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PROJET : Problèmes stratifiés en optimisation dynamique

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Part I

INTRODUCTION

Le contrôle optimal .

Le contrôle optimal consiste à trouver un couple trajectoire-contrôle (x,u), minimisant (ou maximisant) un côut représenté par une fonction g.

Un problème en contrôle optimal est:

$$(P_{S,x_0}) \qquad \begin{cases} \text{Minimize } g(x(T)) \ \ \text{(côut final)} \\ \text{sur le arcs } x \in W^{1,1}([S,T],\mathbb{R}^n) \ \ \text{vérifiant} \\ \dot{x}(t) = f(t,x(t),u(t)), \ \ p.p. \text{ (contrainte dynamique)} \\ u(t) \in U \quad \text{(ensemble des contôles)} \\ x(0) = x_0 \quad \text{(condition initiale)} \end{cases}$$

où $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, est la fonction côut
$f : [S,T] \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ est la dynamique
$U \subset \mathbb{R}^m$ l'ensemble des contrôles
(S, x_0) la donnée initiale.

Processus admissible: Un couple (x, u) vérifiant la contrainte dynamique et la condition initiale est dit admissible au problème P_{S,x_0}.

Un minimiseur: On dit que $(\bar{x}, \bar{u})$ est une **solution du problème/un minimiseur** lorsque pour tout processus admissible (x, u), on a :

$$g(x(T)) \geq g(\bar{x}(T)).$$

Version avec inclusion différentielle: La contrainte dynamique apparente dans le problème P_{S,x_0} peut exister dans d'autres problèmes en contrôle optimal sous forme d'une inclusion différentielle.

Le problème se voit donc sous la forme:

$$(P_{S,x_0}) \qquad \begin{cases} \text{Minimize } g(x(T)) \quad \text{(côut final)} \\ \text{sur le arcs } x \in W^{1,1}([S,T],\mathbb{R}^n) \quad \text{vérifiant} \\ \dot{x}(t) \in F(t,x(t)), \quad p.p. \text{ (contrainte dynamique)} \\ u(t) \in U \quad \text{(ensemble des contôles)} \\ x(0) = x_0 \quad \text{(condition initiale)} \end{cases}$$

où $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ une multifonction, avec $F(t,x) := \{f(t,x,u) : u \in U\}$.

On voit donc qu'une trajectoire admissible au problème P_{S,x_0} qui vérifie une équation différentielle vérifie certainement une inclusion différentielle. La réciproque que pour toute vitesse contenu dans l'ensemble des vitesse il existe un contrôle u dans l'ensemble des contrôles U tel que (x,u) est un processus admissible se voit par le théorème de sélection (Filippov's selection theorem).

Contrôle optimal-Méthodes classiques de résolution: En contrôle optimal on a deux grandes approches:

1.Le principe du maximum (Pontryagin).
▶ Conditions nécessaires du premier ordre pour l'optimalité.

2.Dynamic programming (Bellman).
▶ Relation entre le problème de contrôle optimal et une équation de Hamilton-Jacobi.

On s'interesse ici au **principe du maximum**.

Principe du maximum-Conditions nécessaires d'optimalité

.
On considère l'Hamiltonien non maximisé
$\mathcal{H}(t,p,x,u) = f(t,x,u) \cdot p$

Énoncé:

Soit $(\bar{x}, \bar{u})$ un **minimiseur** pour (P_{S,x_0}). Sous les bonnes hypothèses il existe une fonction $p \in W^{1,1}([0,T], \mathbb{R}^n)$ et $\lambda \geq 0$ (multiplicateurs de Lagrange) tels que :

(i) $(p, \lambda) \neq (0,0)$ **(non trivialité)**

(ii) $-\dot{p}(t) \in \operatorname{co} \partial_x \mathcal{H}(t, \bar{x}(t), p(t), \bar{u}(t))$ p.p. **(système adjoint)**

(iii) $\mathcal{H}(t, \bar{x}(t), p(t), \bar{u}(t)) = \max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), p(t), u)$ **(condition de maximalité ou de Weierstrass)**

(iv) $-p(T) \in \lambda \partial g(\bar{x}(T))$ **(condition de transversalité)**.

