Lois de conservations scalaires hyperboliques stochastiques : existence, unicité et approximation numérique de la solution entropique

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Journées Multiphasiques et Incertitudes, 14 novembre 2017

1. Presentation of the problem and main result
   - The equation
   - Existence and uniqueness
   - Numerical approximation : monotone finite volume schemes
   - The result : existence, uniqueness and convergence of the finite volume approximation

2. Proof : case of a 1D upwind scheme
   - The 1D upwind scheme
   - Convergence of the numerical approximation to a measure-valued solution, up to a subsequence
   - Uniqueness of the measure-valued solution and conclusion

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A stochastic hyperbolic scalar conservation law

- We consider the following hyperbolic scalar conservation law on $\mathbb{R}^d$ with a multiplicative noise:

$$\begin{cases} du + \operatorname{div}_x\big[f(x,t,u)\big]dt &= g(u)dW \quad \text{in } \Omega \times \mathbb{R}^d \times (0,T), \\ u(\omega,x,0) &= u_0(x), \qquad \omega \in \Omega,\ x \in \mathbb{R}^d, \end{cases} \tag{1}$$

- It can be rewritten as

$$\partial_t\Big(u - \underbrace{\int_0^t g(u)dW}_{\text{Itô Integral}}\Big) + \operatorname{div}_x\big[f(x,t,u)\big] = 0,$$

$\rightsquigarrow$ the multiplicative noise can be used to model uncertainties in the model/ small scale phenomenons
$\rightsquigarrow$ goal : applications to fluid mechanics, for example the study of flow in porous media ...

- Questions : existence, uniqueness and numerical approximation of the solution ?

The weak formulation of the stochastic hyperbolic scalar conservation law

- The corresponding weak formulation is: for almost all ω in Ω and for all φ in $\mathcal{D}(Q)$, with $Q = \mathbb{R}^d \times [0, T)$

$$\int_Q u(\omega, x, t)\partial_t\varphi(x,t) + f(x,t,u(\omega,x,t)).\nabla_x\varphi(x,t)dxdt$$
$$+ \int_{\mathbb{R}^d} u_0(x)\varphi(x,0)dx = \int_Q \underbrace{\left(\int_0^t g(u(\omega,x,s))dW(s)\right)}_{\text{Itô Integral}} \partial_t\varphi(x,t)dxdt.$$

- As in the deterministic case, there is no uniqueness of the weak solution in general.
  $\rightsquigarrow$ In the deterministic case, we can use the concept of entropy solution to get uniqueness.
- We now generalize the concept of entropy solution to the stochastic case.

Stochastic entropy solution

- $\mathcal{D}^+\left(\mathbb{R}^d \times [0,T)\right)$ is the subset of nonnegative elements of $\mathcal{D}(\mathbb{R}^d \times [0,T))$
- $\mathcal{A}$ denotes the set of $C^3(\mathbb{R})$ convex functions η such that η'' has compact support
- Φ_η denotes the entropy flux defined for any $a \in \mathbb{R}$ and for any $\eta \in \mathcal{A}$ by $\Phi_\eta(x,t,u) = \displaystyle\int_0^a \eta'(\sigma)\frac{\partial f}{\partial u}(x,t,\sigma)d\sigma$.

Definition

A function u of $\mathcal{N}^2\left(0,T,L^2(\mathbb{R}^d)\right) \cap L^\infty\left(0,T;L^2\left(\Omega \times \mathbb{R}^d\right)\right)$ is a stochastic entropy solution of (1) if P-a.s in Ω, for any $\eta \in \mathcal{A}$ and for any $\varphi \in \mathcal{D}^+\left(\mathbb{R}^d \times [0,T)\right)$

$$\int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)dx + \int_Q \eta(u)\partial_t\varphi(x,t)dxdt + \int_Q \Phi_\eta(x,t,u).\nabla_x\varphi(x,t)dxdt$$

$$+ \int_0^T \int_{\mathbb{R}^d} \eta'(u)g(u)\varphi(x,t)dxdW(t) + \frac{1}{2}\int_Q g^2(u)\eta''(u)\varphi(x,t)dxdt \geqslant 0$$

