A is symmetric, is associated with $\mathbf{W}_A(x, \xi) = (x, \xi + Ax)$.

More on metaplectic operators

The (linear) transformation $\mathbf{K}$ is canonical (preserves σ) or

$$\mathbf{K} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathbf{K}^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.$$

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If $\det B \neq 0$,

$$\mathcal{K}f(x) = \pm(\det(-\mathrm{i}B))^{-1/2} \int \mathrm{e}^{-\pi\mathrm{i}\left(x\cdot B^{-1}Dx-2x\cdot B^{-1}y+y\cdot B^{-1}Ay\right)} f(y)\,\mathrm{d}y.$$

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These operators are also generated by $\exp(-\mathrm{i}tq^w)$ for $q(x,\xi)$ and q^w defined on the next slide.

A quantization respecting the metaplectic group

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The *Weyl quantization* takes the Fourier inversion formula

$$a(x,\xi) = \int \mathrm{e}^{2\pi \mathrm{i}(x,\xi)\cdot(x^*,\xi^*)}\hat{a}(x^*,\xi^*)\,\mathrm{d}x^*\mathrm{d}\xi^*$$

and replaces (x,ξ) with (x,D_x). Since

$$\mathrm{e}^{2\pi \mathrm{i}(x^*,\xi^*)\cdot(x,D_x)} = \mathcal{T}_{(-\xi^*,x^*)},$$

we write

$$a^w(x,D_x) = \int \hat{a}(x^*,\xi^*)\mathcal{T}_{(-\xi^*,x^*)}\,\mathrm{d}x^*\mathrm{d}\xi^*$$

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"Egorov theorem:"

$$\begin{aligned}\mathcal{K}a^w &= \int \hat{a}(x^*,\xi^*)\mathcal{T}_{\mathbf{K}(-\xi^*,x^*)}\,\mathrm{d}x^*\mathrm{d}\xi^* \\ &= (a\circ\mathbf{K}^{-1})^w\mathcal{K}\end{aligned}$$

Explicit computations[1] and the integral kernel

We can obtain an integral kernel for $a^w(x, D_x)u(x) =$

$$\iint e^{-2\pi i(x^*,\xi^*)(x_*,\xi_*)} a(x_*, \xi_*) e^{\pi i x^* \xi^* + 2\pi i x^* x} u(x + \xi^*) \, dx^* \, d\xi^* \, dx_* d\xi_*.$$

Upon making the change of variables $\xi^* + x \to \xi^*$, the exponent becomes

$$2\pi i x^* (\frac{x + \xi^*}{2} - x_*) + 2\pi i (x - \xi^*) \xi_*.$$

We traditionally write (y, ξ) instead of (ξ^*, ξ_*); Fourier inversion in x^*, x^* gives

$$a^w(x, D_x)u(x) = \int e^{2\pi i (x-y)\cdot\xi} a(\frac{x + y}{2}, \xi) u(y) \, dy \, d\xi.$$

[1] not to be read seriously

For polynomials

Most concretely, $x^\alpha D_x^\beta$ can be obtained by expanding $(\frac{x+y}{2})^\alpha \xi^\beta$ and using $x^{\alpha_1} D_x^\beta y^{\alpha_2} \to x^{\alpha_1} D_x^\beta x^{\alpha_2}$:

$$x\xi \to \frac{1}{2}(xD_x + D_x x)$$

and

$$x^3\xi^2 \to \frac{1}{8}(x^3D_x^2 + 3x^2D_x^2x + 3xD_x^2x^2 + D_x^3x^2)$$

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As an integral kernel for a metaplectic operator

The general format for a metaplectic operator quantizing $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, if $b \neq 0$, is

$$\mathcal{M}f(x) = \frac{1}{\sqrt{\mathrm{i}b}} \int \mathrm{e}^{\frac{\pi \mathrm{i}}{b}(dx^2 - 2xy + ay^2)} f(y)\, \mathrm{d}y.$$

(Reminder: $\mathcal{F}$ corresponds to $a = d = 0$ and $b = -c = 1$.)

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The Bargmann transform

$$\mathfrak{B}f(x) = 2^{1/4} \int \mathrm{e}^{-\frac{\pi}{2}x^2 + 2\pi xy - \pi y^2} f(y)\,\mathrm{d}y$$

corresponds to $\mathbf{B} = \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i}/2 & 1/2 \end{pmatrix}$.

