A free energy diminishing DDFV scheme for convection-diffusion equations

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Joint work with C. Cancès (Lille) and S. Krell (Nice)

Outline of the talk

About convection-diffusion equations

Model problem : Fokker-Planck equation

$$\begin{cases} \partial_t u + \mathrm{div}\mathbf{J} = 0, \quad \mathbf{J} = \mathbf{\Lambda}(-\nabla u - u\nabla V), \text{ in } \Omega \times (0,T) \\ + \text{ Neumann boundary conditions} \\ u(\cdot, 0) = u_0 \geq 0 \end{cases}$$

Examples

- Semiconductor models, corrosion models
  - ➠ $\mathbf{\Lambda} = \mathbf{I}$
  - ➠ coupling with a Poisson equation for V
- Porous media flow
  - ➠ $\mathbf{\Lambda}$ bounded, symmetric and uniformly elliptic
  - ➠ $V = gz$

Assumptions : $V \in C^1(\Omega, \mathbb{R}^+)$, $\quad \int_\Omega u_0 > 0.$

Structural properties

$$\begin{cases} \partial_t u + \mathrm{div}\mathbf{J} = 0, \quad \mathbf{J} = \mathbf{\Lambda}(-\nabla u - u\nabla V), \\ u(\cdot, 0) = u_0 \geq 0 \ + \text{Neumann boundary conditions} \end{cases}$$

- Existence and uniqueness of the solution
- Nonnegativity of u and mass conservation
- An energy/energy dissipation relation : $\dfrac{d\mathbb{E}}{dt} + \mathbb{I} = 0$

$$\mathbb{E}(t) = \int_\Omega (H(u) + Vu)dx, \quad (H(s) = s\log s - s + 1)$$

$$\mathbb{I}(t) = \int_\Omega u\mathbf{\Lambda}\nabla(\log u + V) \cdot \nabla(\log u + V)dx$$

- Convergence towards the thermal equilibrium :

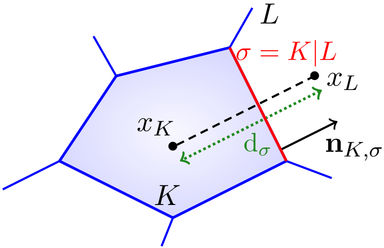
$$u_\infty = \lambda e^{-V} \ (\Longrightarrow \mathbf{J} = 0)$$

$\mathbf{\Lambda} = \mathbf{I}$, TPFA scheme on admissible meshes

$$\begin{cases} \partial_t u + \mathrm{div}\mathbf{J} = 0, \quad \mathbf{J} = -\nabla u - u\nabla V, \text{ in } \Omega \\ u(\cdot, 0) = u_0 \geq 0 \ + \text{Neumann boundary conditions} \end{cases}$$

Classical TPFA scheme

- $\mathcal{T}$: control volumes, $K \in \mathcal{T}$
- $\mathcal{E}$: edges, $\sigma \in \mathcal{E}$
- Δt : time step



$$\begin{cases} \mathrm{m}(K)\dfrac{u_K^{n+1} - u_K^n}{\Delta t} + \displaystyle\sum_{\sigma \in \mathcal{E}_K^{int}} \mathcal{F}_{K,\sigma}^{n+1} = 0 \\ \mathcal{F}_{K,\sigma} \approx \displaystyle\int_\sigma (-\nabla u - u\nabla V) \cdot \mathbf{n}_{K,\sigma} \end{cases}$$

$\mathbf{\Lambda} = \mathbf{I}$, TPFA scheme on admissible meshes

$$\begin{cases} \partial_t u + \mathrm{div}\mathbf{J} = 0, \quad \mathbf{J} = -\nabla u - u\nabla V, \text{ in } \Omega \\ u(\cdot, 0) = u_0 \geq 0 \quad + \text{ Neumann boundary conditions} \end{cases}$$

