Random Schrödinger Operators

Constanza ROJAS-MOLINA
University of Bonn

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Outline

Motivation

Goal : to study the electronic transport in disordered materials and identify if a material is **a conductor or an insulator**

Quantum mechanics setting :

physical state	a vector ψ in a Hilbert space $\mathcal{H}$, with $\|\psi\| = 1$
physical observables	self-adjoint operator H
possible outcomes	$\sigma(H)$ spectrum of the operator H

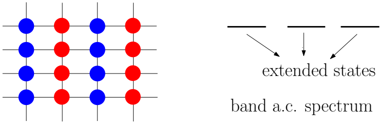
Dynamics of a particle moving in a material : $\psi \in \mathcal{H} = L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, $\|\psi\| = 1$,

$$\partial_t \psi(t,x) = -iH\psi(t,x),$$

$$\psi(t,x) = e^{-itH}\psi(0,x),$$

where $H = H_0 + V$ is a self-adjoint Schrödinger operator on $\mathcal{H}$.

Example : electrons in a crystal, $H = -\Delta + V$ acting on $\ell^2(\mathbb{Z}^d)$, the potential $V\psi(x) = q(x)\psi(x)$, where q is a periodic function.



extended states $\sim \psi(t,x)$ propagate in space as t grows $\sim$ transport

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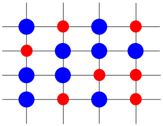
Example : electrons in a disordered crystal



$\psi(t,x)$ do not propagate in space as t grows $\sim$ absence of transport

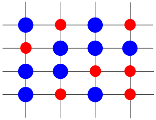
Disordered media

P. W. Anderson 1958 :
if the medium has impurities, there is *no wave propagation*.
"Absence of diffusion in certain random lattices", Phys. Rev. (Nobel 1977)

Anderson model : $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbb{Z}^d)$, with

$$V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \omega_j \delta_j(x),$$

where $\omega = (\omega_j)_{j\in\mathbb{Z}^d}$ is a random variable in a probability space $(\Omega, \mathbb{P})$.



Localization : first rigorous mathematical results in the late 70s, early 80s.

Recall from spectral theory

For a self-adjoint operator H and a vector $\varphi \in \mathcal{H}$, there exists a spectral measure $\mu_{H,\varphi}$ such that

$$\langle \varphi, H\varphi \rangle = \int_{\mathbb{R}} \lambda d\mu_{H,\varphi}(\lambda)$$

or, formally

$$H = \int_{\mathbb{R}} \lambda d\mu_{H,\varphi}(\lambda).$$

For this spectral measure $\mu = \mu_{H,\varphi}$ one has the usual Lebesgue decomposition into three mutually singular parts

$$\mu = \mu^{pp} + \mu^{sc} + \mu^{ac}$$

which induces a decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{ac}$, such that

$$H_{\mathcal{H}_*} = \int_{\mathbb{R}} \lambda d\mu^*_{H,\varphi}(\lambda), \quad * \in pp, sc, ac$$

Then, writing

$$\sigma_*(H) = \sigma(H_{\mathcal{H}_*}), \quad * \in pp, sc, ac$$

we have the following decomposition for the spectrum

$$\sigma(H) = \sigma_{pp}(H) \cup \sigma_{sc}(H) \cup \sigma_{ac}(H)$$

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Going back to the Anderson model $(H_\omega)_{\omega\in\Omega}$,

- We say that the operator H_ω exhibits *spectral localization* in an interval $\mathcal{I}$ if $\sigma(H) \cap I = \sigma_{pp}(H) \cap I$, almost surely.
- We say that H exhibits *Anderson localization* (AL) in I if $\sigma(H) \cap I = \sigma_{pp}(H) \cap I$ with exponentially decaying eigenfunctions, almost surely.

In the late 70s, mathematicians thought that "AL $=$ absence of transport", until the 90s, with the work of del Río-Jitomirskaya-Last-Simon, where they showed that there might be AL with some transport.

Dynamical localization I

- We say that H_ω exhibits *dynamical localization* (DL) in I if there exist constants $C < \infty$ and $c > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} |\langle \delta_y, e^{-itH_\omega}\chi_I(H_\omega)\delta_x\rangle|\right) \leq Ce^{-c|x-y|}$$

Theorem (DL implies absence of transport)

If (DL) holds in $J \subset \mathbb{R}$, then for $\varphi \in \ell^2(\mathbb{Z}^d)$ with compact support we have

weighted space → $\langle X\rangle^{p/2}$; time evolution → e^{-itH_ω}; localization in energy → $\chi_J(H_\omega)$

$$\sup_t \|\langle X\rangle^{p/2} e^{-itH_\omega}\chi_J(H_\omega)\varphi\|_2 < \infty,$$

for every $p \geq 0$, with probability one.

Proof of theorem (DL implies absence of transport)

Recall that $|X|\varphi(n) = |n|\varphi(n)$ for $\varphi \in \ell^2(\mathbb{Z}^d)$. Take $\varphi \in \ell^2_c(\mathbb{Z}^d)$, that is, for some $R > 0$, $\varphi(n) = 0$ for $|n| > R$. Then, using the expression

$$\|x\| = \sum_n |\langle x, \delta_n\rangle|^2$$

$$\begin{aligned}
\left\| |X|^p e^{-itH_\omega}\chi_I(H_\omega)\varphi \right\|^2 &= \sum_{j\in\mathbb{Z}^d} \left|\langle \delta_j, |X|^p e^{-itH_\omega}\chi_I(H_\omega)\varphi\rangle\right|^2 \\
&\leq \sum_j |j|^{2p} \left|\langle \delta_j, e^{-itH_\omega}\chi_I(H_\omega)\varphi\rangle\right|^2 \\
&\leq \sum_j |j|^{2p} \left|\langle \delta_j, e^{-itH_\omega}\chi_I(H_\omega)\varphi\rangle\right| \|\varphi\| \\
&\leq \sum_j |j|^{2p} \|\varphi\| \left|\langle \delta_j, e^{-itH_\omega}\chi_I(H_\omega)\left(\sum_{|k|\leq R}\langle \varphi, \delta_k\rangle \delta_k\right)\rangle\right| \\
&\leq \sum_j \sum_{|k|\leq R} |j|^{2p} \|\varphi\|^2 \left|\langle \delta_j, e^{-itH_\omega}\chi_I(H_\omega)\delta_k\rangle\right|
\end{aligned}$$

$$\left\| |X|^p e^{-itH_\omega} \chi_I(H_\omega)\varphi \right\|^2 \leq \sum_j \sum_{|k|\leq R} |j|^{2p} \|\varphi\|^2 \left| \langle \delta_j, e^{-itH_\omega} \chi_I(H_\omega) \delta_k \rangle \right|$$

Taking the expectation $\mathbb{E}$ in both sides, we get

$$\begin{aligned}
\mathbb{E}\left(\sup_t \left\| |X|^p e^{-itH_\omega} \chi_I(H_\omega)\varphi \right\|^2 \right) \\
&\leq \sum_j \sum_{|k|\leq R} |j|^{2p} \|\varphi\|^2 \, \mathbb{E}\left(\sup_t \left| \langle \delta_j, e^{-itH_\omega} \chi_I(H_\omega) \delta_k \rangle \right| \right) \\
&\leq \sum_j \sum_{|k|\leq R} |j|^{2p} \|\varphi\|^2 \, C e^{-c|j-k|} \qquad (DL) \\
&< \infty
\end{aligned}$$

Finally, if $\mathbb{E}(f) < \infty$, then $f < \infty$ a.s. Therefore, for any $p \geq 0$,

$$\sup_t \left\| |X|^p e^{-itH_\omega} \chi_I(H_\omega)\varphi \right\|^2 < \infty \quad a.s.$$

□

Dynamical localization II

Recall that

- We say that H_ω exhibits *dynamical localization* (DL) in I if there exist constants $C < \infty$ and $c > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} |\langle \delta_y, e^{-itH_\omega}\chi_I(H_\omega)\delta_x\rangle|\right) \leq Ce^{-c|x-y|}$$

Theorem (DL implies pure point spectrum)

If (DL) *holds in an interval I, then* H_ω *has pure point spectrum in I with probability one.*

The proof relies on the RAGE Theorem.

Theorem (Ruelle-Amrein-Georgescu-Enss)

Let H be a s.a. operator on $\ell^2(\mathbb{Z}^d)$, let P_c and P_{pp} be the orthogonal projections onto $\mathcal{H}_c$ and $\mathcal{H}_{pp}$, resp. Let Λ_L be a cube of side L around the origin. Then, for any $\varphi \in \ell^2(\mathbb{Z}^d)$,

$$\|P_c\varphi\|^2 = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\notin\Lambda_L} |e^{-itH}\varphi(x)|^2 \right) dt$$

$$\|P_{pp}\varphi\|^2 = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\in\Lambda_L} |e^{-itH}\varphi(x)|^2 \right) dt$$

Take $\varphi \in \ell_c(\mathbb{Z}^d)$, that is, for some $R > 0$, $\varphi(n) = 0$ for $|n| > R$. From RAGE Theorem we have that

$$\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2 = \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\sum_{x\notin\Lambda_L}|e^{-itH}\chi_I(H_\omega)\varphi(x)|^2\right) dt$$

Note that

$$\sum_{x\notin\Lambda_L}|e^{-itH}\chi_I(H_\omega)\varphi(x)|^2 = \left\|\chi_{\Lambda_L^c}e^{-itH}\chi_I(H_\omega)\varphi\right\|^2 = \left\|\chi_{\Lambda_L^c}e^{-itH}\chi_I(H_\omega)\chi_{\Lambda_R}\varphi\right\|^2$$

$$\leq \left\|\chi_{\Lambda_L^c}e^{-itH}\chi_I(H_\omega)\chi_{\Lambda_R}\right\|\|\varphi\|^2$$

$$\leq \sum_{|x|\geq L}\sum_{|k|\leq R}\left|\langle\delta_x, e^{-itH}\chi_I(H_\omega)\delta_k\rangle\right|\|\varphi\|^2$$

Taking the expectation $\mathbb{E}$ in both sides, and using Fatou's lemma and Fubini, yields

$$\mathbb{E}(\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2)$$

$$\leq \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{|x|\geq L}\sum_{|k|\leq R}\|\varphi\|^2\,\mathbb{E}\left(\left|\langle\delta_x, e^{-itH}\chi_I(H_\omega)\delta_k\rangle\right|\right)$$

$$\mathbb{E}(\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2) \\ \leq \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{|x|\geq L}\sum_{|k|\leq R}\|\varphi\|^2\,\mathbb{E}\left(\left|\langle\delta_x, e^{-itH}\chi_I(H_\omega)\delta_k\rangle\right|\right)$$

Note that by hypothesis (dynamical localization),

$$\mathbb{E}\left(\left|\langle\delta_x, e^{-itH}\chi_I(H_\omega)\delta_k\rangle\right|\right) \leq Ce^{-c|x-k|}$$

uniformly in t, then

$$\mathbb{E}(\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2) \leq C\|\varphi\|^2 \lim_{L\to\infty}\sum_{|x|\geq L}\sum_{|k|\leq R} e^{-c|x-k|}$$

Since the sum in the r.h.s is convergent, the limit when $R\to\infty$ is 0. Then

$$\mathbb{E}(\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2) = 0$$

implies $P_c(H_\omega)\chi_I(H_\omega)\varphi = 0$ for almost every $\omega\in\Omega$ and $\varphi\in\ell_c(\mathbb{Z}^d)$. Since $\ell_c(\mathbb{Z}^d)$ is dense in $\ell^2(\mathbb{Z}^d)$, the result follows. □

Alternative proof (absence of transport implies pure point spectrum).