Existence, uniqueness results : state of the art

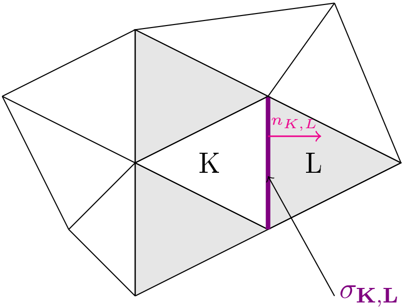
For the equation : $du + \mathrm{div}_x\left[f(u)\right]dt = g(u)dW$

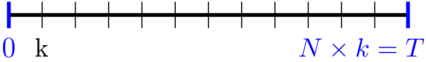
- W. E K. Khanin A. Mazel & Y. Sinai (2000) $\rightsquigarrow$ Existence, uniqueness and invariant measures for stochastic Burgers equation, $d = 1$ on the torus (through Lax-Oleinik formula).
- J. Feng & D. Nualart (2008), G.-Q Chen, Q. Ding & K.H. Karlsen (2012) , I.H. Biswas & A.K. Majee (2014)
  $\rightsquigarrow$ Existence and uniqueness under " an extra property"
- A. Debussche & J. Vovelle (2010)
  $\rightsquigarrow$ Existence and uniqueness of the stochastic entropy solution on the torus with a more general noise (through a kinetic approach)
- C. Bauzet G. Vallet P. Wittbold (2012)
  $\rightsquigarrow$ Existence and uniqueness of the stochastic entropy solution on $\mathbb{R}^d$ (through an entropic approach)
- K. Kobayasi & D. Noboriguchi (2015)
  $\rightsquigarrow$ Existence and uniqueness of the stochastic entropy solution on bounded domains (through a kinetic approach)
- ...

Numerical analysis: state of the art

- H. Holden & N.H. Risebro (1991) : Time-splitting ($d = 1$).
- C. Bauzet (2013) : Time-splitting ($d \geqslant 1$)
- I. Kröker & C. Rohde (2012) : Semi-discrete finite volume discretisation ($d = 1$) $\rightsquigarrow$ No time discretisation and additionnal assumptions.
- K. H. Karlsen & E. B. Storrøsten (preprint) : Analysis of a time-splitting method $\rightsquigarrow$ No spatial discretisation.
- K. Mohamed, M. Seaid& M. Zahri (2013) : Semi-discretisation in space
- C. Bauzet, J.Charrier & T.Gallouët (2016/2017) : Convergence of the finite volume scheme through an entropic approach
- S.Dotti, J.Vovelle (preprint) : Convergence of the finite volume scheme through a kinetic approach (on the torus, general noise)
- T.Funaki, Y.Gao, D.Hillhorst (preprint) : Convergence of the finite volume scheme through an entropic approach (on the torus, general noise)

Mesh and notations





Notations

- $k = T/N$ the time step, $N \in \mathbb{N}^\star$.
- $\mathcal{T}$ an admissible mesh: $|K| \leq h$, $\bar{\alpha} h^d \leqslant |K|$, $|\partial K| \leqslant \frac{1}{\bar{\alpha}} h^{d-1}$, $\quad \forall K \in \mathcal{T}$
- $\mathcal{N}(K)$ the set of control volumes neighbors of $K \in \mathcal{T}$.
- $$f^n_{K,L}(s) = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} f(x,t,u).n_{K,L} d\gamma(x) dt.$$

The scheme

Definition (monotone numerical fluxes)

A family $(F^n_{K,L})$ of functions is said to be a family of monotone numerical fluxes if:

- $F^n_{K,L}(a,b)$ *is nondecreasing with respect to* a *and nonincreasing with respect to* b.
- *There exists* $F_1, F_2 > 0$ *such that for any* $a, b \in \mathbb{R}$ *we have*
  - $|F^n_{K,L}(b,a) - F^n_{K,L}(a,a)| \leq F_1|a-b|$
  - $|F^n_{K,L}(a,b) - F^n_{K,L}(a,a)| \leq F_2|a-b|$
- $F^n_{K,L}(a,a) = f^n_{K,L}(a)$ *for all* $a \in \mathbb{R}$
- $F^n_{K,L}(a,b) = -F_{L,K}(b,a)$ *for all* $a, b \in \mathbb{R}$.