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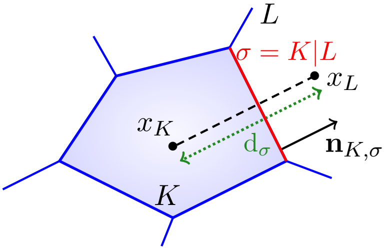


$$\begin{cases} \mathrm{m}(K)\dfrac{u_K^{n+1} - u_K^n}{\Delta t} + \displaystyle\sum_{\sigma \in \mathcal{E}_K^{int}} \mathcal{F}_{K,\sigma}^{n+1} = 0 \\ \mathcal{F}_{K,\sigma} = \dfrac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B\big(V_L - V_K\big)u_K - B\big(-V_L + V_K\big)u_L\Big) \end{cases}$$

$$B_{up}(s) = 1 + s^-, \; B_{ce}(s) = 1 - s/2$$

TPFA + Scharfetter-Gummel fluxes

Definition ❑ SCHARFETTER, GUMMEL, 1969

$$\mathcal{F}_{K,\sigma} = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}\Big(B_{sg}\big(V_L - V_K\big)u_K - B_{sg}\big(-V_L + V_K\big)u_L\Big)$$

$$B_{sg}(x) = \frac{x}{e^x - 1} \quad (B_{sg}(0) = 1).$$

Properties

- Existence, uniqueness of the solution to the scheme
- Preservation of positivity, conservation of mass
- Preservation of the thermal equilibrium :

$$u_K = \lambda \exp(-V_K) \Longrightarrow \mathcal{F}_{K,\sigma} = 0.$$

- Discrete counterpart of the energy/dissipation relation

  ❑ CHATARD, 2011

Motivation

Main drawbacks of the TPFA scheme

- Admissibility of the mesh
- $\mathbf{\Lambda} = \mathbf{I}$

Requirements wanted for a new scheme

- To be applicable on almost-general meshes
- To be applicable for anisotropic equations
- To preserve thermal equilibrium
- To be energy-diminishing
- To ensure the positivity

Outline of the talk

Introduction to DDFV schemes

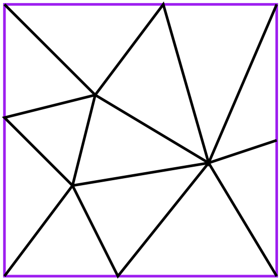
Some (partial) references

- Domelevo, Omnes, 2005
- Coudière, Vila, Villedieu, 1999
- Andreianov, Boyer, Hubert, 2007
- Andreianov, Bendahmane, Karlsen, 2010
- ...

Principles (for diffusion equations)

- Unknowns located at the centers and the vertices of the mesh
- Discrete gradient defined on a diamond mesh
- Discrete divergence defined on primal and dual meshes
- Integration of the equation on primal cells and dual cells
- Discrete-duality formula

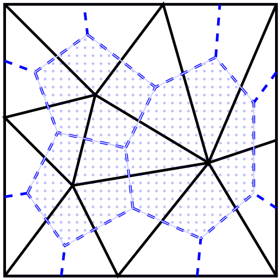
Meshes : primal and dual meshes



$\mathfrak{M}$: primal mesh
$\partial\mathfrak{M}$: exterior primal mesh

➠ approximate values :

$$(u_K)_{K\in\mathfrak{M}\cup\partial\mathfrak{M}}$$



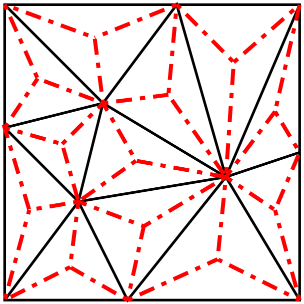
$\mathfrak{M}^*$: interior dual mesh
$\partial\mathfrak{M}^*$: exterior dual mesh

➠ approximate values :

$$(u_{K^*})_{K^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}$$

$$u_{\mathcal{T}} = \left((u_K)_{K\in\mathfrak{M}\cup\partial\mathfrak{M}}, (u_{K^*})_{K^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\right)$$