Problèmes avec structures stratifiés .

La nature de la stratification dans un problème d'optimisation dépend du modèle et des hypothèses sur ce modèle. Chaque type d'un problème en contrôle optimal où on a pas un cas classique ou simple impose une stratification pour le bien résoudre.

Stratification au sens de Clarke La phrase "sous les bonnes hypothèses" introduite dans le principe du maximum signifie par exemple une borne sur l'esemble des vitesses F ou une inégalité de type lipschitz (Figure.1). Dans des cas cette hypothèse n'existe pas. Un ensemble F par exemple définis comme:

$$F(x = (x_1, x_2)) := \{(v_1, v_2) | v_1 < x_1 v_2\}$$

est non borné et ne vérifie pas un relation lipschitz de la forme

$$F(t, x') \subset F(t, x) + k(t)|x - x'|B$$

ce qui exige une stratification par une fonction appelée fonction rayon dépendante du temps qu'on l'intersecte avec l'ensemble des vitesse pour obtenir une borne ou controler la variation. On parle donc des vitesses dans un esemble de la forme $F(t, x) \cap R(t)$.

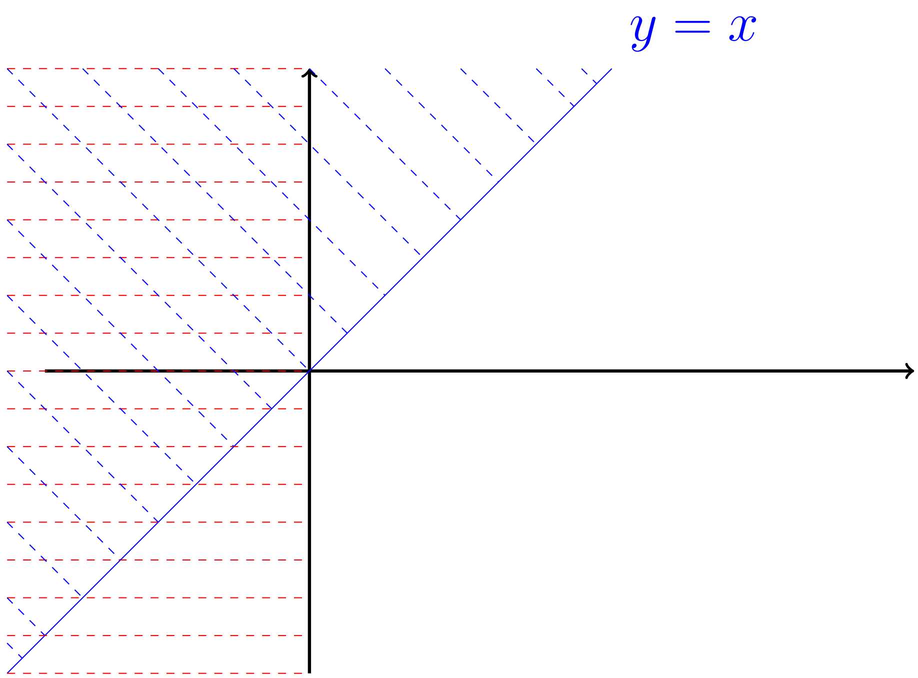
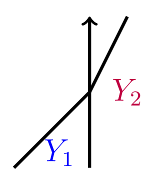


Figure 1:

Stratification avec des Multiprocessus Nous considérons ici une modification du problème de contrôle optimal standard dans lequel les trajectoires d'état sont autorisées à être discontinues à un nombre fini de fois et sur des intervales de temps finis. On parle donc d'un processus de la forme $\{\tau_0^i, \tau_1^i, y_i, u_i\}$ avec $i = 1, ..., k$. Un exemple d'un tel problème est la dérivation de la loi de réfraction de Snell à partir du principe du moindre temps de Fermat.