The scheme

Definition (The scheme)

We consider the following monotone scheme: we define P-as in Ω *and for any* $K \in \mathcal{T}$ *the approximation* u_K^n *by*

$$\begin{cases} \frac{|K|}{k}(u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^n(u_K^n, u_L^n) = |K| g(u_K^n) \frac{W^{n+1} - W^n}{k} \\ u_K^0 = \frac{1}{|K|} \int_K u_0(x)dx, \end{cases}$$

where $W^n = W(nk)$.

The approximate finite volume solution $u_{\mathcal{T},k}$ *is then defined P-a.s in* Ω *on* $\mathbb{R} \times [0,T]$ *by:*

$$u_{\mathcal{T},k}(\omega, x, t) = u_K^n \text{ for } \omega \in \Omega, x \in K \text{ and } t \in \big[nk, (n+1)k\big). \quad (2)$$

Assumptions and result

$$\begin{cases} du + \mathrm{div}_x\left[f(x,t,u)\right]dt &= g(u)dW \quad \text{in } \Omega \times \mathbb{R}^d \times (0,T), \\ u(\omega,x,0) &= u_0(x), \quad \omega \in \Omega,\ x \in \mathbb{R}^d, \end{cases} \tag{3}$$

- $u_0 \in L^2(\mathbb{R}^d)$
- $f \in \mathcal{C}^1(\mathbb{R}^d \times [0,T] \times \mathbb{R})$, $\frac{\partial f}{\partial u}$ is bounded and lipschitz continuous w.r.t. (x,t), uniformly w.r.t. u
- $g : \mathbb{R} \to \mathbb{R}$ is lipschitz continuous with $g(0) = 0$ and g is bounded
- $\mathrm{div}_x[f(x,t,u)] = 0\ \forall (x,t) \in \mathbb{R}^d \times [0,T]$.

Theorem (Bauzet,Castel,C.)

Under these assumptions, there exists a unique stochastic entropy solution to (3), and the finite volume approximation converges to this solution in $L^p_{loc}(\Omega \times \mathbb{R}^d \times [0,T])$ (for $1 \leq p < 2$) as $(h, k/h)$ goes to 0.

$\rightsquigarrow$ if g has compact support and $u_0 \in L^\infty$, we can take $\frac{\partial f}{\partial u}$ unbounded and $\frac{\partial f}{\partial u}$ lipschitz continuous with respect to (x,t), not necesserely uniformly w.r.t. u : it allows to treat the case of Burger's equation.

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The equation and the mesh in a particular case

$$\begin{cases} du + \operatorname{div}_x \left[f(u)\right] dt = g(u) dW & \text{in } \Omega \times \mathbb{R} \times (0,T), \\ u(\omega, x, 0) = u_0(x), & \omega \in \Omega,\ x \in \mathbb{R}, \end{cases} \tag{4}$$

$\rightsquigarrow$ we consider the $1D$ case, with $f(x,t,u) = f(u)$ and we suppose moreover that f is nondecreasing.

Definition (Admissible mesh)

An admissible mesh is $\mathcal{T} = \{K_i, i \in \mathbb{Z}\}$, *where* $K_i = (x_{i-1/2}, x_{i+1/2})$ *for all* $i \in \mathbb{Z}$ *and* $\mathbb{R} = \bigcup_{i\in\mathbb{Z}} [x_{i-1/2}, x_{i+1/2}]$.
We assume that $h = size(\mathcal{T}) = \sup_{i\in\mathbb{Z}} h_i < +\infty$ *and that for some* $\bar{\alpha} > 0$ *we have* $\inf_{i\in\mathbb{Z}} h_i \geq \bar{\alpha} h$.