Meshes : diamond mesh

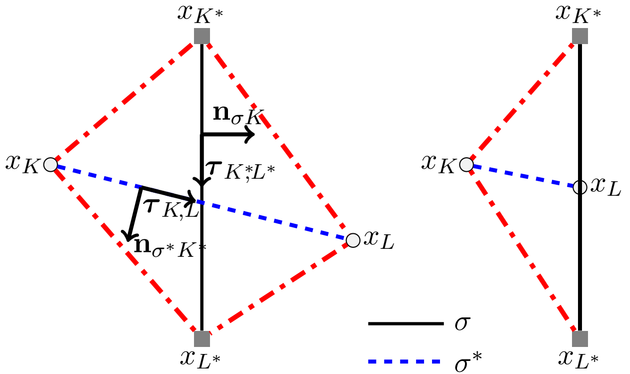


$\mathfrak{D}$: diamond mesh

➟ for the definition of the discrete gradient

$$\nabla^{\mathfrak{D}} \;:\; \mathbb{R}^{\mathcal{T}} \to (\mathbb{R}^2)^{\mathfrak{D}}$$
$$u_{\mathcal{T}} \mapsto (\nabla^{\mathcal{D}} u_{\mathcal{T}})_{\mathcal{D}\in\mathfrak{D}}$$

Discrete gradient operator



$$\nabla^{\mathfrak{D}} u_{\mathcal{T}} = (\nabla^{\mathcal{D}} u_{\mathcal{T}})_{\mathcal{D}\in\mathfrak{D}} \text{ with } \begin{cases} \nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \boldsymbol{\tau}_{K^*,L^*} = \dfrac{u_{L^*} - u_{K^*}}{\mathrm{m}_\sigma}, \\ \nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \boldsymbol{\tau}_{K,L} = \dfrac{u_L - u_K}{\mathrm{m}_{\sigma^*}}. \end{cases}$$

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{2\mathrm{m}_{\mathcal{D}}}\Big(\mathrm{m}_\sigma (u_L - u_K)\mathbf{n}_{\sigma K} + \mathrm{m}_{\sigma^*}(u_{L^*} - u_{K^*})\mathbf{n}_{\sigma^* K^*}\Big).$$

Discrete divergence operator

On the continuous level

$$\int_K \mathrm{div}(\boldsymbol{\xi}(x))\mathrm{d}x = \sum_{\sigma\in\partial K}\int_\sigma \boldsymbol{\xi}(s)\cdot\mathbf{n}_{\sigma K}\mathrm{d}s, \qquad \forall K\in\mathfrak{M}.$$

On the discrete level

$$\forall\ K\in\mathfrak{M},\qquad \mathrm{div}_K\boldsymbol{\xi}_{\mathfrak{D}} = \frac{1}{\mathrm{m}_K}\sum_{\substack{\mathcal{D}\in\mathfrak{D}_K\\ \mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}}} \mathrm{m}_\sigma\ \boldsymbol{\xi}_{\mathcal{D}}\cdot\mathbf{n}_{\sigma K},$$

$$\forall\ K^*\in\mathfrak{M}^*,\quad \mathrm{div}_{K^*}\boldsymbol{\xi}_{\mathfrak{D}} = \frac{1}{\mathrm{m}_{K^*}}\sum_{\substack{\mathcal{D}\in\mathfrak{D}_{K^*}\\ \mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}}} \mathrm{m}_{\sigma^*}\ \boldsymbol{\xi}_{\mathcal{D}}\cdot\mathbf{n}_{\sigma^* K^*},$$

$$\begin{aligned}\mathrm{div}^{\mathcal{T}}:\quad (\mathbb{R}^2)^{\mathfrak{D}} &\to \mathbb{R}^{\mathcal{T}}\\ \boldsymbol{\xi}_{\mathfrak{D}} &\mapsto \Big((\mathrm{div}_K\boldsymbol{\xi}_{\mathfrak{D}})_{K\in\mathfrak{M}}, 0, (\mathrm{div}_{K^*}\boldsymbol{\xi}_{\mathfrak{D}})_{K^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\Big)\end{aligned}$$