Take $\varphi \in \ell_c(\mathbb{Z}^d)$, that is, for some $R > 0$, $\varphi(n) = 0$ for $|n| > R$. From RAGE Theorem we have that

$$\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2 = \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\sum_{x\notin\Lambda_L}|e^{-itH}\chi_I(H_\omega)\varphi(x)|^2\right)dt$$

Note that

$$\sum_{x\notin\Lambda_L}|e^{-itH}\chi_I(H_\omega)\varphi(x)|^2 \le \sum_{x\notin\Lambda_L}\frac{1}{|x|^{2p}}|\,|X|^p e^{-itH}\chi_I(H_\omega)\varphi(x)|^2$$

$$\le \|\,|X|^p e^{-itH}\chi_I(H_\omega)\varphi(x)\|^2 \sum_{x\notin\Lambda_L}\frac{1}{|x|^{2p}}$$

Therefore,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \|\,|X|^p e^{-itH}\chi_I(H_\omega)\varphi(x)\|^2 dt < C$$

Which leaves

$$\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2 \le C\lim_{L\to\infty}\sum_{x\notin\Lambda_L}\frac{1}{|x|^{2p}} = 0$$

□

Summary

- Transport of electrons in materials is studied by looking at dynamical properties of Schrödinger operators.
- There is a relation between spectral and dynamical properties, but they are not equivalent !
- Disordered materials are represented by random Schrödinger operators
- Random Schrödinger operators exhibit localization in some regions of the spectrum
- The *right* notion of localization is dynamical localization (physically relevant)

What P.W. Anderson observed in '58 is...
dynamical localization.

Proof of RAGE Theorem

Theorem (Ruelle-Amrein-Georgescu-Enss)

Let H be a s.a. operator on $\ell^2(\mathbb{Z}^d)$, let P_c and P_{pp} be the orthogonal projections onto $\mathcal{H}_c$ and $\mathcal{H}_{pp}$, resp. Let Λ_L be a cube of side L around the origin. Then, for any $\varphi \in \ell^2(\mathbb{Z}^d)$,

$$\|P_c\varphi\|^2 = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\notin\Lambda_L} |e^{-itH}\varphi(x)|^2 \right) dt$$

$$\|P_{pp}\varphi\|^2 = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\in\Lambda_L} |e^{-itH}\varphi(x)|^2 \right) dt$$

Proof :

- Characterization of $\psi \in \mathcal{H}_{pp}$
- Characterization of $\psi \in \mathcal{H}_{ac}$
- Characterization of $\psi \in \mathcal{H}_c$
- Proof of Theorem

Characterization of $\psi \in \mathcal{H}_{pp}$

Theorem

Let H be a self-adjoint operator in $\ell^2(\mathbb{Z}^d)$. Take $\varphi \in \mathcal{H}_{pp}$ and let $\Lambda_L := [-L, L]^d \cap \mathbb{Z}^d$. Then

$$\lim_{L\to\infty} \sup_t \left(\sum_{x \in \Lambda_L} \left| e^{-itH} \varphi(x) \right|^2 \right) = \|\varphi\|^2$$

and

$$(*) \qquad \lim_{L\to\infty} \sup_t \left(\sum_{x \notin \Lambda_L} \left| e^{-itH} \varphi(x) \right|^2 \right) = 0$$

Proof :

1) the case φ is an eigenfunction
2) φ is a finite linear combination of eigenfunctions
3) $\varphi \in \mathcal{H}_{pp}$

Since e^{-itH} is unitary, for all t we have

$$\|\varphi\|^2 = \left\|e^{-itH}\varphi\right\|^2 = \sum_{x\in\mathbb{Z}^d} \left|\langle\delta_x, e^{-itH}\varphi\rangle\right|^2$$
$$= \sum_{x\in\Lambda_L} \left|(e^{-itH}\varphi)(x)\right|^2 + \sum_{x\notin\Lambda_L} \left|(e^{-itH}\varphi)(x)\right|^2$$

1) Let φ be an eigenfunction with eigenvalue E, $(e^{-itH}\varphi)(x) = e^{-itE}\varphi(x)$, so $\left|(e^{-itH}\varphi)(x)\right| = |\varphi(x)|$ uniformly on t. Therefore, since $\varphi \in \ell^2(\mathbb{Z}^d)$,

$$\sum_{x\notin\Lambda_L} \left|(e^{-itH}\varphi)(x)\right|^2 = \sum_{x\notin\Lambda_L} |\varphi(x)|^2 \to 0, \text{ when } L \to \infty$$

Next, note that $(*)$ can be written as

$$\left\|\chi_{\Lambda_L^c} e^{-itH}\varphi\right\| \to_{L\to\infty} 0 \quad \text{uniformly in t}$$

2) Let φ be the finite linear combination of eigenfunctions φ_k $\varphi = \sum_{k=1}^N a_k \varphi_k$. Then

$$\left\|\chi_{\Lambda_L^c} e^{-itH}\varphi\right\| = \left\|\sum_{k=1}^N a_k \chi_{\Lambda_L^c} e^{-itH}\varphi_k\right\| \leq \sum_{k=1}^N |a_k| \left\|\chi_{\Lambda_L^c} e^{-itH}\varphi_k\right\|$$

$$= \sum_{k=1}^N |a_k| \left\|\chi_{\Lambda_L^c} e^{-itE}\varphi_k\right\|$$

$$= \sum_{k=1}^N |a_k| \left\|\chi_{\Lambda_L^c}\varphi_k\right\|$$

Since $\varphi_k \in \ell^2(\mathbb{Z}^d)$, $\left\|\chi_{\Lambda_L^c}\varphi_k\right\| \to 0$. So we can take L large enough depending on N in order to make the r.h.s. as small as we want, uniformly in t.

3) Let $\varphi \in \mathcal{H}_{pp}$. There exists a sequence of linear combinations of eigenfunctions $\varphi_N := \sum_{k=1}^{N} a_k \varphi_k$ such that, given $\varepsilon > 0$, $\|\varphi - \varphi_N\| < \varepsilon$ for N large enough. Then

$$\left\|\chi_{\Lambda_L^c} e^{-itH} \varphi\right\| \leq \left\|\chi_{\Lambda_L^c} e^{-itH} (\varphi - \varphi_N)\right\| + \left\|\chi_{\Lambda_L^c} e^{-itH} (\varphi_N)\right\|$$
$$\leq \left\|e^{-itH} (\varphi - \varphi_N)\right\| + \left\|\chi_{\Lambda_L^c} e^{-itH} (\varphi_N)\right\|$$

By taking N large enough, $\|\varphi - \varphi_N\| < \varepsilon/2$, while by taking L large enough, depending on N, we have $\left\|\chi_{\Lambda_L^c} e^{-itH} (\varphi_N)\right\| < \varepsilon/2$, therefore

$$\left\|\chi_{\Lambda_L^c} e^{-itH} \varphi\right\| < \varepsilon \quad \text{uniformly in } t$$

which yields

$$\left\|\chi_{\Lambda_L^c} e^{-itH} \varphi\right\| \to_{L \to \infty} 0$$

□

Characterization of $\psi \in \mathcal{H}_{ac}$

Theorem

Let H be a self-adjoint operator in $\ell^2(\mathbb{Z}^d)$. Take $\varphi \in \mathcal{H}_{ac}$ and let Λ_L be a finite set in $\mathbb{Z}^d$. Then

$$\lim_{t\to\infty}\left(\sum_{x\in\Lambda_L}\left|e^{-itH}\varphi(x)\right|^2\right)=0$$

and

$$\lim_{t\to\infty}\left(\sum_{x\notin\Lambda_L}\left|e^{-itH}\varphi(x)\right|^2\right)=\|\varphi\|^2$$

Note that

$$\langle \psi, e^{-itH}\varphi \rangle = \int e^{-it\lambda} d\mu_{\psi,\varphi}(\lambda),$$

where $d\mu_{\psi,\varphi}(\lambda)$ is the spectral measure associated to ψ and φ in $\ell^2(\mathbb{Z}^d)$. If $\varphi \in \mathcal{H}_{ac}$, then $d\mu_{\psi,\varphi}$ is a.c. with respect to the Lebesgue measure, i.e., there exists a function $g \in L^1(\mathbb{R}, d\lambda)$ such that

$$d\mu_{\psi,\varphi}(\lambda) = g(\lambda) d\lambda.$$

Then,

$$\langle \psi, e^{-itH}\varphi \rangle = \int e^{-it\lambda} g(\lambda) d\lambda$$

which is the Fourier transform of g. By the Riemann-Lebesgue Lemma, the r.h.s. tends to 0 in absolute value, as $t \to \infty$.
Taking $\psi = \delta_x$, we get

$$\left|(e^{-itH}\varphi)(x)\right| = \left|\langle \delta_x, e^{-itH}\varphi \rangle\right| \to_{t\to\infty} 0$$

Taking now the vector $\chi_{\Lambda_L} = \sum_{x\in\Lambda_L} \delta_x \in \ell^2(\mathbb{Z}^d)$ we get the desired result. □

Characterization of $\varphi \in \mathcal{H}_c$

Now, we want an expression for $\varphi \in \mathcal{H}_c$, not just $\mathcal{H}_{ac}$. The following will be useful,

Theorem (Wiener)

Let μ be a bounded Borel measure on $\mathbb{R}$. Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\hat{\mu}(t)|^2 \, dt = \sum_{x \text{ atom of } \mu} |\mu(\{x\})|^2,$$

where $\hat{\mu}(t) = \int e^{-it\lambda} d\mu(\lambda)$ is the Fourier transform of the measure μ. If μ is continuous, the r.h.s. is 0.

Theorem

Let H be a self-adjoint operator in $\ell^2(\mathbb{Z}^d)$. Take $\varphi \in \mathcal{H}_c$ and let Λ_L be a finite set in $\mathbb{Z}^d$. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\in\Lambda_L} \left|e^{-itH}\varphi(x)\right|^2 \right) = 0$$

and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\notin\Lambda_L} \left|e^{-itH}\varphi(x)\right|^2 \right) = \|\varphi\|^2$$

Proof : for $\varphi \in \mathcal{H}_c$, for any $x \in \mathbb{Z}^d$, the measure $\mu_{\delta_x,\varphi}$ is continuous. Using Wiener, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left| \int e^{-it\lambda} d\mu_{\delta_x,\varphi}(\lambda) \right|^2 dt = 0$$

Note that $\left|\int e^{-it\lambda} d\mu_{\delta_x,\varphi}(\lambda)\right|^2 = \left|\langle \delta_x, e^{-itH}\varphi\rangle\right|^2 = \left|e^{-itH}\varphi(x)\right|^2$. Taking the vector $\chi_{\Lambda_L} = \sum_{x\in\Lambda_L} \delta_x$ gives the claim. □

Proof of RAGE Theorem

Theorem (Ruelle-Amrein-Georgescu-Enss)

Let H be a s.a. operator on $\ell^2(\mathbb{Z}^d)$, let P_c and P_{pp} be the orthogonal projections onto $\mathcal{H}_c$ and $\mathcal{H}_{pp}$, resp. Let Λ_L be a cube of side L around the origin. Then, for any $\varphi \in \ell^2(\mathbb{Z}^d)$,

$$\|P_c\varphi\|^2 = \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\sum_{x\notin\Lambda_L}|e^{-itH}\varphi(x)|^2\right)dt$$

$$\|P_{pp}\varphi\|^2 = \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\sum_{x\in\Lambda_L}|e^{-itH}\varphi(x)|^2\right)dt$$

Proof :

$$\|P_c\varphi\|^2 = \left\|e^{-itH}P_c\varphi\right\|^2 = \sum_{x\in\Lambda_L}\left|e^{-itH}P_c\varphi(x)\right|^2 + \sum_{x\notin\Lambda_L}\left|e^{-itH}(\varphi - P_{pp}\varphi)(x)\right|^2$$

$$\sum_{x\notin\Lambda_L}\left|e^{-itH}(\varphi - P_{pp}\varphi)(x)\right|^2 = \sum_{x\notin\Lambda_L}\left|e^{-itH}\varphi\right|^2 + \sum_{x\notin\Lambda_L}\left|e^{-itH}P_{pp}\varphi\right|^2 + \sum_{x\notin\Lambda_L} 2Re(e^{-itH}\varphi(x))\overline{(e^{-itH}P_{pp}\varphi(x))}$$

Using Cauchy-Schwarz, one can show that

$$|\mathcal{E}| := \left|\sum_{x\notin\Lambda_L} 2Re(e^{-itH}\varphi(x))\overline{(e^{-itH}P_{pp}\varphi(x))}\right| \leq \|\varphi\|^2\left(\sup_t \sum_{x\notin\Lambda_L}\left|(e^{-itH}P_{pp}\varphi(x))\right|^2\right)^{1/2}$$

Recalling the characterization for $P_{pp}\varphi$, taking $\lim_{L\to\infty}$, the r.h.s. tends to 0.