The 1D upwind scheme in the case f nondecreasing

Definition

Let $k > 0$ be the time step, such that $T = Nk$. We consider the following upwind scheme: we define P-as in Ω the u_i^n by

$$\begin{cases} \frac{u_i^{n+1}-u_i^n}{k} + \frac{f(u_i^n)-f(u_{i-1}^n)}{h_i} = g(u_i^n)\frac{W^{n+1}-W^n}{k} & 0 \leq n \leq N-1, \forall i \in \mathbb{Z}, \\ u_i^0 = \frac{1}{h_i}\int_{K_i} u_0(x)dx, & \forall i \in \mathbb{Z}, \end{cases}$$

where $W^n = W(nk)$.
The approximate finite volume solution $u_{\mathcal{T},k}$ is then defined P-a.s in Ω on $\mathbb{R} \times [0, T]$ by:

$$u_{\mathcal{T},k}(x,t) = u_i^n \text{ for } i \in \mathbb{Z}, t \in [nk, (n+1)k)$$

The result in the 1D upwind scheme

Theorem

The equation (4) admits a unique entropic solution and the approximate solution $u_{\mathcal{T},k}$ converges to this solution in $L^p_{loc}(\Omega \times [0,T] \times \mathbb{R}^d)$ for any $p < 2$ as $(h, k/h)$ tends to 0.

$\rightsquigarrow$ two difficulties coming from the stochastic framework:

1. In the deterministic framework, one works classically in $L^\infty([0,T] \times \mathbb{R}^d)$ and it is very easy to get L^∞ estimates, whereas in the stochastic case u and $u_{\mathcal{T},k}$ do not belong to L^∞.
$\rightsquigarrow$ a natural space is then $L^2([0,T] \times \mathbb{R}^d)$ (because of Itô calculus), but it raises some difficulties.
2. In the deterministic framework, one works classically with Kruzkov entropies, whereas in the stochastic case, we cannot use them because Itô formula requires smoothness of the test functions.
$\rightsquigarrow$ we work with smooth entropies, but it also raises some difficulties.

Idea of the proof : deterministic case

1. Stability result on the numerical approximation
   $\Rightarrow$ convergence of a subsequence
   $\rightsquigarrow$ compacity result (Young measures)
2. The limit of the subsequence is a generalized solution
3. UNIQUENESS of the generalized solution = entropic solution
   $\rightsquigarrow$ Kato inequality : comparaison between two generalized solutions
4. Uniqueness $\Rightarrow$ CONVERGENCE of the scheme
   $+$ EXISTENCE of the solution

Idea of the proof : existence and uniqueness in the stochastic case

1. Stability result on the viscous approximation
   $\Rightarrow$ convergence of a subsequence
   $\rightsquigarrow$ compacity result (Young measures)
2. The limit of the subsequence is a generalized solution
3. UNIQUENESS of the generalized solution = entropic solution
   $\rightsquigarrow$ Kato inequality : comparaison between a generalized solutions and a viscous approximation
4. Uniqueness $\Rightarrow$ EXISTENCE of the solution

Idea of the proof : previous work

1. Stability result on the numerical approximation
   $\Rightarrow$ convergence of a subsequence
   $\rightsquigarrow$ compacity result (Young measures)
2. The limit of the subsequence is a generalized solution
3. UNIQUENESS of the generalized solution = entropic solution
   $\rightsquigarrow$ Kato inequality : comparaison between the viscous approximation and a generalized solution
4. Uniqueness $\Rightarrow$ CONVERGENCE of the scheme
   + EXISTENCE of the solution