Discrete duality property

Scalar products and norms

$$\llbracket v_{\mathcal{T}}, u_{\mathcal{T}} \rrbracket_{\mathcal{T}} = \frac{1}{2}\left(\sum_{K\in\mathfrak{M}} \mathrm{m}_K u_K v_K + \sum_{K^*\in\overline{\mathfrak{M}^*}} \mathrm{m}_{K^*} u_{K^*} v_{K^*} \right),$$

$$(\boldsymbol{\xi}_{\mathfrak{D}}, \boldsymbol{\varphi}_{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\, \boldsymbol{\xi}_{\mathcal{D}} \cdot \boldsymbol{\varphi}_{\mathcal{D}},$$

On the continuous level : the Green formula

$$\int_\Omega \mathrm{div}\boldsymbol{\xi}\ v = -\int_\Omega \boldsymbol{\xi} \cdot \nabla v \quad + \int_{\partial\Omega} \boldsymbol{\xi} \cdot \mathbf{n}\ v$$

On the discrete level : the discrete duality formula

$$\llbracket \mathrm{div}^{\mathcal{T}} \boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}} \rrbracket_{\mathcal{T}} = -(\boldsymbol{\xi}_{\mathfrak{D}}, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathfrak{D}} \quad + \langle \gamma^{\mathfrak{D}}(\boldsymbol{\xi}_{\mathfrak{D}}) \cdot \mathbf{n}, \gamma^{\mathcal{T}}(v_{\mathcal{T}}) \rangle_{\partial\Omega},$$

DDFV scheme for an anisotropic diffusion equation

$$\begin{cases} -\operatorname{div} \boldsymbol{\Lambda} \nabla u = f \\ + \text{ boundary conditions} \end{cases}$$

The DDFV scheme

$$\begin{cases} -\operatorname{div}^{\mathcal{T}} \Big(\boldsymbol{\Lambda}_{\mathfrak{D}} \nabla^{\mathfrak{D}} u_{\mathcal{T}} \Big) = f_{\mathcal{T}} \\ + \text{ boundary conditions} \end{cases}$$

"Variational" formulation

$$(\boldsymbol{\Lambda}_{\mathfrak{D}} \nabla^{\mathfrak{D}} u_{\mathcal{T}}, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathfrak{D}} = [\![f_{\mathcal{T}}, v_{\mathcal{T}}]\!]_{\mathcal{T}} \quad \forall v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$$

DDFV scheme for an anisotropic diffusion equation

$$\begin{cases} -\operatorname{div} \mathbf{\Lambda} \nabla u = f \\ + \text{ boundary conditions} \end{cases}$$

The DDFV scheme

$$\begin{cases} -\operatorname{div}^{\mathcal{T}} \Big(\mathbf{\Lambda}_{\mathfrak{D}} \nabla^{\mathfrak{D}} u_{\mathcal{T}} \Big) = f_{\mathcal{T}} \\ + \text{ boundary conditions} \end{cases}$$

"Variational" formulation

$$\underbrace{(\mathbf{\Lambda}_{\mathfrak{D}} \nabla^{\mathfrak{D}} u_{\mathcal{T}}, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathfrak{D}}}_{(\nabla^{\mathfrak{D}} u_{\mathcal{T}}, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathbf{\Lambda},\mathfrak{D}}} = [\![f_{\mathcal{T}}, v_{\mathcal{T}}]\!]_{\mathcal{T}} \quad \forall v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$$

DDFV scheme for an anisotropic diffusion equation

$$\begin{cases} -\operatorname{div}\mathbf{\Lambda}\nabla u = f \\ + \text{ boundary conditions} \end{cases}$$

The DDFV scheme

$$\begin{cases} -\operatorname{div}^{\mathcal{T}}\left(\mathbf{\Lambda}_{\mathfrak{D}}\nabla^{\mathfrak{D}}u_{\mathcal{T}}\right) = f_{\mathcal{T}} \\ + \text{ boundary conditions} \end{cases}$$