We get

$$\|P_c\varphi\|^2 = \sum_{x\in\Lambda_L} \left|e^{-itH}P_c\varphi(x)\right|^2 + \sum_{x\notin\Lambda_L} \left|e^{-itH}\varphi\right|^2 + \sum_{x\notin\Lambda_L} \left|e^{-itH}P_{pp}\varphi\right|^2 + \mathcal{E}$$

We take $\frac{1}{T}\int_0^T$ in both sides and note that $\frac{1}{T}\int_0^T \|P_c\varphi\|^2 = \|P_c\varphi\|^2$,

$$\|P_c\varphi\|^2 = \frac{1}{T}\int_0^T \sum_{x\in\Lambda_L} \left|e^{-itH}P_c\varphi(x)\right|^2 + \frac{1}{T}\int_0^T \sum_{x\notin\Lambda_L} \left|e^{-itH}\varphi\right|^2 + \frac{1}{T}\int_0^T \sum_{x\notin\Lambda_L} \left|e^{-itH}P_{pp}\varphi\right|^2 + \mathcal{E}$$

Taking $\lim_{L\to\infty}\lim_{T\to\infty}$, we can use the characterizations obtained for $\mathcal{H}_c$ ▸ ac and $\mathcal{H}_{pp}$ ▸ pp and the fact that the error goes to 0, to finally obtain

$$\|P_c\varphi\|^2 = \lim_{L\to\infty}\lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{x\notin\Lambda_L} \left|e^{-itH}\varphi\right|^2$$

□

References

- W. Kirsch, *An invitation to Random Schrödinger Operators*, in Random Schrödinger Operators, Panoramas et Syntheses Vol. 25, 2008 (SMF).
- G. Stolz, *An introduction to the mathematics of Anderson localization*, Contemporary Mathematics 551, 2010.

Recall that in RAGE Theorem, given a self-adjoint operator H, we have the following expression for any $\varphi \in \ell^2(\mathbb{Z}^d)$,

$$\|P_c\varphi\|^2 = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\notin\Lambda_L} |e^{-itH}\varphi(x)|^2 \right) dt.$$

To prove this we used :

Theorem (Wiener)

Let μ be a bounded Borel measure on $\mathbb{R}$. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\hat{\mu}(t)|^2 \, dt = \sum_{x \text{ atom of } \mu} |\mu(\{x\})|^2 ,$$

where $\hat{\mu}(t) := \int e^{-it\lambda} d\mu(\lambda)$ is the Fourier transform of the measure μ. In particular, if μ is continuous, the r.h.s. is 0.

Note that if $d\mu = \mu_{\delta_x,\varphi}$ is the spectral measure of H associated to the vectors δ_x and φ, we have that

$$\hat{\mu}(t) = \int e^{-it\lambda} d\mu_{\delta_x,\varphi}(\lambda) = \langle \delta_x, e^{-itH}\varphi\rangle = e^{-itH}\varphi(x),$$

so Wiener's theorem gives that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left|e^{-itH}\varphi(x)\right|^2 dt = \sum_{\lambda \text{ atom of } \mu} \left|\mu_{\delta_x,\varphi}(\{\lambda\})\right|^2.$$

Moreover, if $\varphi \in \mathcal{H}_c$ for H, then $\mu_{\delta_x,\varphi}$ is also continuous measure (it has no atoms). Indeed, for any $u \in \mathbb{R}$:

$$\mu_{\delta_x,\varphi}(\{u\}) = \langle \delta_x, \chi_{\{u\}}\varphi\rangle \le \|\delta_x\| \left\|\chi_{\{u\}}\varphi\right\| = \mu_\varphi(\{u\}) = 0.$$

Therefore,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left|e^{-itH}\varphi(x)\right|^2 dt = 0.$$

Proof of Wiener's Theorem

Theorem (Wiener)

Let μ be a bounded Borel measure on $\mathbb{R}$. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\, dt = \sum_{x \text{ atom of } \mu} |\mu(\{x\})|^2,$$

where $\hat{\mu}(t) := \int e^{-it\lambda} d\mu(\lambda)$ is the Fourier transform of the measure μ.

Proof :

$$\begin{aligned}
\frac{1}{T}\int_0^T \left|\int e^{-it\lambda} d\mu(\lambda)\right|^2 dt &= \frac{1}{T}\int_0^T \left(\int e^{-it\lambda} d\mu(\lambda)\right)\overline{\left(\int e^{-itv} d\mu(v)\right)} dt \\
&= \frac{1}{T}\int_0^T \left(\int e^{-it\lambda} d\mu(\lambda)\right)\left(\int \overline{e^{-itv}}\, d\overline{\mu}(v)\right) dt \\
&= \frac{1}{T}\int_0^T \left(\int e^{-it\lambda} d\mu(\lambda)\right)\left(\int e^{itv} d\overline{\mu}(v)\right) dt
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{T}\int_0^T \left|\int e^{-it\lambda} d\mu(\lambda)\right|^2 dt &= \frac{1}{T}\int_0^T \left(\int e^{-it\lambda} d\mu(\lambda)\right)\left(\int e^{itv} d\overline{\mu}(v)\right) dt \\
&= \frac{1}{T}\int_0^T \int\int e^{-it(\lambda-v)} d\mu(\lambda)\, d\overline{\mu}(v)\, dt \\
&= \int\int \frac{1}{T}\int_0^T e^{-it(\lambda-v)} dt\, d\mu(\lambda)\, d\overline{\mu}(v)
\end{aligned}
$$

- Note that

$$
\left|\frac{1}{T}\int_0^T e^{-it(\lambda-v)} dt\right| \leq 1,
$$

- If $\lambda \neq v$,

$$\frac{1}{T}\int_0^T e^{-it(\lambda-v)}dt = -\frac{1}{T}\ \frac{e^{-it(\lambda-v)}}{i(\lambda-v)}\Bigg|_0^T = \frac{1}{iT(\lambda-v)}(1-e^{-iT(\lambda-v)}).$$

Then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-it(\lambda-v)}dt = 0.$$

- If $\lambda = v$,

$$\frac{1}{T}\int_0^T e^{-it(\lambda-v)}dt = 1.$$

Therefore, we have that the function

$$f(T,\lambda,v) := \frac{1}{T}\int_0^T e^{-it(\lambda-v)}dt$$

is such that $|f| \leq 1$, $f(T,\lambda,v) \to 0$ for $\lambda \neq v$ and $f = 1$ for $\lambda = v$. Therefore, pointwise, when $T \to \infty$

$$f(T,\lambda,v) \to \chi_{\{(x,y);x=y\}}(\lambda,v).$$

Next we use Lebesgue's dominated convergence theorem to show

$$\begin{aligned}\lim_{T\to\infty}\int\int f(T,\lambda,u)d\mu(\lambda)d\overline{\mu}(v) &= \int\int \chi_{\{(x,y);x=y\}}(\lambda,v)d\mu(\lambda)d\overline{\mu}(v)\\ &= \int \mu(\{v\})d\overline{\mu}(v)\\ &= \sum_{v \text{ atom of } \mu} |\mu(\{v\})|^2.\end{aligned}$$

□

Previously on...

Last time we saw that electronic transport in disordered materials is studied using a random Schrödinger operator of the form

$$H_\omega = -\Delta + V_\omega, \quad \omega \in \Omega$$

where $(\Omega, \mathcal{B}, \mathbb{P})$ is a certain probability space.

At very strong disorder, there is no propagation of waves. The material is therefore an insulator. Mathematically, this is described by the notion of **dynamical localization**.

Absence of transport in the material represented by H_ω is described as : for any $\varphi \in \ell_c(\mathbb{Z}^d)$,

$$\sup_t \left\| |X|^p e^{-itH_\omega} \chi_I(H_\omega)\varphi \right\| < \infty$$

for all $p \geq 0$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Types of localization

- We say that the operator H_ω exhibits *spectral localization* in an interval I if $\sigma(H) \cap I = \sigma_{pp}(H) \cap I$, a.s.
- We say that H exhibits *Anderson localization* (AL) in I if $\sigma(H) \cap I = \sigma_{pp}(H) \cap I$ with exponentially decaying eigenfunctions, a.s.
- We say that H_ω exhibits *dynamical localization* (DL) in I if there exist constants $C < \infty$ and $c > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} |\langle \delta_y, e^{-itH_\omega} \chi_I(H_\omega)\delta_x \rangle| \right) \leq Ce^{-c|x-y|}$$

$$DL \Rightarrow \text{absence of transport}$$

$$DL \Rightarrow \text{AL} \Rightarrow \text{pp spectrum}$$

The Anderson model

Ergodic properties and spectrum

Some definitions from probability

- We consider a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, where $\mathcal{B}$ is a σ-algebra and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{B})$.
- Given a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, a random variable is a measurable function $X : \Omega \to \mathbb{R}$.
- The probability distribution of X is the measure μ defined by

$$\mu(A) = \mathbb{P}(\{\omega \in \Omega; X(\omega) \in A\}).$$

- The support of the measure μ is given by

$$\mathrm{supp}\,\mu := \{x \in \mathbb{R}; \mu([x-\varepsilon, x+\varepsilon]) > 0, \forall \varepsilon > 0\}.$$

- If for any $A \in \mathcal{B}$, $\mathbb{P}(Y(\omega) \in A) = \mathbb{P}(X(\omega) \in A) = \mu(A)$, we say X and Y are *identically distributed*.
- A collection of random variables $\{X_i\}_{i \in \mathbb{Z}^d}$ is called a *stochastic process*.