Idea of the new proof

1. Stability result on the ~~viscous~~ numerical approximation
   $\Rightarrow$ convergence of a subsequence
   $\rightsquigarrow$ compacity result (Young measures)
2. The limit of the subsequence is a generalized solution
3. UNIQUENESS of the generalized solution = entropic solution
   $\rightsquigarrow$ Kato inequality : comparaison between the ~~viscous~~ numerical approximation and a generalized solution
4. Uniqueness $\Rightarrow$ CONVERGENCE of the scheme
   + EXISTENCE of the solution

$\rightsquigarrow$ morevoer we hope that this proof of uniquenness is the first step to get strong error estimates

Step 1: stability result

- In the deterministic case, it is easy to prove an $L^\infty(\Omega \times [0,T] \times \mathbb{R})$ bound for $u_{\mathcal{T},k}$: $u_m \leq u_0(x) \leq u_M$ a.e. $\Rightarrow u_m \leq u_{\mathcal{T},k} \leq u_M$
  $\rightsquigarrow$ In the stochastic case, we cannot get such bounds. We work in $L^2(\Omega \times [0,T] \times \mathbb{R})$, which is a natural space to deal with the noise (because of Itô calculus).
- We denote by C_f the lipschitz constant of f and by C_g the lipschitz constant of g.

Proposition (stability estimate)

Under the CFL condition $k \leq \frac{\bar{\alpha} h}{C_f}$, *we have:*

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}))} \leq e^{C_g^2 T/2} \|u_0\|_{L^2(\mathbb{R})}$$

$$\|u_{\mathcal{T},k}\|^2_{L^2(\Omega\times[0,T]\times\mathbb{R})} \leq T e^{C_g^2 T} \|u_0\|^2_{L^2(\mathbb{R})}$$

Step 2: convergence to an entropy process

- We deduce from the stability estimate that, up to a subsequence, $u_{\mathcal{T},k}$ converges to an entropy process $\mathbf{u} \in L^2(\Omega \times Q \times (0,1))$ in the sense of Young measures.
- More precisely, given a Caratheodory function $\Psi : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\Psi(\cdot, u_{\mathcal{T},k})$ is uniformly integrable we have, up to a subsequence,

$$\mathbf{E}\left[\int_{[0,T]\times\mathbb{R}} \Psi(\cdot, u_{\mathcal{T},k}) dx dt\right] \to \mathbf{E}\left[\int_{[0,T]\times\mathbb{R}} \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot,\alpha)) dx d\alpha dt\right]$$

$\rightsquigarrow$ it is a compacity result.

Step 2: convergence to an entropy process

Definition (Measure-valued stochastic entropy solution)

A function $\mathbf{u}$ of $\mathcal{N}_w^2\big(0,T,L^2(\mathbb{R}\times(0,1))\big)\cap L^\infty\big(0,T;L^2(\Omega\times\mathbb{R}\times(0,1))\big)$ is a measure-valued entropy solution of (1), if P-a.s in Ω, for any $\eta\in\mathcal{A}$ and for any $\varphi\in\mathcal{D}^+(\mathbb{R}\times[0,T))$

$$\int_{\mathbb{R}}\eta(u_0)\varphi(x,0)dx+\int_{[0,T]\times\mathbb{R}}\int_0^1\eta(\mathbf{u}(x,t,\alpha))\partial_t\varphi(x,t)d\alpha dxdt$$
$$+\int_{[0,T]\times\mathbb{R}}\int_0^1\Phi_\eta(\mathbf{u}(x,t,\alpha))\varphi_x(x,t)d\alpha dxdt$$
$$+\int_0^T\int_{\mathbb{R}^d}\int_0^1\eta'(\mathbf{u}(x,t,\alpha))g(\mathbf{u}(x,t,\alpha))\varphi(x,t)d\alpha dxdW(t)$$
$$+\frac{1}{2}\int_{[0,T]\times\mathbb{R}}\int_0^1 g^2(\mathbf{u}(x,t,\alpha))\eta''(\mathbf{u}(x,t,\alpha))\varphi(x,t)d\alpha dxdt\geq 0.$$

Step 2: convergence to an entropy process

It remains to prove that:

- $\mathbf{u}$ is a measure-valued entropy solution of (1)
- the equation (1) admits a unique stochastic measure-valued entropy solution, which is a stochastic entropy solution.