Consequences of the Egorov theorem

Writing $q_0(x,\xi) = \pi(x^2+\xi^2)$, we expect $\mathfrak{B}Q_0\mathfrak{B}^*$ to be the quantization of

$$(q_0 \circ \mathbf{B}^{-1})(x,\xi) = \pi\left((x/2+\mathrm{i}\xi)^2 + (\mathrm{i}x/2+\xi)^2\right) \\ = 2\pi\mathrm{i}x\xi.$$

And it is true that

$$\mathfrak{Q}_0 := \mathfrak{B}Q_0\mathfrak{B}^* = \frac{2\pi\mathrm{i}}{2}(xD_x + D_x x) = x\partial_x + \frac{1}{2}.$$

Formal consequences of $\mathfrak{Q}_0 = x \cdot \partial_x + \frac{1}{2}$

We obtain the "Hermite functions"

$$\mathfrak{Q}_0 f(x) = (k + \frac{1}{2}) f(x) \iff f(x) = Cx^k$$

and the Schrödinger evolution $\mathrm{e}^{-\mathrm{i}t\mathfrak{Q}_0}$

$$(\mathrm{i}\partial_t - \mathfrak{Q}_0) F(t,x) = 0 \iff F(t,x) = \mathrm{e}^{-\mathrm{i}t/2} F(0, \mathrm{e}^{-\mathrm{i}t} x)$$

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But:

- The transformation **B** is complex,
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This is solved by saying $\mathfrak{B}f$ is *holomorphic* in a *weighted* space:

$$\|f\|_{L^2(\mathbb{R})} = \|\mathrm{e}^{-\frac{\pi}{2}|x|^2} \mathfrak{B}f(x)\|_{L^2(\mathbb{C})}.$$

Other points of view

We can also view the Bargmann transform as the wave packet decomposition: if $x = x_0 + \mathrm{i}\xi_0$,

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We can also formally view the Bargmann transform as the Schrödinger evolution

$$\mathfrak{B}f(x) = e^{\frac{\pi}{4}(x^2 - D_x^2)}.$$

Some computations[2] around the ground state

If $\varphi_0(x) = 2^{1/4}\mathrm{e}^{-\pi x^2}$ is a normalized Gaussian,

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Note that $(x\partial_x + \frac{1}{2})1 = \frac{1}{2}1$ and

$$\|\mathrm{e}^{-\frac{\pi}{2}|x|^2}1\|_{L^2(\mathbb{C})} = \int \mathrm{e}^{-\pi(x_1^2+x_2^2)}\,\mathrm{d}x_1\mathrm{d}x_2 = 1 = \|\varphi_0\|_{L^2(\mathbb{R})}.$$

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Some computations[3] around the Hermite functions

To normalize the Hermite functions, we note that

$$\begin{aligned}\langle x^j e^{-\frac{\pi}{2}|x|^2}, x^k e^{-\frac{\pi}{2}|x|^2}\rangle &= \int x^j \bar{x}^k e^{-\pi|x|^2}\, dx_1\, dx_2 \\ &= \int_0^{2\pi} e^{i(j-k)\theta}\, d\theta \int r^{j+k+1} e^{-\pi r^2} dr \\ &= \frac{k!}{\pi^k}\delta(j-k).\end{aligned}$$

So the real-side Hermite functions $\{h_k\}$ are

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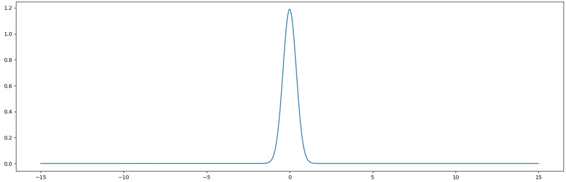
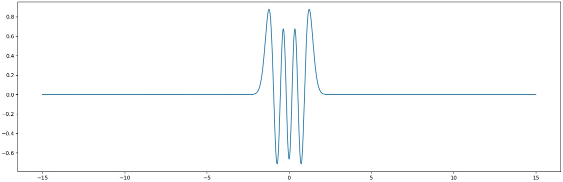
Visualizing the Hermite functions

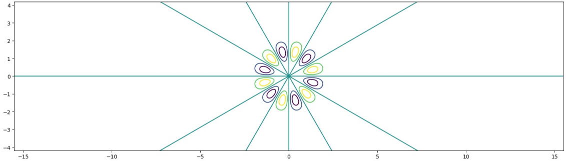
The first Hermite function is a Gaussian:



Visualizing the Hermite functions

We compare the Hermite function to $\Im e^{-\frac{\pi}{2}|x|^2}\mathfrak{B}h_k(x)$ when $k = 6$:





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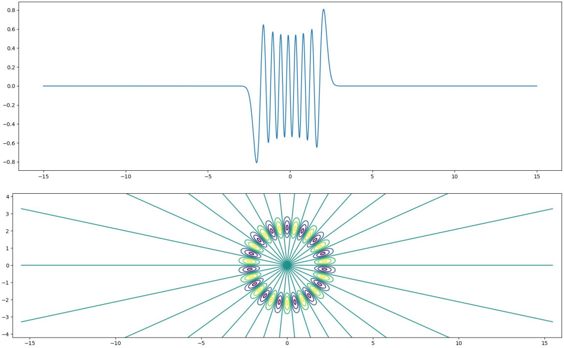

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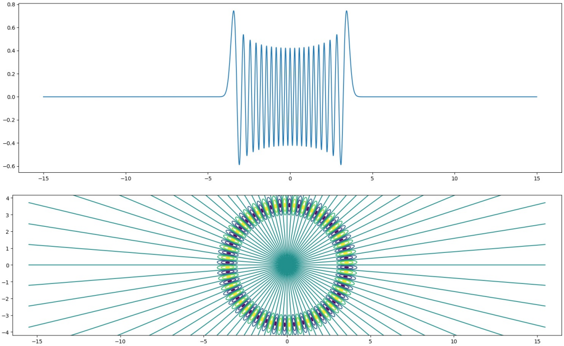

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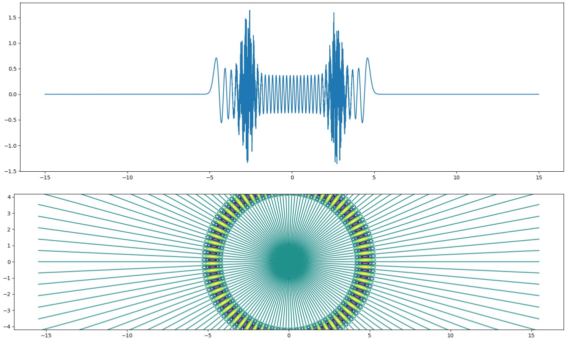
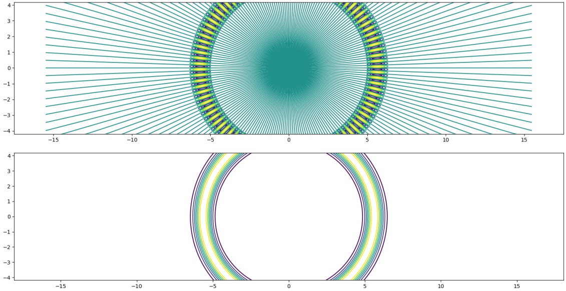


Visualizing the Hermite functions

When numerical error makes it difficult to analyze h_k, the Bargmann transform is computable and understandable, $k = 100$:



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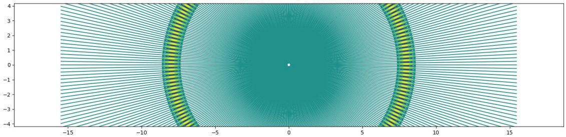

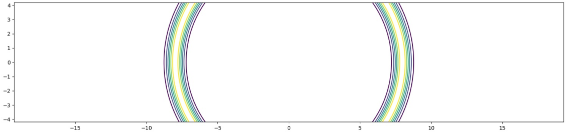

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This allows us to identify

$$\mathscr{S}(\mathbb{R}) \overset{\mathfrak{B}}{\to} \{f \in \mathrm{Hol}(\mathbb{C}) \;:\; \forall k \in \mathbb{N},\;\; (1+|x|)^k f(x)\mathrm{e}^{-\frac{\pi}{2}|x|^2} \in L^2(\mathbb{C})\},$$
$$\mathscr{S}'(\mathbb{R}) \overset{\mathfrak{B}}{\to} \{f \in \mathrm{Hol}(\mathbb{C}) \;:\; \exists k \in \mathbb{N},\; (1+|x|)^{-k} f(x)\mathrm{e}^{-\frac{\pi}{2}|x|^2} \in L^2(\mathbb{C})\}.$$