- A collection of random variables $\{X_n\}$ is called *independent* if, for any finite subset $\{n_1,...n_k\} \subset \mathbb{Z}^d$ and abritrary Borel sets $A_1,...,A_k \subset \mathbb{R}$,

$$\mathbb{P}(X_{n_1}(\omega) \in A_1,...,X_{n_k}(\omega) \in A_k) = \prod_{j=1}^{k} \mathbb{P}(X_{n_j}(\omega) \in A_j).$$

- If the collection of random variables $\{X_n\}$ is independent and identically distributed (i.i.d.), we have

$$\mathbb{P}(X_1(\omega) \in A,...,X_k(\omega) \in A) = \prod_{j=1}^{k} \mu(A).$$

- We will often consider $(\Omega,\mathcal{B},\mathbb{P}) = \left(\mathbb{R}^{\mathbb{Z}^d},\mathcal{B}_{\mathbb{R}},\bigotimes_{n\in\mathbb{Z}^d}\mu\right)$, where $\mathbb{R}^{\mathbb{Z}^d} := \bigotimes_{j\in\mathbb{Z}^d}\mathbb{R}$ and write $\omega := (\omega_n)_{n\in\mathbb{Z}^d}$ instead of $\{X_n(\omega)\}_{n\in\mathbb{Z}^d}$.

The Anderson model

$$H_\omega = -\Delta + \sum_{j \in \mathbb{Z}^d} \omega_j P_{\delta_j} \quad \text{on } \ell^2(\mathbb{Z}^d),$$

where $P_{\delta_j} = \langle \delta_j, \cdot \rangle \delta_j$.

- $-\Delta$ is the discrete Laplacian

$$-\Delta\varphi(n) = \sum_{m \sim n} \varphi(m) - \varphi(n),$$

- ω_j are i.i.d. random variables, with probability distribution μ with compact support $\mathbb{A}$.
- $\Omega := \mathbb{A}^{\mathbb{Z}^d} \ni \omega := (\omega_j)$. The probability space is the product space $(\Omega, \mathcal{B}, \mathbb{P})$ with the product σ-algebra of Borel sets $\mathcal{B}$ and the product probability measure

$$\mathbb{P} = \bigotimes_{j \in \mathbb{Z}^d} \mu.$$

Analogously, we can define the Anderson model on $\ell^2(\Gamma)$, for Γ a countable set. For ex., on a tree with branching number K, called the Bethe lattice $\mathbb{B}$.

The Anderson model

$$H_\omega = -\Delta + \underbrace{\sum_{j\in\mathbb{Z}^d} \omega_j P_{\delta_j}}_{V_\omega} \quad \text{on } \ell^2(\mathbb{Z}^d),$$

where $P_{\delta_j} = \langle \delta_j, \cdot \rangle \delta_j$. This operator acts in the following way

$$\begin{aligned}(H_\omega \varphi)(n) &= -\Delta\varphi + V_\omega(n)\varphi(n)\\ &= -\Delta\varphi + \omega_n \varphi(n).\end{aligned}$$

Since $\operatorname{supp}\mu$ is compact, the potential V_ω is bounded. Moreover, V_ω is self-adjoint on $\ell^2(\mathbb{Z}^d)$.

Since $-\Delta$ and V_ω are self-adjoint, the operator $H_\omega = -\Delta + V_\omega$ is self-adjoint in $\ell^2(\mathbb{Z}^d)$.

Definition

The map $\Omega \ni \omega \mapsto H_\omega \in \mathcal{L}(\mathcal{H})$ is measurable if for any $\varphi, \psi \in \mathcal{H}$, the map $\Omega \ni \omega \mapsto \langle \varphi, H_\omega \psi \rangle \in \mathbb{C}$ is measurable.

- The Anderson model $\omega \mapsto H_\omega$ on $\ell^2(\mathbb{Z}^d)$ is measurable.

Note that H_ω represents the *family* of operators $(H_\omega)_{\omega \in \Omega}$.

Definition

H_ω is called *ergodic* if there exists an ergodic group of transformations $(\tau_\gamma)_{\gamma\in\Gamma}$ acting on Ω associated to a family of unitary operators $(U_\gamma)_{\gamma\in\Gamma}$ on $\mathcal{H}$ s.t.

$$H_{\tau_\gamma(\omega)} = U_\gamma H_\omega U_\gamma^* \quad \text{for all}\, \gamma \in \Gamma.$$

• The Anderson model H_ω on $\ell^2(\mathbb{Z}^d)$ is ergodic with respect to $\mathbb{Z}^d$. That is, with respect to the translations $\tau_\gamma(\omega) = (\omega_{n+\gamma})_{n\in\mathbb{Z}^d}$ and $U_\gamma\varphi(n) = \varphi(n-\gamma)$ with $\gamma \in \mathbb{Z}^d$.

The Anderson model H_ω on $\ell^2(\mathbb{B})$ is ergodic w.r.t. a certain family of transformations in $\mathbb{B}$ (see Acosta-Klein'92).

• The Anderson model H_ω on $\ell^2(\mathbb{Z}^d)$ is ergodic with respect to $\mathbb{Z}^d$.
Indeed, recall the family $\{\tau_\gamma\}_{\gamma\in\mathbb{Z}^d}$ of translations on Ω given by

$$\tau_\gamma(\omega) = (\omega_{n-\gamma})_{n\in\mathbb{Z}^d},$$

and the family of unitary operators U_γ acting on $\ell^2(\mathbb{Z}^d)$ defined by

$$U_\gamma\varphi(n) = \varphi(n-\gamma), \quad \gamma\in\mathbb{Z}^d.$$

Note that U_γ^* is given by $U_\gamma^*\varphi(n) = \varphi(n+\gamma) = U_{-\gamma}$. Then

$$\begin{aligned}
U_\gamma H_\omega U_{-\gamma}\varphi(n) &= U_\gamma(-\Delta)U_{-\gamma}\varphi(n) + U_\gamma\left(V_\omega\, U_{-\gamma}\right)\varphi(n) \\
&= -\Delta\,\varphi(n) + \left(V_\omega\, U_{-\gamma}\varphi\right)(n-\gamma) \\
&= -\Delta\,\varphi(n) + V_\omega(n-\gamma)\left(U_{-\gamma}\varphi\right)(n-\gamma) \\
&= -\Delta\,\varphi(n) + V_\omega(n-\gamma)\varphi(n).
\end{aligned}$$

Recall that V_ω acts in the following way : $V_\omega\varphi(n) = \omega_n\varphi(n)$, for all $n\in\mathbb{Z}^d$.
Therefore $V_\omega(n-\gamma)\varphi(n) = \omega_{n-\gamma}\varphi(n) = V_{\tau_\gamma(\omega)}\varphi(n)$, and so

$$U_\gamma H_\omega U_{-\gamma}\varphi = H_{\tau_\gamma(\omega)}.$$

□

Spectrum

Theorem (Kunz-Souillard'80)

Let $H_\omega = -\Delta + V_\omega$ *be the Anderson model on* $\ell^2(\mathbb{Z}^d)$*. Then*

$$(*) \qquad \sigma(H_\omega) = \sigma(-\Delta) + \operatorname{supp}\mu \quad \textit{a.s.}$$

Remarks :

a) *For the Anderson model* H_ω *on* $\ell^2(\mathbb{Z}^d)$*,* $\sigma(-\Delta) = [-2d, 2d]$*.*

b) *For the Anderson model* H_ω *on* $\ell^2(\mathbb{B})$*,* $(*)$ *remains valid. In that case,* $\sigma(-\Delta_{\mathbb{B}}) = [-2\sqrt{K}, 2\sqrt{K}]$*, where* K *is the branching number of* $\mathbb{B}$*.*

See S. Golénia's course

The following will be crucial in our proof.

W. Kirsch describes this result as *"Whatever can happen, will happen, in fact, infinitely often".*

Proposition

There exists Ω_0 *such that :*
for any $\omega \in \Omega_0$*, any compact set* $\Lambda \subset \mathbb{Z}^d$*, any sequence* $\{q_i\}_{i\in\Lambda}$ *with* $q_i \in \operatorname{supp}\mu$ *and any* $\varepsilon > 0$*,*
there exists a sequence $\{\gamma_j\}_{j\in\mathbb{Z}^d} \subset \mathbb{Z}^d$ *with* $\|\gamma_j\| \to \infty$ *such that*

$$\sup_{n\in\Lambda} |V_\omega(n+\gamma_j) - q_n| < \varepsilon.$$

Now we can prove the theorem

Theorem (Kunz-Souillard'80)

Let $H_\omega = -\Delta + V_\omega$ *be the Anderson model on* $\ell^2(\mathbb{Z}^d)$*. Then*

$$\sigma(H_\omega) = \sigma(-\Delta) + \mathrm{supp}\,\mu \quad \textit{a.s.}$$

Proof :

- $\sigma(H_\omega) \subset \sigma(-\Delta) + \mathrm{supp}\,\mu$

One can show that $\sigma(V_\omega) = \mathrm{supp}\,\mu$ almost surely. One can also show that for a bounded operator B, and self-adjoint operator A,

$$\sigma(A+B) \subset \sigma(A) + [-\|B\|, \|B\|].$$

This, applied to V_ω and $-\Delta$ gives

$$\sigma(H_\omega) \subset \mathrm{supp}\,\mu + [-2\mathrm{d}, 2\mathrm{d}].$$

• $\sigma(-\Delta) + \operatorname{supp}\mu \subset \sigma(\mathrm{H}_\omega)$

We will use Weyl's criterion for the spectrum of the operator :

$$E \in \sigma(H) \iff \exists(\varphi_n) \subset \ell_c^2(\mathbb{Z}^d),\ \|\varphi_n\| = 1\ s.t.\ \|(H-E)\varphi_n\| \underset{n\to\infty}{\to} 0$$

Let $E \in \sigma(-\Delta) + \operatorname{supp}\mu$, that is,

$$E = E_0 + E_1 \text{ with } E_0 \in \sigma(-\Delta) \text{ and } E_1 \in \operatorname{supp}\mu$$

There exists a Weyl sequence (φ_j) for $-\Delta$ and E_0 s.t. $\varphi_j \in \ell_c(\mathbb{Z}^d)$, $\|\varphi_j\| = 1$ and

$$\|(-\Delta - E_0)\varphi_j\| \underset{j\to\infty}{\to} 0$$

Then

$$\begin{aligned}\|(H_\omega - E)\varphi_j\| &= \|(-\Delta + V_\omega - (E_0 + E_1))\varphi_j\| \\ &\leq \underbrace{\|(-\Delta - E_0)\varphi_j\|}_{\to 0} + \|(V_\omega - E_1)\varphi_j\|\end{aligned}$$

Note that for a fixed ω, $\|(V_\omega - E_1)\varphi_j\|$ is not necessarily small.

Fix j, φ_j and $\varepsilon := 1/j$. Note that $E_1 \in \operatorname{supp}\mu$, so we can apply the *"whatever can happen will happen"*-Proposition to

$$\Lambda = \operatorname{supp}\varphi_j, \quad \text{and} \quad \{q_i\}_{i\in\Lambda}, q_i = E_1, \forall i \in \Lambda$$

This says that for almost every $\omega \in \Omega$, there exists a sequence $\{\gamma_k^{(j)}\}_k \subset \mathbb{Z}^d$ with $\left\|\gamma_k^{(j)}\right\| \to \infty$ with k, such that

$$\sup_{n\in\operatorname{supp}\varphi_j} \left| V_\omega(n+\gamma_k^{(j)}) - E_1 \right| < \frac{1}{j}$$

Since $\left\|\gamma_k^{(j)}\right\| \to \infty$ with k, for every φ_j we can pick a k_j, $\gamma_{k_j}^{(j)}$ such that the sequence $\{\varphi_j(\cdot - \gamma_{k_j}^{(j)})\}_{j\in\mathbb{Z}^d}$ is orthogonal.