$\rightsquigarrow$ it will enable us to deduce that the whole sequence $u_{\mathcal{T},k}$ converges to u in $L^1_{loc}(\Omega \times [0,T] \times \mathbb{R})$.

Step 3: the weak BV estimate

Proposition (Weak BV estimate)

Under the stronger CFL condition: $k \leq \frac{(1-\xi)\bar{\alpha}h}{L_f}$ for some $\xi \in (0,1)$, we have for any $R > 0$ and $T_1 > 0$ the existence of a constant C such that

$$\sum_{i=i_0}^{i_1} \sum_{n=0}^{N_1} k|f(u_i^n) - f(u_{i-1}^n)| \leq Ch^{-1/2},$$

where $i_0, i_1 \in \mathbb{Z}$ and $N_1 \in \mathbb{N}$ are such that $-R \in \bar{K}_{i_0}, R \in \bar{K}_{i_1}$ and $T_1 \in (N_1 k, (N_1+1)k]$.

$\rightsquigarrow$ Note that in the linear case it means that the discrete BV norm is locally bounded by $h^{-1/2}$.

Step 4: continuous entropy inequalities

Using the stability estimate and the weak BV estimate, we deduce:

Proposition (Continuous entropy inequalities)

We have, P-a.s. in Ω, for any $\eta \in \mathcal{A}$ and for any $\varphi \in \mathcal{D}^+\big(\mathbb{R} \times [0,T)\big)$:

$$\int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)dx + \int_{[0,T]\times\mathbb{R}^d} \eta(u_{\mathcal{T},k})\varphi_t(x,t)dxdt$$
$$+ \int_{[0,T]\times\mathbb{R}^d} \Phi_\eta(u_{\mathcal{T},k})\varphi_x(x,t)dxdt + \int_0^T \int_{\mathbb{R}^d} \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi(x,t)dxdW(t)$$
$$+ \frac{1}{2}\int_{[0,T]\times\mathbb{R}^d} \eta''(u_{\mathcal{T},k})g^2(u_{\mathcal{T},k})\varphi(x,t)dxdt \geqslant R^{h,k,\eta},$$

where for any P-measurable set A, $E\big[\mathbb{1}_A R^{h,k,\eta}\big] \to 0$ as $h \to 0$ with $\frac{k}{h} \to 0$.

$\rightsquigarrow$ it remains to pass to the limit in the Young measure sense to establish that $\mathbf{u}$ is a measure-valued stochastic entropy solution

Step 5 : $\mathbf{u}$ is a measure-valued stochastic entropy solution

Proposition

For any P-measurable set A, any $\eta \in \mathcal{A}$ and any $\varphi \in \mathcal{D}^+(\mathbb{R}^\times[0,T])$

$$\mathbf{E}\Big[\mathbb{1}_A \int_{\mathbb{R}^d} \eta(u_0)\varphi(x,0)dx\Big]$$
$$+\mathbf{E}\Big[\mathbb{1}_A \int_{[0,T]\times\mathbb{R}^d}\int_0^1 \eta(\mathbf{u}(x,t,\alpha))\varphi_t(x,t)d\alpha dx dt\Big]$$
$$+\mathbf{E}\Big[\mathbb{1}_A \int_{[0,T]\times\mathbb{R}^d}\int_0^1 \Phi_\eta(\mathbf{u}(x,t,\alpha))\varphi_x(x,t)d\alpha dx dt\Big]$$
$$+\mathbf{E}\Big[\mathbb{1}_A \int_0^T\int_{\mathbb{R}^d}\int_0^1 \eta'(\mathbf{u}(x,t,\alpha))g(\mathbf{u}(x,t,\alpha))\varphi(x,t)d\alpha dx dW(t)\Big]$$
$$+\mathbf{E}\Big[\mathbb{1}_A \frac{1}{2}\int_{[0,T]\times\mathbb{R}^d}\int_0^1 \eta''(\mathbf{u}(x,t,\alpha))g^2(\mathbf{u}(x,t,\alpha))\varphi(x,t)d\alpha dx dt\Big] \geqslant 0.$$

