Bilinear Rubio de Francia inequalities

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Atelier d'Analyse Harmonique 2018, 30th March 2018

Joint work with Frédéric Bernicot

Classical Littlewood-Paley theory

For I interval, let Δ_I be the freq. proj. $\widehat{\Delta_I f} = \mathbb{1}_I \hat{f}$.

Littlewood-Paley inequality

$$\Big\|\Big(\sum_k |\Delta_{[2^k,2^{k+1}]} f|^2\Big)^{1/2}\Big\|_{L^p} \sim \|f\|_{L^p} \qquad 1 < p < \infty$$

Can work as a substitute of $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ when $p \neq 2$ (can analyse multipliers piece by piece → e.g. Marcinkiewicz multiplier theorem)
Hugely important for all harmonic analysis

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Generalizations

Carleson in '67 was the first to generalize to non-dyadic freq. intervals:

$$\left\| \left(\sum_n |\Delta_{[n,n+1]} f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p} \qquad 2 \leqslant p < \infty.$$

False for $p < 2$ (!) (frequencies no longer incomparable...)

Rubio de Francia proved the general case: let $\{I_k\}_k$ be disjoint arbitrary intervals, then

- $\left\| \left(\sum_k |\Delta_{I_k} f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}$, $2 \leqslant p < \infty$
- for $r > 2$, $\left\| \left(\sum_k |\Delta_{I_k} f|^r \right)^{1/r} \right\|_{L^p} \lesssim \|f\|_{L^p}$, $r' < p < \infty$

Proof works by interpolation between $p = 2$ (Plancherel) and a suitable substitute for $p = \infty$ (e.g. $\|Gf\|_{BMO} \lesssim \|f\|_{L^\infty}$, G smoothed out square function...)

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Bilinear analogues

Question: Suppose $m_j(\xi,\eta)$ are (reasonable) multipliers with disjoint supports in $\widehat{\mathbb{R}^2}$. Under what conditions on the m_j's and their supports can we have

$$\Big\|\Big(\sum_j |T_{m_j}(f,g)|^2\Big)^{1/2}\Big\|_{L^s} \lesssim \|f\|_{L^p}\|g\|_{L^q} \quad ?$$

and for which range of p, q, s?

First results: **Strips**

- (Lacey, '96): $T_j(f,g)(x) = \int \hat{f}(\xi)\hat{g}(\eta)\chi_{[-1,1]}(\xi-\eta-j)e^{2\pi i(\xi+\eta)x}\,d\xi\,d\eta$, $p,q \geqslant 2,\ s=2,\ 1/p+1/q=1/2$;
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Bilinear Rubio de Francia for squares

Benea and Bernicot proved a bilinear generalization for squares: for ω a square, define bilinear frequency projection

$$\pi_\omega(f,g)(x) := \int_{\mathbb{R}} \widehat{f}(\xi)\widehat{g}(\eta)\chi_\omega(\xi,\eta)e^{2\pi i(\xi+\eta)x}\, d\xi\, d\eta.$$

Theorem [Benea,Bernicot,'16]

Let $\Omega = \{\omega\}$ be a family of disjoint squares in $\widehat{\mathbb{R}^2}$ and $r > 2$. Then

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It suffices to assume all ω dyadic and study

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We have resolved our quantity into **wavepackets**:

$|\omega_j|^{-1/2} \check{\chi}_{\omega_j}(|\omega_j|^{-1} n - \cdot)$ is L^2-norm.d, smooth, freq. supported in ω_j, concentrated in $|\omega_j|[n, n+1]$ and decays rapidly outside it. It's a wavepacket!

Tiles

Introduce then tiles

$$P = (\omega_1 \times I_P, \omega_2 \times I_P, \omega_3 \times I_P), \qquad I_P = |\omega_j|^{-1}[n, n+1],$$

and the wavepackets

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- find good collections of tiles $\mathbb{Q}$ such that you can estimate $\Lambda_{\mathbb{Q}}(f, g, \mathbf{h})$ explicitely by

  $$|\Lambda_{\mathbb{Q}}(f, g, \mathbf{h})| \lesssim \text{“}\mathrm{Avg}_1 f\text{”} \cdot \text{“}\mathrm{Avg}_2 g\text{”} \cdot \text{“}\mathrm{Avg}_3 \|\mathbf{h}\|_{\ell^{r'}}\text{”} \cdot |\text{time support of } \mathbb{Q}|;$$

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where we have defined **Sizes**

$$\mathrm{Size}^1_f(\mathbb{P}) := \sup_{P\in\mathbb{P}} \frac{|\langle f, \phi^1_P\rangle|}{|I_P|^{1/2}}, \quad \mathrm{Size}^3_{\mathbf{h}}(\mathbb{P}) := \sup_{\substack{\mathcal{C}\subset\mathbb{P} \\ \text{column}}} \Big(\frac{1}{|I_{top}|} \sum_{\omega\in\mathcal{C}} \int_{I_{top}} M h^{r'}_\omega\Big)^{\frac{1}{r'}}$$