We define a new sequence $\tilde{\varphi}_j := \varphi_j(\cdot - \gamma_{k_j}^{(j)})$.

Note that for $\tilde{\varphi}_j := \varphi_j(\cdot - \gamma^{(j)}_{k_j})$ we have

$$\begin{aligned}
\|(V_\omega - E_1)\tilde{\varphi}_j\|^2 &= \sum_{n \in \operatorname{supp}\tilde{\varphi}_j} |(V_\omega(n) - E_1)\tilde{\varphi}_j(n)|^2 \\
&= \sum_{n \in \operatorname{supp}\tilde{\varphi}_j} \left|(V_\omega(n) - E_1)\varphi_j(n - \gamma^{(j)}_{k_j})\right|^2, \quad m = n - \gamma^{(j)}_{k_j} \\
&= \sum_{m \in \operatorname{supp}\varphi_j} \left|(V_\omega(m + \gamma^{(j)}_{k_j}) - E_1)\varphi_j(m)\right|^2 \\
&\leq \sup_{m \in \operatorname{supp}\varphi_j} \left|(V_\omega(m + \gamma^{(j)}_{k_j}) - E_1)\right|^2 \sum_{m \in \operatorname{supp}\varphi_j} |\varphi_j(m)|^2 \\
&\leq 1/j^2
\end{aligned}$$

Therefore,

$$\|(H_\omega - E)\tilde{\varphi}_j\| \leq \|(-\Delta - E_0)\tilde{\varphi}_j\| + \|(V_\omega - E_1)\tilde{\varphi}_j\| \underset{j\to\infty}{\to} 0.$$

That is, $\tilde{\varphi}_j$ is a Weyl sequence for H_ω and E, therefore $E \in \sigma(H_\omega)$. □

Proof of Proposition "Whatever can happen will happen"

We will need the following fundamental tool :

Lemma (Borel-Cantelli)

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of measurable sets. Define

$$A_\infty := \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\} = \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_n.$$

1) *If $\sum_n \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A_\infty) = 0$.*

2) *If $A_1, A_2, \dots A_n ..$ are independent and $\sum_n \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(A_\infty) = 1$.*

Proposition (Whatever can happen, will happen)

There exists Ω_0 *such that :*
for any $\omega \in \Omega_0$*, any compact set* $\Lambda \subset \mathbb{Z}^d$*, any sequence* $\{q_i\}_{i\in\Lambda}$ *with* $q_i \in \operatorname{supp}\mu$ *and any* $\varepsilon > 0$*,*
there exists a sequence $\{\gamma_j\}_{j\in\mathbb{Z}^d} \subset \mathbb{Z}^d$ *with* $\|\gamma_j\| \to \infty$ *such that*

$$\sup_{n\in\Lambda} |V_\omega(n+\gamma_j) - q_n| < \varepsilon.$$

Proof : Fix a compact set $\Lambda \subset \mathbb{Z}^d$, a sequence $\{q_i\}_{i\in\Lambda}$ with $q_i \in \operatorname{supp}\mu$ and $\varepsilon > 0$. Define

$$A := \{\omega \in \Omega : \sup_{n\in\Lambda} |V_\omega(n) - q_n| < \varepsilon\}.$$

Since $q_n \in \operatorname{supp}\mu$,

$$\mathbb{P}(A) > 0.$$

Now take a sequence $\gamma_j \in \mathbb{Z}^d$ such that $\|\gamma_m - \gamma_k\| > \operatorname{diam}(\Lambda)$ for $m \neq k$ and define

$$A_j := \{\omega \in \Omega : \sup_{n\in\Lambda} |V_\omega(n+\gamma_j) - q_n| < \varepsilon\}.$$

Since the $V_\omega(n)$ are i.i.d., A_j are independent and

$$\mathbb{P}(A_j) = \mathbb{P}(A) > 0 \quad \forall j,$$

therefore

$$\sum_j \mathbb{P}(A_j) = \infty.$$

$$A_j := \{\omega \in \Omega : \sup_{n\in\Lambda} |V_\omega(n+\gamma_j) - q_n| < \varepsilon\}, \quad \sum_j \mathbb{P}(A_j) = \infty.$$

Then, we can use the Borel-Cantelli lemma, and deduce that for

$$A_\infty(\Lambda, \{q_i\}, \varepsilon) := \{\omega \in \Omega : \ \omega \in A_j \text{ for infinitely many } j\},$$

we have

$$\mathbb{P}(A_\infty(\Lambda, \{q_i\}, \varepsilon)) = 1.$$

Now, we want to take all possible sets Λ. The space F of all finite subsets of $\mathbb{Z}^d$ is countable, then

$$\mathbb{P}\left(\bigcap_{\Lambda\in F} A_\infty(\Lambda, \{q_i\}, \varepsilon) \right) = 1.$$

We also want to consider all possible sequences $\{q_i\}$ with $q_i \in \operatorname{supp}\mu$. We can extract a countable dense subset Q of $\operatorname{supp}\mu$ and get

$$\mathbb{P}\left(\bigcap_{q_i \in Q} \bigcap_{\Lambda \in F} A_\infty(\Lambda, \{q_i\}, \varepsilon)\right) = 1.$$

We also want to have the estimate to hold for $\varepsilon > 0$ as small as we want. We can take $\varepsilon = 1/k$ with $k \in \mathbb{N}$, and define

$$\Omega_0 := \bigcap_{k \in \mathbb{N}} \bigcap_{q_i \in Q} \bigcap_{\Lambda \in F} A_\infty(\Lambda, \{q_i\}, \frac{1}{k})$$

and get $\mathbb{P}(\Omega_0) = 1$. This is the set Ω_0 we were looking for. □

Ergodic properties I

Recall that H_ω is *ergodic* if there exists an ergodic group of transformations $(\tau_\gamma)_{\gamma\in\Gamma}$ acting on Ω associated to a family of unitary operators $(U_\gamma)_{\gamma\in\Gamma}$ on $\mathcal{H}$ s.t.

$$H_{\tau_\gamma(\omega)} = U_\gamma H_\omega U_\gamma^* \quad \text{for all}\, \gamma \in \Gamma.$$

As a consequence of egodicity, we have

Theorem (Pastur'80, Kunz-Souillard'80, Kirsch-Martinelli '82)

If H_ω is an ergodic operator, there exist closed sets Σ, $\Sigma_{pp}, \Sigma_{ac}, \Sigma_{sc} \subset \mathbb{R}$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\Sigma = \sigma(H_\omega)$$

$$\Sigma_{pp} = \sigma_{pp}(H_\omega),\ \Sigma_{ac} = \sigma_{ac}(H_\omega),\ \Sigma_{sc} = \sigma_{sc}(H_\omega).$$

Ergodic properties II

Eigenvalue counting function : Let $\{\Lambda_L\}_{L\in\mathbb{N}}$ be a sequence of concentric cubes in $\mathbb{Z}^d$. Consider the restriction $H_\omega\restriction_{\Lambda_L} := \chi_{\Lambda_L} H_\omega \chi_{\Lambda_L}$. We define, for $E \in \mathbb{R}$,

$$N_L^\omega(E) := \frac{1}{\mathrm{vol}(\Lambda_L)} \sharp\{\text{e.v. of } H_\omega\restriction_{\Lambda_L} \leq E\}.$$

The Integrated Density of States (IDS) is defined as

$$N(E) := \lim_{L\to\infty} N_L^\omega(E).$$

- For the Anderson model H_ω on $\ell^2(\mathbb{Z}^d)$,
  - Existence : the limit exists for $\mathbb{P}$-a.e. $\omega \in \Omega$, and is deterministic.
  - Almost-sure spectrum : for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\overline{\{E :\ E \text{ is a growth point of } N\}} = \sigma(H_\omega)$$

Ergodic properties II

Eigenvalue counting function : Let $\{\Lambda_L\}_{L\in\mathbb{N}}$ be a sequence of concentric cubes in $\mathbb{Z}^d$. Consider the restriction $H_\omega\restriction_{\Lambda_L} = \chi_{\Lambda_L} H_\omega \chi_{\Lambda_L}$. We define, for $E \in \mathbb{R}$,

$$N_L^\omega(E) := \frac{1}{\mathrm{vol}(\Lambda)} \sharp\{\text{e.v. of } H_\omega\restriction_{\Lambda_L} \leq E\}.$$

The Integrated Density of States (IDS) is defined as

$$N(E) := \lim_{L\to\infty} N_L^\omega(E).$$

- For the Anderson model H_ω on $\ell^2(\mathbb{B})$,
  - ∗ Existence : the limit exists for $\mathbb{P}$-a.e. $\omega \in \Omega$, and is deterministic (for a particular μ, see Acosta-Klein'92).
  - ∗ Almost-sure spectrum : for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\overline{\{E : E \text{ is a growth point of } N\}} = \sigma(H_\omega)$$

Lifshitz tails

Let $E_0 = \inf \sigma(-\Delta + V_0)$, with V_0 periodic. The Integrated Density of States (IDS) for $H = -\Delta + V_0$ behaves as

$$N(E) \sim (E - E_0)^{d/2}, \quad E \searrow E_0.$$

On the other hand, the IDS for the Anderson model $H_\omega = -\Delta + V_\omega$, behaves near $E_0 = \inf \Sigma$ as

$$N(E) \sim e^{-(E-E_0)^{-d/2}} \quad E \searrow E_0 \quad \text{Lifshitz tails}$$

(see H. Najar's talk last Friday)

- For the Anderson model H_ω on $\ell^2(\mathbb{Z}^d)$,
  - ∗ The IDS decays exponentially near the bottom of the spectrum $\Rightarrow$ localization.
- For the Anderson model H_ω on $\ell^2(\mathbb{B})$,
  - ∗ The IDS decays exponentially near the bottom of the spectrum $\not\Rightarrow$ localization (see Hocker–Escuti - Schumacher'14).

Summary

We saw that the Anderson model H_ω in $\ell^2(\mathbb{Z}^d)$ is ergodic. That is, there exists an ergodic group of transformations $(\tau_\gamma)_{\gamma\in\Gamma}$ acting on Ω associated to a family of unitary operators $(U_\gamma)_{\gamma\in\Gamma}$ on $\mathcal{H}$ s.t.

$$H_{\tau_\gamma(\omega)} = U_\gamma H_\omega U_\gamma^* \quad \text{for all}\, \gamma \in \Gamma.$$

- ergodicity $\Rightarrow$ the spectrum of H_ω is deterministic.
  That is, there exists $\Sigma \subset \mathbb{R}$, such that

$$\sigma(H_\omega) = \Sigma \text{ for } \mathbb{P}\text{-a.e.}\, \omega \in \Omega.$$

- ergodicity $\Rightarrow$ the pp/sc/ac spectrum of H_ω is deterministic.
- For H_ω in $\ell^2(\mathbb{Z}^d)$, we can compute the exact set in $\mathbb{R}$ which corresponds to the deterministic spectrum.

- ergodicity $\Rightarrow$ existence of Integrated Density of States. Moreover, this function does not depend on $\omega \in \Omega$.
- The IDS gives another way to prove that the spectrum is deterministic.
- In some cases, the IDS gives also information on the localization region !

Reference

- W. Kirsch, *An invitation to Random Schrödinger Operators*, in Random Schrödinger Operators, Panoramas et Syntheses Vol. 25, 2008 (SMF).