Smoothness estimates

Let's suppose that $f \in \mathscr{S}'(\mathbb{R})$ such that

$$e^{-\frac{\pi}{2}|x|^2}|\mathfrak{B}f(x)| \leq C.$$

Then, using integration in polar coordinates and Stirling's formula, we can show that

$$\begin{aligned}|\langle \mathfrak{B}f(x)e^{-\frac{\pi}{2}|x|^2}\mathfrak{B}f(x), \sqrt{\frac{\pi^k}{k!}}x^k\rangle| &\leq C\sqrt{\frac{\pi^k}{k!}}\int |x|^k e^{-\frac{\pi}{2}|x|^2}\,dx_1\,dx_2 \\ &\sim 2C(2\pi k)^{1/4}.\end{aligned}$$

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$$\sim 2C(2\pi k)^{1/4}.$$

If $p > 3/4$, we can conclude that

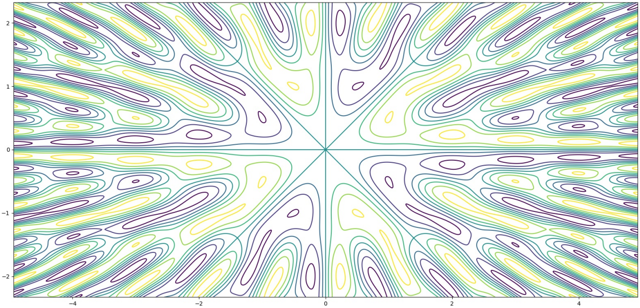
$$\|\langle Q^{-p}f, h_k\rangle\| \lesssim k^{-1/2},$$

so

$$Q^{-p}f \in L^2(\mathbb{R}).$$

Application to Dirac comb

We can apply this to obtain non-optimal results on the Dirac comb $u(x) = \sum_{k\in\mathbb{Z}} \delta(x-k)$, where $\Im \mathfrak{B} u(x) e^{-\frac{\pi}{2}|x|^2}$ is



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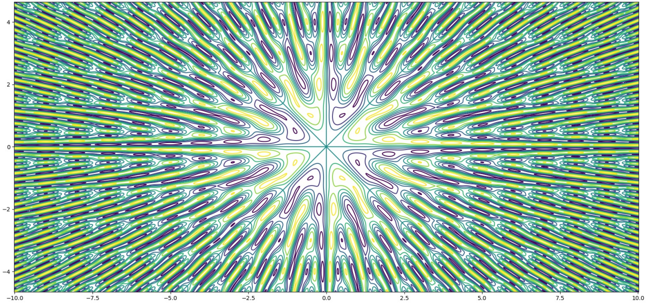

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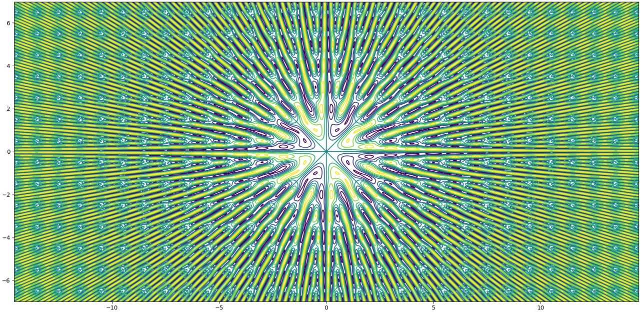

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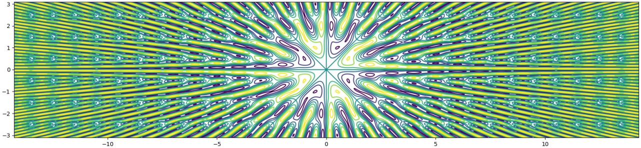
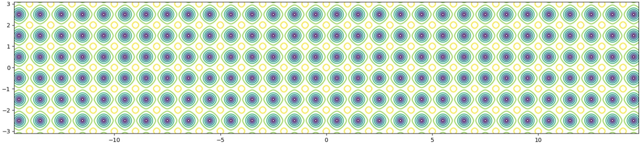
The lattice of symmetries in phase space become more evident with the absolute value:





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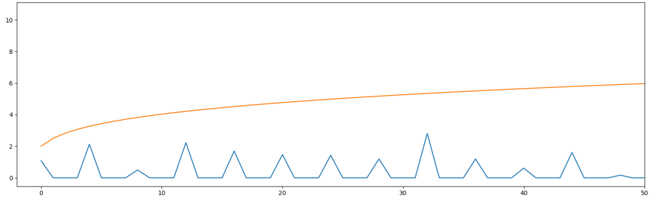
Of course, the reality is much more complicated (NB: these functions particularly like giving numerical errors).