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These are our averages! They are good averages indeed:

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Energies

We need to control collections that are uniform in size:

$$\mathrm{Energy}_f(\mathbb{P}) := \sup 2^n \Big(\sum_{\mathcal{C}} |I_{\mathcal{C}}| \Big)^{1/2},$$

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These quantities are good too!

$$\underbrace{\mathrm{Energy}_f(\mathbb{P}) \lesssim \|f\|_{L^2}}_{\text{(by orthogonality)}}, \qquad \underbrace{\mathrm{Energy}_{\mathbf{h}}(\mathbb{P}) \lesssim \|\mathbf{h}\|_{L^{r'}(\ell^{r'})}}_{\text{(by disjointness of supports)}}$$

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Now we use the good bounds for Size and Energy: we can assume

$$|f| \leqslant \mathbb{1}_F, \quad |g| \leqslant \mathbb{1}_G, \quad \Big(\sum_{\omega} |h_\omega|^{r'}\Big)^{1/r'} \leqslant \mathbb{1}_H;$$

for interpolation purposes, we can also throw away $\ll |H|$ of H, so assume also that for any P there is $x \in I_P$ s.t.

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What about rectangles?

Assume $\frac{|R_2|}{|R_1|} =: \mathrm{ecc}(R) \gg 1$, dyadic rectangles $\mathscr{R} = \{R\}$.

First problem: we have **two scales** for each R!

We do the same reductions we did as for squares, but now everything has to be w.r.t. the smallest scale, $|R_2|^{-1}$: in the end we study

$$\Lambda(f, g, \mathbf{h}) = \sum_{R \in \mathscr{R}} \sum_{n} |R_1|^{1/2} \langle f, |R_1|^{-1/2} \widecheck{\chi_{R_1}}(|R_2|^{-1} n - \cdot) \rangle$$
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[see drawing on the board] ⇒ Bad combinatorics!

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[see drawing on the board] $\Rightarrow$ Bad combinatorics!

The way around is: reduce to a trilinear form with better (algebraic) structure

- Morally, $|\langle f, |R_1|^{-1/2}\widecheck{\chi_{R_1}}(|R_2|^{-1}n - \cdot)\rangle| \approx |\langle f, |R_1|^{-1/2}\widecheck{\chi_{R_1}}(|R_1|^{-1}\lfloor n/\mathrm{ecc}(R)\rfloor - \cdot)\rangle|$
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We need tiles that are adapted to the two scales $|R_1|, |R_2|$: let

$$I^n := I + n|I|,$$

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Then they are of the form

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similar to square estimates, but non-trivial to get there...

Energy estimates

$\text{Energy}_f^{\mathfrak{n}}(\mathbb{P})$ is defined as before but the notion of (shifted) column disjointness is different (shifted tiles $R_1 \times I_P^{\mathfrak{n}}$ are disjoint instead)

We have as a consequence a slightly worse estimate:

$$\text{Energy}_f^{\mathfrak{n}}(\mathbb{P}) \lesssim \log(\mathfrak{n}) \|f\|_{L^2}$$

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The rest of the argument mimicks the one for squares, with some important differences:

- we freeze g and consider only columns, not rows;
- we have to be careful in dealing with the different structure of the (shifted) columns.

Running stopping-times as before and doing similar computations we end up with

$$|\Lambda^{n}_{\mathbb{P}}(f, g, h)| \lesssim (\log(n))^{O(1)}|F|^{1/p}|G|^{1/r}|H|^{1/s'}$$

for $2 < p < r$, $|f| \leqslant \mathbb{1}_F, |g| \leqslant \mathbb{1}_G, (\sum_R |h_R|^{r'})^{1/r'} \leqslant$ "$\mathbb{1}_H$".
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$$\sup_R |\pi_{R_1} f \cdot \pi_{R_2} g| \leqslant \mathcal{C}f \cdot \mathcal{C}g,$$

where $\mathcal{C}$ is the Carleson operator; so it's bounded for all $1 < p, q < \infty$. There is an interpolation argument for vector-valued situations (due to Silva) that allows us to interpolate between $r_0 = \infty$ and r_1 close to 2 to obtain for any $r > 2$ that

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Non-smooth rectangles

We can replace χ_R by $\mathbb{1}_R$:

$$\sqcap_R(f,g)(x) := \int \widehat{f}(\xi)\widehat{g}(\eta)\mathbb{1}_R(\xi,\eta)e^{2\pi i(\xi+\eta)x}\,d\xi\,d\eta$$

This is more singular because of the discontinuity at the boundary (same phenomenon as for the Bilinear Hilbert transform).
We can't quite prove the same inequalities (for now...) but at least we can say

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For all $\varepsilon > 0$ and finite family $\mathscr{R}$ of disjoint dyadic rectangles

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