The Anderson model

Results on localization and spectral type

$$\text{Let} \quad H_{\omega,\lambda} = -\Delta + \lambda V_{\omega}, \quad \lambda \in (0,\infty).$$

Now that we know that $H_{\omega,\lambda}$ has a deterministic spectrum, and the spectral types *pp, sc, ac* are also deterministic, we can ask :

For which energies in $\sigma(H_{\omega,\lambda})$ and strength of the disorder λ do we have localization, and for which energies and values of λ do we have delocalization ?

For the Anderson model on $\ell^2(\mathbb{Z}^d)$ there is a very good understanding of the region of localization (and in particular, the *pure point* part) in spectral band edges or at high disorder :

$$\sigma(H_{\omega,\lambda}) = \sigma_{pp}(H_{\omega,\lambda}) \cup \sigma_c(H_{\omega,\lambda}).$$

Unfortunately, the delocalization problem is still open.
However, for the Anderson model on $\ell^2(\mathbb{B})$ there is more information on delocalization.

Between the regions of localization and delocalization, there is a transition :

- spectral : transition between pp spectrum and ac spectrum.
- dynamical : transition between localization (absence of quantum transport) and delocalization (non-null quantum transport). Also called *metal-insulator* transport transition or Anderson transition.

Absence of quantum transport in the material represented by $H_{\omega,\lambda}$ is described as : for any $\varphi \in \ell_c(\mathbb{Z}^d)$,

$$\sup_t \left\| |X|^p e^{-itH_{\omega,\lambda}} \chi_I(H_{\omega,\lambda})\varphi \right\| < \infty$$

for all $p \geq 0$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$.
Recall that

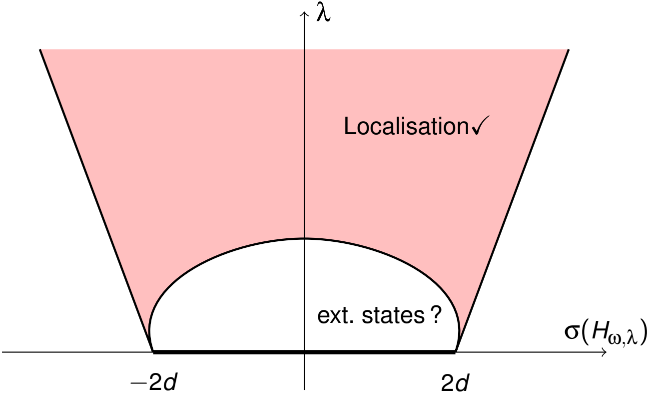
$$(DL) \Rightarrow \text{absence of transport} \Rightarrow \text{pp spectrum.}$$

Presence of quantum transport

$$\left\| |X|^p e^{-itH_{\omega,\lambda}} \chi_I(H_{\omega,\lambda})\varphi \right\| \to \infty \text{ as } t \to \infty$$

Phase diagram for $H_{\omega,\lambda}$ on $\ell^2(\mathbb{Z}^d)$, with $d \geq 2$

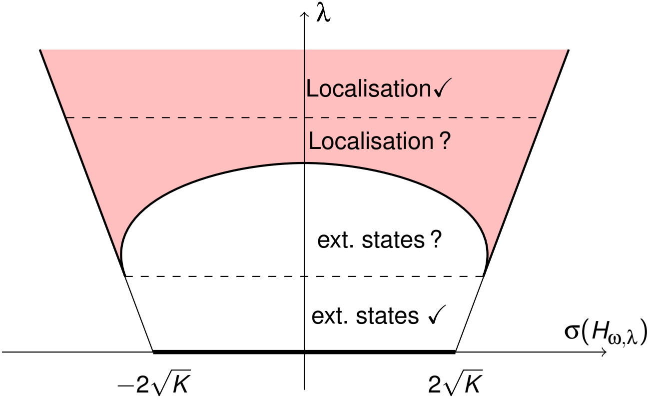
Transport (Anderson) transition : passage from *localized* to *extended states*.



Phase diagram for $H_{\omega,\lambda}$ on $\ell^2(\mathbb{B})$

$\mathbb{B}$: Bethe lattice with branching number $K+1$.

Transport (Anderson) transition : passage from *localized* to *extended states*.



How to prove localization ?

- Show the decay of the resolvent

$$G_{\omega,\lambda}(x,y;E+i\varepsilon) := \langle \delta_x, (H_{\omega,\lambda} - (E+i\varepsilon))^{-1}\delta_y\rangle,$$

when $\varepsilon \to 0$, for $E \in I$, for some open subset $I \subset \sigma(H_\omega)$, and $x, y \in \mathbb{Z}^d$.
This usually holds for I contained in the spectral edges.

- Use this decay to obtain

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} |\langle \delta_x, e^{-itH_{\omega,\lambda}}\chi_I(H_{\omega,\lambda})\delta_y\rangle|\right) \le Ce^{-c|x-y|}.$$

For example, one can use that, for $s \in (0,1)$ there exists C_s such that

$$\mathbb{E}\left(\sup_{f\in\mathcal{C}(\mathbb{R}),|f|\le 1} |\langle \delta_x, f(H)\chi_I(H)\delta_y\rangle|\right) \le C_s \liminf_{|\varepsilon|\to 0} \int_I \mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;E+i\varepsilon)\right|^s\right) dE.$$

There are other ways to link the resolvent to the spectrum.

For example, the Simon-Wolff Criterion : Let $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbb{Z}^d)$, such that the probability distribution of the random variables, μ, is absolutely continuous. Then, if for Lebesgue-a.e. $E \in I$ and $\mathbb{P}$-a.e. ω

$$\lim_{\varepsilon \to 0} \sum_{y \in \mathbb{Z}^d} \left| \langle \delta_y, (H_\omega - (E + i\varepsilon))^{-1} \delta_x \rangle \right|^2 < \infty,$$

then the spectral measure associated with δ_x is pure point in I for $\mathbb{P}$-a.e. ω.

For more examples, see S. Golénia's course.

Note that the resolvent $(H_{\omega,\lambda} - E)^{-1}$ is not defined for $E \in \sigma(H_{\omega,\lambda})$! The methods to prove localization need to deal with this problem.

Non-exhaustive list of results

Results on localization for the Anderson model on $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$

- $d = 1$: localization in the whole spectrum.
  Golsheid-Molchanov-Pastur '77, Kotani '82, Carmona '82, Simon '84, Damanik-Sims-Stolz '01 *(Bernoulli)*.

⚠ *It is conjectured that in $d = 2$ there is localization in the whole spectrum. So far, the methods only give localization at the edges of the spectrum. This is an open problem !*

Results on localization for the Anderson model on $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$

- $d \geq 2$: localization at the edges of the spectrum.
  - Multiscale Analysis (MSA)
    (Weak version) Prove that for some interval $I \subset \mathbb{R}$ the following holds : for some $\alpha > 1$, $p > 2d$ and $\gamma > 0$ and for all $E \in I \subset \mathbb{R}$, there is a sequence of cubes Λ_{L_k}, $L_{k+1} = L_k^\alpha$, $L_k \nearrow \mathbb{Z}^d$,

$$\mathbb{P}\left(\left| \langle \delta_x, (H_{\omega,\lambda} \restriction_{\Lambda_{L_k}} - E)^{-1} \delta_y \rangle \right| \leq e^{-\gamma L_k} \right) \geq 1 - \frac{1}{L_k^p}.$$

    Fröhlich-Spencer '83, von Dreifus-Klein '89, Combes-Hislop '94, Germinet-De Bièvre '98, Damanik-Stollmann '01, Germinet-Klein '01-'11, Bourgain-Kenig '06 *(Bernoulli)*.

⚠ *It is conjectured that in $d \geq 3$ there is a metal-insulator transition. This is an open problem !*

Results on localization for the Anderson model on $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$

- $d \geq 2$: localization at the edges of the spectrum.

  - Fractional Moment Method (FMM)

    Prove that for $I \subset \mathbb{R}$, the following holds : there exists $s \in (0,1)$ and $0 < c, C < \infty$ such that

$$\mathbb{E}\left(\left|\langle\delta_x, (H_{\omega,\lambda} - (E + i\varepsilon))^{-1}\delta_y\rangle\right|^s\right) \leq Ce^{-c\|x-y\|}$$

    uniformly in $E \in I$, $\varepsilon > 0$ and $x, y \in \mathbb{Z}^d$.

    Aizenman-Molchanov '93, Aizenman'96, Graf,
    Aizenman-Elgart-Hundertmark-Schenker '01,
    Aizenman-Elgart-Naboko-Schenker-Stolz '03.

It is conjectured that in $d \geq 3$ there is a metal-insulator transition. This is an open problem !

Results for the Anderson model on graphs (ex. $\ell^2(\mathbb{B})$)

- Localization
  Aizenman-Molchanov '93, Aizenman'94, Tautenhahn'11.
  Exner-Helm-Stollmann'08, Schubert'14, Hislop-Post'08
- Delocalization and ac spectrum, $\ell^2(\mathbb{B})$
  Klein '96- '98, Aizenman-Sims-Warzel'06, Froese-Hasler-Spitzer'06,'07,
  Halasan'09, Aizenman-Warzel'06–'16.
- Integrated Density of States.
  Acosta-Klein'92, Hoecker–Escuti-Schumacher'12 ($\mathbb{B}$), Antunović-Veselić'08

Results for the Anderson model on quantum graphs :
Klopp-Pankrashkin'08,'09, Aizenman-Sims-Warzel'06, Sabri'12.
Percolation graphs : Kirsch-Müller'06, Müller-Stollmann'.

For more results, see works by the "Chemnitz school" :
P. Stollmann, I. Veselić, D. Lenz,
and M. Keller, M. Tautenhahn, C. Schubert, C. Schumacher, etc.

Fractional Moment Method

Proof of localization at high disorder

Reference :
We follow closely Section 4 in G. Stolz's notes *An introduction to the mathematics of Anderson localization*, Contemporary Mathematics 551, 2010.

Recall that

- H_ω exhibits *dynamical localization* (DL) in I if there exist constants $C < \infty$ and $c > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} |\langle \delta_y, e^{-itH_\omega}\chi_I(H_\omega)\delta_x\rangle|\right) \leq Ce^{-c|x-y|}$$

Previously, we saw that $DL \Rightarrow$ AL $\Rightarrow$ pp spectrum, and $DL \Rightarrow$ absence of transport.
Absence of transport : for any $\varphi \in \ell_c(\mathbb{Z}^d)$,

$$\sup_t \left\| |X|^p e^{-itH_\omega}\chi_I(H_\omega)\varphi \right\| < \infty$$

for all $p \geq 0$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Goal : to prove (DL) for $H_\omega = -\Delta + \lambda V_\omega$, for large λ.

Goal : to prove (DL) for $H_\omega = -\Delta + \lambda V_\omega$, for large λ.

Theorem

Let $I \subset \mathbb{R}$ be a bounded open interval. If there exists $s \in (0,1)$, $0 < c$, $C < \infty$ such that

$$(*) \qquad \mathbb{E}\left(\left| \langle \delta_x, (H_{\omega,\lambda} - (E + i\varepsilon))^{-1} \delta_y \rangle \right|^s \right) \leq Ce^{-c\|x-y\|}$$

uniformly in $E \in I$, $\varepsilon > 0$ and $x, y \in \mathbb{Z}^d$. Then $H_{\omega,\lambda}$ exhibits dynamical localization in I.

Therefore, our goal becomes

Goal : to prove (*) for $H_\omega = -\Delta + \lambda V_\omega$, for large λ.