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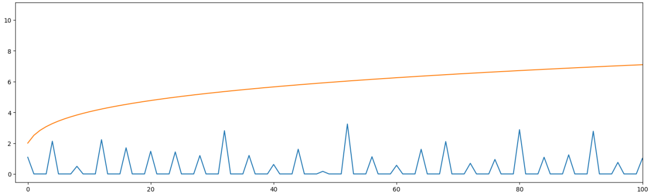

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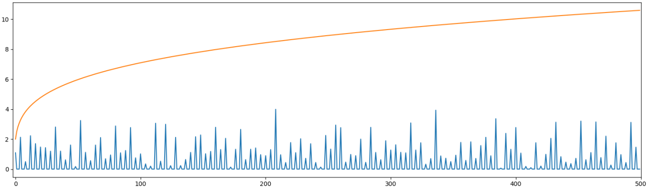

Gap with interpolation

What's more, we showed

$$\|Q_0^{1/2} f\|_{L^2(\mathbb{R})} \sim \|(1 + |x|^2)\mathfrak{B} f \mathrm{e}^{-\frac{\pi}{2}|x|^2}\|_{L^2(\mathbb{C})}.$$

If this extends to Q_0^{-p}, we'd have

$$p > \frac{1}{2}, |\mathfrak{B} f(x)| \mathrm{e}^{-\frac{\pi}{2}|x|^2} \in L^\infty \implies Q_0^{-p} f \in L^2(\mathbb{R}).$$

As a composition of metaplectic operators

Can we represent the Bargmann transform using more familiar operators?

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We have the metaplectic operators

- $\mathcal{F}$, the Fourier transform;
- $\mathcal{V}_G$, change of variables;
- and $\mathcal{W}_A$, multiplication by a Gaussian.

To construct the Bargmann transform, we need a way to pass from $\mathbb{R}_x$ to $\mathbb{C} \sim \mathbb{R}_x \times \mathbb{R}_y$ and to add a holomorphy condition.

Doubling variables

A natural way to extend a function, preserving the L^2 norm, is to take the tensor product with a normalized Gaussian:

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Everything which follows is an attempt to turn $D_y - \mathrm{i}y$ into $D_{\bar{z}} = \frac{1}{2}(D_x + \mathrm{i}D_y)$.

Main ideas

Without going into details...

- A change of variables turns $D_y - \mathrm{i}y$ into $D_y - \mathrm{i}(y - x)$.

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Note that we have many points of flexibility!

Different Bargmann transforms for different operators

Johannes Sjöstrand [1974] showed that, for many quadratic forms (q^w for q quadratic), one can find $\mathfrak{B}_q$ such that

$$\mathfrak{B}_q q^w \mathfrak{B}_q^* = Gx \cdot \partial_x + \frac{1}{2} \operatorname{tr} G$$

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for a matrix G in Jordan normal form.

Example: For the operator $\pi(\mathrm{e}^{\mathrm{i}\theta} x^2 + \mathrm{e}^{-\mathrm{i}\theta} D_x^2)$, we have the same operator $x \cdot \partial_x + \frac{1}{2}$, but with the weight

$$\mathrm{e}^{-\frac{\pi}{\cos\theta}|x|^2 - \Re(\pi \mathrm{i} \mathrm{e}^{\mathrm{i}\theta} \tan\theta x^2)}$$

Eigenfunctions and evolution for $Gx \cdot D_x + \frac{1}{4\pi \mathrm{i}} \operatorname{tr} G$

[Sjöstrand, 1974] The generalized eigenfunctions of q^w are

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As for Schrödinger,

$$\mathrm{e}^{-2\pi \mathrm{i} t(Gx\cdot D_x + \operatorname{tr} G/4\pi \mathrm{i})} f(x) = \mathrm{e}^{-t\frac{1}{2}\operatorname{tr} G} f(\mathrm{e}^{-tG}x),$$

so boundedness and compactness depend on $\Phi(\mathrm{e}^{tG}x) - \Phi(x)$
[Aleman, V. 2018]