Step 6 : Kato inequatity

Proposition

Let ν be a measure-valued stochastic entropy solution, then for any $\varphi \in \mathcal{D}^+(\mathbb{R} \times (0,T))$ we have

$$\mathbf{E}\Big[\int_{\mathbb{R}}\int_0^T\int_0^1\int_0^1 (|\nu(x,t,\alpha) - \mathbf{u}(x,t,\beta)|\varphi_t(x,t) \\ - \psi(\nu(x,t,\alpha), \mathbf{u}(x,t,\alpha))\varphi_x(x,t))d\alpha d\beta dt dx\Big] \geq 0,$$

where $\psi(a,b) = sgn(a-b)[f(a) - f(b)]$: entropy flux associated to Krushkov entropy.

$\rightsquigarrow$ *we deduce that* $\nu(x,t,\alpha) = \mathbf{u}(x,t,\beta) = \int_0^1 \nu(x,t,\alpha)d\alpha = \mathbf{u}(x,t)$.

Proof of Kato inequatity : Kruzkov's doubling of variable

- Example of one the the five groups of terms : the term with times derivatives
  First contribution :

$$\mathbf{E}\left[\int \eta_\delta(\nu(x,t,\alpha)-\kappa)\varphi_t(x,t)\bar{\rho}_n(t-s)\rho_m(x-y)\rho_l(u_{\mathcal{T},k}(y,s)-\kappa)dtdxd\alpha d\kappa dsdy\right]$$
$$+\mathbf{E}\left[\int \eta_\delta(\nu(x,t,\alpha)-\kappa)\varphi(x,t)\bar{\rho}_n'(t-s)\rho_m(x-y)\rho_l(u_{\mathcal{T},k}(y,s)-\kappa)dtdxd\alpha d\kappa dsdy\right]$$

  Second contribution :

$$-\mathbf{E}\left[\int \eta_\delta(u_{\mathcal{T},k}(y,s)-\kappa)\varphi(x,t)\bar{\rho}_n'(t-s)\rho_m(x-y)\rho_l(\nu(x,t,\alpha)-\kappa)dsdyd\kappa d\alpha dtdx\right]$$

$\rightsquigarrow \mathbf{E}[\int_{\mathbb{R}}\int_0^T\int_0^1\int_0^1(|\nu(x,t,\alpha)-\mathbf{u}(x,t,\beta)|\varphi_t(x,t)dtdsd\alpha d\beta]$

- To pass to the limit in each term : we take $n=h^{-5}, k=h^{21}$, we let $h\to 0$, then $l\to+\infty$, then $\delta\to 0$, then $m\to+\infty$.

Sketch of the proof in the general case

1. $L^2(\Omega \times [0,T] \times \mathbb{R}^d)$ stability estimate + Weak BV estimate.
2. $u_{\mathcal{T},k}$ converges to an entropy process $\mathbf{u}$ in the sense of Young measures (up to a subsequence)
3. Decomposition of numerical monotone flux as a convex combination of a modified Lax-Friedrich flux and a Godunov flux.
4. Continuous entropy inequalities for the numerical approximation by considering separately
   - the case of flux-splitting schemes
   - the case of the Godunov scheme
5. We pass to the limit : $\mathbf{u}$ is a stochastic measure-valued entropy solution.
6. Uniqueness result of the stochastic measure-valued entropy solution, which is moreover a stochastic entropy solution.
   $\rightsquigarrow$ Kruzkov's doubling of variable : we compare a measure-vauled solution to the numerical approximation

Thank you for your attention !