"Variational" formulation

$$\underbrace{(\mathbf{\Lambda}_{\mathfrak{D}}\nabla^{\mathfrak{D}}u_{\mathcal{T}}, \nabla^{\mathfrak{D}}v_{\mathcal{T}})_{\mathfrak{D}}}_{(\nabla^{\mathfrak{D}}u_{\mathcal{T}}, \nabla^{\mathfrak{D}}v_{\mathcal{T}})_{\mathbf{\Lambda},\mathfrak{D}}} = [\![f_{\mathcal{T}}, v_{\mathcal{T}}]\!]_{\mathcal{T}} \quad \forall v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$$

with $(\boldsymbol{\xi}_{\mathfrak{D}}, \boldsymbol{\varphi}_{\mathfrak{D}})_{\mathbf{\Lambda},\mathfrak{D}} = \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\, \boldsymbol{\xi}_{\mathcal{D}} \cdot \mathbf{\Lambda}_{\mathcal{D}}\boldsymbol{\varphi}_{\mathcal{D}}, \quad \mathbf{\Lambda}_{\mathcal{D}} = \frac{1}{\mathrm{m}_{\mathcal{D}}}\int_{\mathcal{D}} \mathbf{\Lambda}.$

Structure of the scalar product of two discrete gradients

Discrete gradient

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{2\mathrm{m}_{\mathcal{D}}}\Big(\mathrm{m}_{\sigma}(u_L - u_K)\mathbf{n}_{\sigma K} + \mathrm{m}_{\sigma^*}(u_{L^*} - u_{K^*})\mathbf{n}_{\sigma^* K^*}\Big).$$

$$\delta^{\mathcal{D}} u_{\mathcal{T}} = \begin{pmatrix} u_K - u_L \\ u_{K^*} - u_{L^*} \end{pmatrix}$$

Scalar product

$$\begin{aligned}(\nabla^{\mathfrak{D}} u_{\mathcal{T}}, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathbf{\Lambda},\mathfrak{D}} &= \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\ \nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \mathbf{\Lambda}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}}, \\ &= \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\ \delta^{\mathcal{D}} u_{\mathcal{T}} \cdot \mathbb{A}_{\mathcal{D}} \delta^{\mathcal{D}} v_{\mathcal{T}}.\end{aligned}$$

Local matrices $\mathbb{A}_{\mathcal{D}}$

➟ Uniform bound on $\mathrm{Cond}_2(\mathbb{A}_{\mathcal{D}})$

Outline of the talk

Nonlinear formulation of the problem

$$\begin{cases} \partial_t u + \text{div}\mathbf{J} = 0, \quad \mathbf{J} = -u\mathbf{\Lambda}\nabla(\log u + V), \text{ in } \Omega \times (0,T) \\ u(\cdot,0) = u_0 \geq 0 + \text{Neumann boundary conditions} \end{cases}$$

How to approximate the current ?

- $V_{\mathcal{T}}$ given. For $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we define $g_{\mathcal{T}} = \log u_{\mathcal{T}} + V_{\mathcal{T}}$.
- $\nabla^{\mathfrak{D}} g_{\mathcal{T}}$ has a sense.
- Reconstruction of u on the diamond mesh, $r^{\mathfrak{D}}(u_{\mathcal{T}})$

$$r^{\mathcal{D}}(u_{\mathcal{T}}) = \frac{1}{4}(u_K + u_L + u_{K^*} + u_{L^*}) \quad \forall \mathcal{D} \in \mathfrak{D}$$

- We can define a discrete current on the diamond mesh :

$$\mathbf{J}_{\mathfrak{D}} = -r^{\mathfrak{D}}(u_{\mathcal{T}})\mathbf{\Lambda}_{\mathfrak{D}}\nabla^{\mathfrak{D}} g_{\mathcal{T}}.$$

The scheme

$$\begin{cases} \partial_t u + \mathrm{div}\mathbf{J} = 0, \quad \mathbf{J} = -u\mathbf{\Lambda}\nabla(\log u + V), \text{ in } \Omega \times (0,T) \\ u(\cdot,0) = u_0 \geq 0 + \text{Neumann boundary conditions} \end{cases}$$