Decomposition

The projections onto eigenfunctions (= Taylor series)

$$\Pi_\alpha g(x) = \frac{\partial^\alpha g(0)}{\alpha!} x^\alpha$$

and the low-high energy decomposition (= truncated Taylor series) $\Pi_{|\alpha|\leq N}$ grow (at most) exponentially rapidly in operator norm,

$$\|\Pi_\alpha\|, \|\Pi_{|\alpha|\leq N}\| \leq C\mathrm{e}^{CN}, \quad \forall |\alpha| \leq N,$$

see [Hitrik, Sjöstrand, V. 2013], [§3, V. 2013]

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see [Hitrik, Sjöstrand, V. 2013], [§3, V. 2013]
These estimates are elementary from $\frac{1}{C}|x|^2 \leq \Phi(x) \leq C|x|^2$ and the exponent is optimal for the non-self-adjoint harmonic oscillator, [§3, V. 2013]

Resolvents with low-high decomposition

The low-high decomposition allows us [Hitrik, Sjöstrand, V. 2013] to obtain

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The proof is

- a low-high energy decomposition (with exponential error),
- ellipticity on high energies,
- and straightening to $\Phi(x) = \frac{\pi}{2}|x|^2$ on low energies (with exponential error).

Brief explanation of process

The process works for quadratic forms such that, for good matrices,

$$q(x,\xi) = B(\xi - A_-^* x)\cdot(\xi - A_+ x),$$

happily $\Re q$ positive definite is sufficient.

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$$\mathbf{K}\{(x, A_+ x)\} = \{(z, 0)\},$$
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Brief explanation of process

The process works for quadratic forms such that, for good matrices,

$$q(x,\xi) = B(\xi - A_-^* x) \cdot (\xi - A_+ x),$$

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But the weight $\mathrm{e}^{-\pi|x|^2}$ *will change!*

Some purely self-adjoint concerns

If $q_1(x,\xi), q_2(x,\xi)$ are *real-valued* positive definite quadratic forms, each is unitarily equivalent to $C(x^2 + D_x^2)$ — but not at the same time! We can therefore ask whether

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By bridging the gap with

$$q_3(x,\xi) = (\text{creation of } q_1)(\text{annihilation of } q_2)$$

we can show that

- For t_1, t_2 small it is necessary that $t_1 \leq Ct_2$ and sufficient that $t_1 \leq \frac{1}{C} t_2$ and
- unless the same transformation works for q_1 and q_2, there exists $t_1^c > 0$ such that there exists a t_2 for every $0 < t_1 < t_1^c$ and if $t_1 > t_1^c$ then $e^{t_1 q_1^w} e^{-t_2 q_2^w}$ is never bounded (because the ground state of q_2 leaves L^2 under $e^{t_1^c q_1^w}$).

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Summary

What I hoped to say is:

- what are translations in phase space,
- how their transformation properties are useful,
- what the Bargmann transform is,
- how the Bargmann transform simplifies the search for the Hermite functions,
- how the spectral decomposition of Q_0 corresponds to "decomposing into annuli in phase space",
- how there are different Bargmann transforms which work better for different operators,
- and how we still have decomposition (= Taylor series) into low and high energies (= distance from the origin).

Other techniques

Other techniques include:

- Approximating an operator by multiplication ([Cordoba-Fefferman 1978], Martinez, Sjöstrand, and many others.)
- "Adapting" a Bargmann space by changing the weight, which can for instance solve a Hamilton-Jacobi equation. ([Herau-Sjöstrand-Stolk 2005], many other works of Sjöstrand, Hitrik, Pravda-Starov.)
- For quadratic operators, solving differential equations for the phase of the Weyl symbol of the Schrödinger evolution (Howe, Robert, Combescure, Derezinski and Karczmarczyk, Graefe and Schuman...)
- Studying complex canonical transformations as a holomorphic extension of the metaplectic group ([Hörmander 1983, 1995], also Howe).

Some open questions

- When are simple estimates for spectral projections optimal?
- Optimal exponential rate of resolvent growth.
- How rapidly does the resolvent norm decay when restricted to large energies?
- Geometric understanding of what happens to Gaussians of different shapes under certain operators.
- “Beyond ellipticity”: can one make more rigorous claims like $\mathfrak{B} = \mathrm{e}^{\frac{\pi}{4}(x^2 - D_x^2)}$?

Merci!

Thanks for listening!