In the rest of this lecture, we will focus in showing

Theorem

Let $s \in (0,1)$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, there are constants $0 < c$, $C < \infty$ such that

$$(*) \qquad \mathbb{E}\left(\left| \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle \right|^s \right) \leq Ce^{-c\|x-y\|}$$

uniformly in $x, y \in \mathbb{Z}^d$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

We assume the random variables ω_n have an absolutely continuous probability distribution, with a continuous density, i.e., there exists $\rho \in \mathcal{C}(\mathbb{R})$ s.t.

$$d\mu(x) = \rho(x)dx$$

The proof relies on two results :

- An a priori bound on the fractional moment of the resolvent :

$$\mathbb{E}\left(\left|\langle\delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^s\right) \leq C(s, \lambda, \rho).$$

- A decoupling lemma : for ρ there exists a constant $C < \infty$ s.t., uniformly in α and $\beta \in \mathbb{C}$,

$$\int \frac{1}{|v-\beta|^s}\rho(v)dv \leq C \int \frac{|v-\alpha|^s}{|v-\beta|^s}\rho(v)dv$$

The *a priori* bound

Since the random variables ω_n have a probability density ρ, compactly supported and bounded, we can write

$$\mathbb{E}(\cdot) := \int_{\Omega} (\cdot) d\mathbb{P} = \int_{\mathbb{A}} \dots \int_{\mathbb{A}} (\cdot) \dots g(\omega_n) d\omega_n \dots$$

Lemma (A priori bound)

There exists a constant $C = (s, \rho) < \infty$ *such that*

$$\mathbb{E}\left(\left| \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle \right|^s \right) \leq \frac{C(s,\rho)}{\lambda^s},$$

for all $x, y \in \mathbb{Z}^d$ *and* $\lambda > 0$.

Proof : we will start by showing that

$$\mathbb{E}_{x,y}\left(\left| \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle \right|^s \right) \leq \frac{C(s,\rho)}{\lambda^s}.$$

We will use the conditional expectation with $(\omega_n)_{n\neq x,y}$ fixed.

$$\mathbb{E}_{x,y}(\cdot) = \int_{\mathbb{A}}\int_{\mathbb{A}} (\cdot)\rho(\omega_x)\,\rho(\omega_y)\,d\omega_x\,d\omega_y.$$

Note that if we are able to show

$$\mathbb{E}_{x,y}\left(\left|\langle\delta_x,(H_{\omega,\lambda}-z)^{-1}\delta_y\rangle\right|^s\right) \leq \frac{C(s,\rho)}{\lambda^s},$$

the r.h.s does not depend on $(\omega_n)_{n\notin\{x,y\}}$ anymore. We can then take the $\mathbb{E}$ with respect to the rest of the r.v. and obtain

$$\mathbb{E}\left(\left|\langle\delta_x,(H_{\omega,\lambda}-z)^{-1}\delta_y\rangle\right|^s\right) \leq \frac{C(s,\rho)}{\lambda^s},$$

which is the desired result.

Proof of the a priori bound

Goal : to obtain an upper bound for

$$\mathbb{E}_{x,y}\left(\left|\langle \delta_x, (H_{\omega,\lambda}-z)^{-1}\delta_y\rangle\right|^s\right),\ x,y\in\mathbb{Z}^d.$$

We split the proof in two cases : i) when $x=y$ and ii) when $x\neq y$.

i) Case $x=y$ (rank-one perturbation)
Recall that

$$H_{\omega,\lambda} = -\Delta + \sum_{n\in\mathbb{Z}^d}\omega_n P_n, \quad P_n := \langle \delta_n, \cdot\rangle\delta_n.$$

Write $\omega = (\hat{\omega}, \omega_x)$, where $\hat{\omega} = (\omega_n)_{n\neq x}$. Then

$$H_{\omega,\lambda} = H_{\hat{\omega},\lambda} + \lambda\omega_x P_x$$

Using the resolvent identity, we get

$$\left(H_{\omega,\lambda}-z\right)^{-1} = \left(H_{\hat{\omega},\lambda}-z\right)^{-1} - \lambda\omega_x\left(H_{\hat{\omega},\lambda}-z\right)^{-1}P_x\left(H_{\omega,\lambda}-z\right)^{-1}$$

$$\left(H_{\omega,\lambda}-z\right)^{-1}=\left(H_{\hat{\omega},\lambda}-z\right)^{-1}-\lambda\omega_x\left(H_{\hat{\omega},\lambda}-z\right)^{-1}P_x\left(H_{\omega,\lambda}-z\right)^{-1}$$

Now we take matrix-elements i.e. compute $\langle\delta_x,\cdot\rangle$ in both sides :

$$\begin{aligned}\langle\delta_x,\left(H_{\omega,\lambda}-z\right)^{-1}\delta_x\rangle &= \langle\delta_x,\left(H_{\hat{\omega},\lambda}-z\right)^{-1}\delta_x\rangle\\ &\quad -\lambda\omega_x\langle\delta_x,\left(H_{\hat{\omega},\lambda}-z\right)^{-1}\delta_x\rangle\langle\delta_x,\left(H_{\omega,\lambda}-z\right)^{-1}\delta_x\rangle\end{aligned}$$

In abbreviated form :

$$G_{\omega,\lambda}(x,x;z)=G_{\hat{\omega},\lambda}(x,x;z)-\lambda\omega_x G_{\hat{\omega},\lambda}(x,x;z)G_{\omega,\lambda}(x,x;z).$$

If we write $\alpha=\alpha(\hat{\omega},x,z):=\left(G_{\hat{\omega},\lambda}(x,x;z)\right)^{-1}$, then

$$G_{\omega,\lambda}(x,x;z)=\frac{1}{\alpha+\lambda\omega_x}.$$

Here, α is well-defined, because $\frac{\mathrm{Im}\,G_{\hat{\omega},\lambda}(x,x;z)}{\mathrm{Im}\,z}>0$.

$$G_{\omega,\lambda}(x,x;z) = \frac{1}{\alpha + \lambda\omega_x},$$

where $\alpha \in \mathbb{C}$ and does not depend on ω_x !
Suppose $\operatorname{supp}\rho \subset [-M, M]$. Then

$$\begin{aligned}\mathbb{E}_x\left(\left|G_{\omega,\lambda}(x,x;z)\right|^s\right) &= \int_{-M}^{M} \frac{1}{\left|\alpha + \lambda\omega_x\right|^s} \rho(\omega_x)\, d\omega_x \\ &\leq \frac{\|\rho\|_\infty}{\lambda^s} \int_{-M}^{M} \frac{1}{\left|\alpha\lambda^{-1} + \omega_x\right|^s} d\omega_x.\end{aligned}$$

The r.h.s is integrable, independent of α and λ. Therefore,

$$\mathbb{E}_x\left(\left|G_{\omega,\lambda}(x,x;z)\right|^s\right) \leq \frac{C(\rho, s)}{\lambda^s}.$$

which is the desired bound for $x = y$.

ii) Case $x \neq y$ (rank-two perturbation)
Recall that

$$H_{\omega,\lambda} = -\Delta + \sum_{n \in \mathbb{Z}^d} \omega_n P_n, \quad P_n := \langle \delta_n, \cdot \rangle \delta_n.$$

Write $\omega = (\hat{\omega}, \omega_x, \omega_y)$, with $\hat{\omega} = (\omega_n)_{n \notin \{x,y\}}$, then

$$H_{\omega,\lambda} = H_{\hat{\omega},\lambda} + \lambda \omega_x P_x + \lambda \omega_y P_y.$$

Writing $P = P_x + P_y$ and using the resolvent identity, we get

$$\left(H_{\omega,\lambda} - z\right)^{-1} = \left(H_{\hat{\omega},\lambda} - z\right)^{-1} - \left(H_{\omega,\lambda} - z\right)^{-1} \left(\lambda \omega_x P_x + \lambda \omega_y P_y\right) \left(H_{\hat{\omega},\lambda} - z\right)^{-1}$$

Now, we want to determine the matrix-elements (omit z for convenience)

$$\begin{pmatrix} G_{\omega,\lambda}(x,x) & G_{\omega,\lambda}(x,y) \\ G_{\omega,\lambda}(y,x) & G_{\omega,\lambda}(y,y) \end{pmatrix}$$

in terms of

$$\begin{pmatrix} G_{\hat{\omega},\lambda}(x,x) & G_{\hat{\omega},\lambda}(x,y) \\ G_{\hat{\omega},\lambda}(y,x) & G_{\hat{\omega},\lambda}(y,y) \end{pmatrix}$$

Using

$$\left(H_{\omega,\lambda}-z\right)^{-1}=\left(H_{\hat{\omega},\lambda}-z\right)^{-1}-\left(H_{\omega,\lambda}-z\right)^{-1}\left(\lambda\omega_x P_x+\lambda\omega_y P_y\right)\left(H_{\hat{\omega},\lambda}-z\right)^{-1}.$$

we can compute each matrix element, for ex.

$$G_{\omega,\lambda}(x,x)=G_{\hat{\omega},\lambda}(x,x)-\lambda\omega_x G_{\omega,\lambda}(x,x)G_{\hat{\omega},\lambda}(x,x)-\lambda\omega_y G_{\omega,\lambda}(x,y)G_{\hat{\omega},\lambda}(y,x).$$

After some computations... we get

$$\begin{aligned}\begin{pmatrix} G_{\omega,\lambda}(x,x) & G_{\omega,\lambda}(x,y) \\ G_{\omega,\lambda}(y,x) & G_{\omega,\lambda}(y,y)\end{pmatrix} &= \left[\begin{pmatrix} G_{\hat{\omega},\lambda}(x,x) & G_{\hat{\omega},\lambda}(x,y) \\ G_{\hat{\omega},\lambda}(y,x) & G_{\hat{\omega},\lambda}(y,y)\end{pmatrix}+\lambda\begin{pmatrix}\omega_x & 0 \\ 0 & \omega_y\end{pmatrix}\right]^{-1} \\ &=: \left[G_{\hat{\omega}}+\lambda\begin{pmatrix}\omega_x & 0 \\ 0 & \omega_y\end{pmatrix}\right]^{-1}\end{aligned}$$

Since $G_{\omega,\lambda}(x,y;z)$ is one element of the matrix, we can bound it by the norm of the matrix

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right)\leq\mathbb{E}_{x,y}\left(\left\|\left[G_{\hat{\omega}}+\lambda\begin{pmatrix}\omega_x & 0 \\ 0 & \omega_y\end{pmatrix}\right]^{-1}\right\|^s\right).$$

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{1}{\lambda^s}\mathbb{E}_{x,y}\left(\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}} + \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\right]^{-1}\right\|^s\right)$$

$$= \frac{1}{\lambda^s}\int\int\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}} + \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\right]^{-1}\right\|^s \rho(\omega_x)\,\rho(\omega_y)\,d\omega_x\,d\omega_y$$

$$\leq \frac{\|\rho\|_\infty^2}{\lambda^s}\int_{-M}^{M}\int_{-M}^{M}\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}} + \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\right]^{-1}\right\|^s d\omega_x\,d\omega_y,$$

Now, we would like to decouple the matrix with elements ω_x, ω_y, and isolate each term. For this, we do a change of variables

$$u = \frac{\omega_x + \omega_y}{2}, \quad v = \frac{\omega_x - \omega_y}{2},$$

and get

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{2\,\|\rho\|_\infty^2}{\lambda^s}\int_{-M}^{M}\int_{-M}^{M}\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}} + \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix} + u\mathbb{I}_{2x2}\right]^{-1}\right\|^s du\,dv$$

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{2\,\|\rho\|_\infty^2}{\lambda^s}\int_{-M}^{M}\int_{-M}^{M}\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}} + \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix} + u\mathbb{I}_{2x2}\right]^{-1}\right\|^s du\,dv$$

Note that the matrix

$$\frac{1}{\lambda}G_{\hat{\omega}} + \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix}$$

has either positive or negative imaginary part.

