Harmonic polynomials and its properties

Let $\mathbb{R}$ be the field of reals and $\mathbb{R}[X_1, \ldots, X_n]$ the polynomial ring over $\mathbb{R}$ with variables $X_1, \ldots, X_n$ and write $|X|^2 = \sum_{i=1}^n X_i^2$. Recall that a polynomial $h \in \mathbb{R}[X_1, \ldots, X_n]$ is called to be *harmonic* if $\Delta h = 0$, where Δ the Laplacian.

The aim of this talk to sketch a proof of the following result :

Theorem : *Every polynomial $p \in \mathbb{R}[X_1, \ldots, X_n]$ can be uniquely written in the form*

$$p = h_0 + |X|^2 h_2 + \cdots + |X|^{2k} h_k$$

for some $k \geq 0$, where $h_0, \ldots, h_k$ are harmonic polynomials.

For $n = 2$, we easily deduce from that the only polynomials invariant with respect to the action of the group $O(2)$ are generated by $|X|^2 = X_1^2 + X_2^2$.