"Classical" formulation

$$\frac{u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^{n}}{\Delta t} + \mathrm{div}^{\mathcal{T}}(J_{\mathfrak{D}}^{n+1}) = 0, \; J_{\mathfrak{D}}^{n+1} = -r^{\mathfrak{D}}[u_{\mathcal{T}}^{n+1}]\mathbf{\Lambda}^{\mathfrak{D}}\nabla^{\mathfrak{D}} g_{\mathcal{T}}^{n+1},$$

$$\mathrm{m}_\sigma J_{\mathcal{D}}^{n+1} \cdot \mathbf{n} = 0, \qquad \forall\, \mathcal{D} = \mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}.$$

Compact form

$$\left[\!\left[\frac{u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^{n}}{\Delta t}, \psi_{\mathcal{T}} \right]\!\right]_{\mathcal{T}} + T_{\mathfrak{D}}(u_{\mathcal{T}}^{n+1}; g_{\mathcal{T}}^{n+1}, \psi_{\mathcal{T}}) = 0, \quad \forall \psi_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}},$$

$$T_{\mathfrak{D}}(u_{\mathcal{T}}^{n+1}; g_{\mathcal{T}}^{n+1}, \psi_{\mathcal{T}}) = \sum_{\mathcal{D} \in \mathfrak{D}} r^{\mathcal{D}}(u_{\mathcal{T}}^{n+1})\, \delta^{\mathcal{D}} g_{\mathcal{T}}^{n+1} \cdot \mathbb{A}^{\mathcal{D}} \delta^{\mathcal{D}} \psi_{\mathcal{T}},$$

$$g_{\mathcal{T}}^{n+1} = \log(u_{\mathcal{T}}^{n+1}) + V_{\mathcal{T}}.$$

Key discrete properties

$$\left[\!\left[\frac{u_{\mathcal{T}}^{n+1}-u_{\mathcal{T}}^{n}}{\Delta t},\psi_{\mathcal{T}}\right]\!\right]_{\mathcal{T}}+T_{\mathfrak{D}}(u_{\mathcal{T}}^{n+1};g_{\mathcal{T}}^{n+1},\psi_{\mathcal{T}})=0,\quad \forall\psi_{\mathcal{T}}\in\mathbb{R}^{\mathcal{T}},$$

$$T_{\mathfrak{D}}(u_{\mathcal{T}}^{n+1};g_{\mathcal{T}}^{n+1},\psi_{\mathcal{T}})=\sum_{\mathcal{D}\in\mathfrak{D}} r^{\mathcal{D}}(u_{\mathcal{T}}^{n+1})\,\delta^{\mathcal{D}}g_{\mathcal{T}}^{n+1}\cdot\mathbb{A}^{\mathcal{D}}\delta^{\mathcal{D}}\psi_{\mathcal{T}},$$

$$g_{\mathcal{T}}^{n+1}=\log(u_{\mathcal{T}}^{n+1})+V_{\mathcal{T}}.$$

A priori estimates

- Mass conservation : $[\![u_{\mathcal{T}}^{n},1_{\mathcal{T}}]\!]_{\mathcal{T}}=\int_{\Omega}u_0 \quad \forall n\geq 0$
- Energy/dissipation property
  - Discrete free energy : $\mathbb{E}_{\mathcal{T}}^{n}=[\![H(u_{\mathcal{T}}^{n}),1_{\mathcal{T}}]\!]_{\mathcal{T}}+[\![V_{\mathcal{T}},u_{\mathcal{T}}^{n}]\!]_{\mathcal{T}}$
  - Discrete dissipation : $\mathbb{I}_{\mathcal{T}}^{n}=T_{\mathfrak{D}}\left(u_{\mathcal{T}}^{n};g_{\mathcal{T}}^{n},g_{\mathcal{T}}^{n}\right), \qquad \forall n\geq 1$

$$\frac{\mathbb{E}_{\mathcal{T}}^{n+1}-\mathbb{E}_{\mathcal{T}}^{n}}{\Delta t}+\mathbb{I}_{\mathcal{T}}^{n+1}\leq 0,\qquad \forall n\geq 0.$$