Therefore we can use the following result :

Lemma : *For all* 2x2 *matrices A such that either* $\mathrm{Im}A \geq 0$ *or* $\mathrm{Im}A \leq 0$*, one has*

$$\int_{-M}^{M}\left\|(A+u\mathbb{I})^{-1}\right\|^s du \leq C(M,s).$$

For a proof, see G. Stolz's notes.

We obtain

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq 4M\,\|\rho\|_\infty^2\,C(M,s)\frac{1}{\lambda^s}$$

□

Remarks

In the last proof we obtained the following

$$\begin{pmatrix} G_{\omega,\lambda}(x,x) & G_{\omega,\lambda}(x,y) \\ G_{\omega,\lambda}(y,x) & G_{\omega,\lambda}(y,y) \end{pmatrix} = \left[\begin{pmatrix} G_{\hat{\omega},\lambda}(x,x) & G_{\hat{\omega},\lambda}(x,y) \\ G_{\hat{\omega},\lambda}(y,x) & G_{\hat{\omega},\lambda}(y,y) \end{pmatrix} + \lambda \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix} \right]^{-1}$$

This is a special case of a more general result, called *the Krein formula*.

Theorem (Krein formula)

Let H be a self-adjoint operator on some Hilbert space $\mathcal{H}$. If

$$H = H_0 + W,$$

with W a finite rank operator satisfying

$$W = PWP$$

for some finite-dimensional orthogonal projection P, then, for z with $\mathrm{Im} z \neq 0$, we have

$$\left[P(H-z)^{-1}P\right] = \left[W + \left[P(H_0-z)^{-1}P\right]^{-1}\right]^{-1}$$

where the inverse is taken on the restriction to the range of P.

Let us recall that we want to prove the following

Theorem

Let $s \in (0,1)$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, there are constants $0 < c$, $C < \infty$ such that

$$(*) \qquad \mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^s\right) \leq Ce^{-c\|x-y\|}$$

uniformly in $x, y \in \mathbb{Z}^d$ and $z \in \mathbb{C}\setminus\mathbb{R}$.

Ingredients of the proof :

- The a priori bound on the fractional moment of the resolvent :

$$\mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^s\right) \leq C(s,\lambda,\rho).$$

- A decoupling lemma : for ρ there exists a constant $C' < \infty$ s.t., uniformly in α and $\beta \in \mathbb{C}$,

$$\int \frac{1}{|v-\beta|^s}\rho(v)dv \leq C\int \frac{|v-\alpha|^s}{|v-\beta|^s}\rho(v)dv$$

Proof of Theorem

Suppose $x \neq y$. Then $\langle \delta_x, \delta_y \rangle = 0$ and

$$\begin{aligned}
\langle \delta_x, \delta_y \rangle &= \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} (H_{\omega,\lambda} - z)\, \delta_y \rangle \\
&= \left\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} (-\Delta \delta_y - (V_\omega - z)\delta_y) \right\rangle \\
&= \left\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \left(-\sum_{u \sim y} \delta_u - (\lambda \omega_y - z)\delta_y \right) \right\rangle \\
&= \left\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \left(-\sum_{u \sim y} \delta_u \right) \right\rangle + (\lambda \omega_y - z) \left\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \right\rangle \\
&= -\sum_{u \sim y} G_{\omega,\lambda}(x, u; z) + (\lambda \omega_y - z) G_{\omega,\lambda}(x, y; z).
\end{aligned}$$

One can compute that

$$G_{\omega,\lambda}(x, y; z) = \frac{a}{\lambda \omega_y - b},$$

where a and b do not depend on ω_y.

$$\begin{aligned}
\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) &= \frac{1}{\lambda^s}\mathbb{E}\left(\frac{|a|^s}{\left|\omega_y - \frac{b}{\lambda}\right|^s}\right) \\
&\leq \frac{C'}{\lambda^s}\mathbb{E}\left(\frac{\left|\omega_y - \frac{z}{\lambda}\right|^s |a|^s}{\left|\omega_y - \frac{b}{\lambda}\right|^s}\right) \qquad \text{decoupling lemma} \\
&= \frac{C'}{\lambda^s}\mathbb{E}\left(|\lambda\omega_y - z|^s \left|G_{\omega,\lambda}(x,y;z)\right|^s\right)
\end{aligned}$$

where we used that

$$G_{\omega,\lambda}(x,y;z) = \frac{a}{\lambda\omega_y - b}.$$

Recall that we had shown that

$$(\lambda\omega_y - z)G_{\omega,\lambda}(x,y;z) = \sum_{u\sim y} G_{\omega,\lambda}(x,u;z).$$

Therefore, using that $(\sum_n |a_n|)^s \leq \sum_n |a_n|^s$, we get

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{C'}{\lambda^s}\sum_{u\sim y}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right).$$

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{C'}{\lambda^s}\sum_{u\sim y}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right).$$

If none of the points u is equal to x, we can iterate this argument.

$$\begin{aligned}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) &\leq \frac{C'}{\lambda^s}\sum_{u\sim y}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right)\\ &\leq \frac{C'}{\lambda^s}(\sharp\text{of neighbors})\max_{u,u\sim y}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right)\\ &\leq \left(\frac{C'}{\lambda^s}\right)^2(\sharp\text{of neighbors})\sum_{u'\sim u}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u';z)\right|^s\right)\end{aligned}$$

iterating this argument, at each step we get a factor

$$\left(\frac{C'}{\lambda^s}\right)(\sharp\text{of neighbors})$$

We can iterate this argument at most $\|x-y\|$ times,

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \left(\left(\frac{C'}{\lambda^s}\right)^2 (\sharp\text{of neighbors})\right)^{\|x-y\|} \sup_{u\in\mathbb{Z}^d} \mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right)$$

We can bound the r.h.s using the a priori bound and get

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{C(\rho,s)}{\lambda^s} \left(\left(\frac{C'}{\lambda^s}\right)^2 (\sharp\text{of neighbors})\right)^{\|x-y\|}$$

Finally, we take λ large enough such that

$$\left(\left(\frac{C'}{\lambda^s}\right)^2 2d\right) < 1.$$

Then, we have

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{C(\rho,s)}{\lambda^s} e^{-C(C',\lambda,s,d)\|x-y\|}.$$

□

We have shown

Theorem

Let $s \in (0,1)$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, there are constants $0 < c$, $C < \infty$ such that

$$(*) \qquad \mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^s\right) \leq Ce^{-c\|x-y\|}$$

uniformly in $x, y \in \mathbb{Z}^d$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

With this result, we can prove dynamical localization, and pure point spectrum. For a proof of dynamical localization, see Section 5 in G. Stolz's notes.

Theorem (The Simon-Wolff Criterion, Simon-Wolff'86)

Let Γ be a countable set of points. Let $H_\omega = -\Delta + V_\omega$ on $\ell^2(\Gamma)$, such that the probability distribution of the random variables, μ, is absolutely continuous. Then, for any Borel set I :

- *If for Lebesgue-a.e. $E \in I$ and $\mathbb{P}$-a.e. ω*

$$\lim_{\varepsilon \to 0} \sum_{y \in \Gamma} \left| \langle \delta_y, (H_\omega - (E + i\varepsilon))^{-1} \delta_x \rangle \right|^2 < \infty,$$

  then for $\mathbb{P}$-a.e. ω, the spectral measure of H associated to δ_x is pure point in I.

- *If for Lebesgue-a.e. $E \in I$ and $\mathbb{P}$-a.e. ω*

$$\lim_{\varepsilon \to 0} \sum_{y \in \Gamma} \left| \langle \delta_y, (H_\omega - (E + i\varepsilon))^{-1} \delta_x \rangle \right|^2 = \infty,$$

  then for $\mathbb{P}$-a.e. ω, the spectral measure of H associated to δ_x is continuous in I.

To prove pp spectrum, we would like to use the Simon-Wolff Criterion. Recall our result, which holds for any given $s \in (0,1)$, in the whole spectrum with λ large enough, uniformly on $z = E + i\varepsilon$, $\varepsilon > 0$,

$$\mathbb{E}\left(\left|\langle\delta_x,(H_{\omega,\lambda}-z)^{-1}\delta_y\rangle\right|^s\right) \leq Ce^{-c\|x-y\|}$$

Then

$$\mathbb{E}\left(\sum_y \left|\langle\delta_x,(H_{\omega,\lambda}-z)^{-1}\delta_y\rangle\right|^s\right) \leq \sum_y \mathbb{E}\left(\left|\langle\delta_x,(H_{\omega,\lambda}-z)^{-1}\delta_y\rangle\right|^s\right) < \infty.$$

which implies that

$$\sum_y \left|\langle\delta_x,(H_{\omega,\lambda}-z)^{-1}\delta_y\rangle\right|^s < \infty \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Because the bound is uniform on ε, we an take the limit when $\varepsilon \to 0$.

We use the inequality : *If* $s \in (0,1)$*,*

$$\left(\sum_n |a_n|\right)^s \leq \sum_n |a_n|^s .$$

Take $s = 1/4$,

$$\left(\sum_y \left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle\right|^2\right)^{\frac{1}{4}} \leq \sum_y \left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle\right|^{\frac{1}{2}} < \infty$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Therefore, by the Simon-Wolff Criterion, the spectral measure associated to H_ω and δ_x is pure point in the deterministic spectrum of H_ω, for $\mathbb{P}$-a.e. $\omega \in \Omega$. Since this holds for every δ_x, one can deduce that

$$\sigma(H_\omega) = \sigma_{pp}(H_\omega) \quad \text{for } \mathbb{P}\text{-a.e.}\, \omega \in \Omega.$$

Summary

We have seen for $H_{\omega,\lambda} = -\Delta + \lambda V_\omega$ that

- For any given $s \in (0,1)$, for large values of λ,

$$(*) \qquad \mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^s\right) \leq Ce^{-c\|x-y\|}$$

  uniformly in $x, y \in \mathbb{Z}^d$ and $z \in \mathbb{C} \setminus \mathbb{R}$.
- The last expression implies the summability of the terms $\left|G_{\omega,\lambda}(x,y;E+i0)\right|^2$, almost surely, with $E \in \mathbb{R}$.
- The Simon-Wolff theorem relates the summability of the resolvent with the pure point spectrum or the continuous spectrum.
- The operator $H_{\omega,\lambda} = -\Delta + \lambda V_\omega$, for large values of λ exhibits localization in the whole spectrum.
- In the proof, it was crucial that one can isolate the dependence of the resolvent on the random variables corresponding to one or two sites ω_x.
- The other ingredient was the regularity of the probability distribution μ.

Summary II

We have seen so far,

- The Anderson model is used to study electronic transport in a disordered medium.
- There are different notions of localization.
- The Anderson model is an example of an ergodic operator, and it has a deterministic spectrum, which we can compute explicitly.
- the Integrated Density of States exists and gives information on the deterministic spectrum.

  These results are also valid for the Anderson model on graphs
- There are two methods to prove localization for dimension $d \geq 2$: the Multiscale Analysis and the Fractional Moment Method.

Thank you !

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