Consequences

- Decay of the free energy + bounds
- Further a priori estimates related to Fisher information
- Positivity + lower bound of the approximate solution
- Existence of a solution to the scheme
- Compactness of a sequence of approximate solutions
- Convergence (if penalization term)

Outline of the talk

Test case

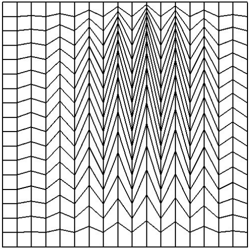
Data and solution

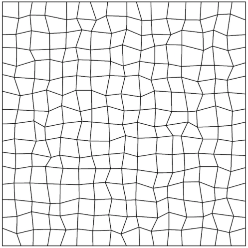
- $\Omega = (0,1)^2$, and $V(x_1, x_2) = -x_2$.
- Exact solution, with $\alpha = \pi^2 + \frac{1}{4}$,

$$u_{\text{ex}}((x_1, x_2), t) = e^{-\alpha t + \frac{x_2}{2}} \left(\pi \cos(\pi x_2) + \frac{1}{2} \sin(\pi x_2) \right) + \pi e^{(x_2 - \frac{1}{2})}$$

- Initial condition : $u_0(x) = u_{\text{ex}}(x, 0)$.

Meshes





Convergence with respect to the grid

On Kershaw meshes

M	dt	erru	ordu	N_{max}	N_{mean}	Min u^n
1	2.0E-03	7.2E-03	—	9	2.15	1.010E-01
2	5.0E-04	1.7E-03	2.09	8	2.02	2.582E-02
3	1.2E-04	7.2E-04	2.20	7	1.49	6.488E-03
4	3.1E-05	4.0E-04	2.11	7	1.07	1.628E-03
5	3.1E-05	2.6E-04	1.98	7	1.04	1.628E-03

On quadrangle meshes

M	dt	erru	ordu	N_{max}	N_{mean}	Min u^n
1	4.0E-03	2.1E-02	—	9	2.26	1.803E-01
2	1.0E-03	5.1E-03	2.08	9	2.04	5.079E-02
3	2.5E-04	1.3E-03	2.06	8	1.96	1.352E-02
4	6.3E-05	3.3E-04	2.09	8	1.22	3.349E-03
5	1.2E-05	7.7E-05	1.70	7	1.01	8.695E-04

Long time behavior

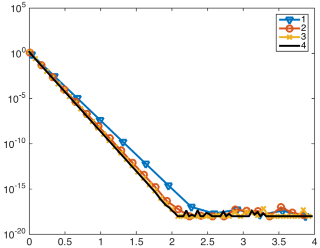
Discrete stationary solution

$$u_K^\infty = \rho e^{-V(x_K)}, \; u_{K^*}^\infty = \rho^* e^{-V(x_{K^*})}$$

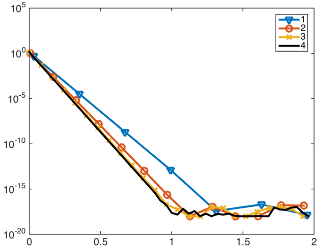
$$\rho, \; \rho^* \text{ such that } \sum_{K \in \mathfrak{M}} u_K^\infty m_K = \sum_{K \in \overline{\mathfrak{M}}^*} u_{K^*}^\infty m_{K^*} = \int_\Omega u_0(x) dx.$$

Relative energy

$$\mathbb{E}_\mathcal{T}^n - \mathbb{E}_\mathcal{T}^\infty = \left[\!\left[u_\mathcal{T}^n \log\left(\frac{u_\mathcal{T}^n}{u_\mathcal{T}^\infty}\right) - u_\mathcal{T}^n + u_\mathcal{T}^\infty, 1_\mathcal{T} \right]\!\right]_\mathcal{T}, \quad n \geq 0$$



Kershaw



quadrangles