$k = 2$:



Notation

B closed unit ball in Euclidean space

$|x|$ Euclidean norm of x

$d_C(x)$ Euclidean distance of x from C

int C interior of C

N_C^P Proximal normal cone to C at x

N_C^L, N_C Limiting normal cone to C at x

epi f Epigraph of f

$\partial^P f(x)$ Proximal subdifferential of f at x

$\partial^L f(x)$ Limiting subdifferential of f at x

Gr F Graph of F

$\nabla f(x)$ Gradient vector of f at x

$\Psi_C(x)$ Indicator fuction of the set C

$\mathcal{H}$,H Unmaximized hamiltonian

dom f Domain of f

$\bar{C}$ Closure of C

Part II
MAXIMUM PRINCIPLE

This chapter focus on optimality condition in a smooth case where the dynamic constraint is smooth with respect to the state varialble. We will start by a general case of the problem then go back to the proof in the snooth case. The optimal control studied here is

$$
(P) \qquad \begin{cases} \text{Minimize g(x(S),x(T))} \\ \text{over } x \in W^{1,1}([S,T];\mathbb{R}^n) \\ \text{ and mesurable function } u : [S,T] \longrightarrow \mathbb{R}^m \text{ satisfying} \\ \dot{x}(t) = f(t, x(t), u(t)) \ \text{ a.e.,} \\ u(t) \in U(t) \ \text{ a.e.,} \\ (x(S), x(T)) \in C \end{cases}
$$

the data for which comprise an interval $[S,T]$, functions $g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and $f : [S,T] \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, a nonempty multifunction U :$[S,T] \rightsquigarrow \mathbb{R}^m$, and a closed set $C \subset \mathbb{R}^n \times \mathbb{R}^n$.

A mesurable function $u : [S,T] \longrightarrow \mathbb{R}^m$ that satisfies

$$u(t) \in U(t) \quad a.e.$$

is called a control function.The set of all control functions is written $\mathcal{U}$.
A process (x, u) comprises a control function u together with an arc $x \in W^{1,1}([S,T];\mathbb{R}^n)$ which is a solution to the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)) \quad a.e.$$

A state trajectory x is the first component of some process (x, u). A process (x, u) is said to be feasible for (P) if the state trajectory x satisfies the endpoint constraint

$$(x(S), x(T)) \in C.$$

1.Definition Take a feasible process $(\bar{x}, \bar{u})$.

$(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer if there exist $\delta > 0$ such that

$$g(x(S), x(T)) \geq g(\bar{x}(S), \bar{x}(T)),$$

for all feasible processes (x, u) which satisfy

$$||x - \bar{x}||_{W^{1,1}} \leq \delta.$$

$(\bar{x}, \bar{u})$ is a strong local minimixer if there exist $\delta > 0$ such that

$$g(x(S), x(T)) \geq g(\bar{x}(S), \bar{x}(T)),$$

for all feasible processes (x, u) which satisfy

$$||x - \bar{x}||_{L^\infty} \leq \delta.$$

2.THE MAXIMUM PRINCIPLE .

Denote by $\mathcal{H} : [S, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ the unmaximied Hmiltonian function

$$\mathcal{H}(t, x, p, u) := p.f(t, x, u)$$

.

Theorem 2.1(The maximum Principle) Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer for (P). Assume that, for some $\delta > 0$, the following hypotheses are satisfied.

$(H1)$ For fixed x, $f(., x, .)$ is $\mathcal{L} \times \mathcal{B}^m$ mesurable. There exist an $\mathcal{L} \times \mathcal{B}^m$ mesurable function $k : [S, T] \times \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $t \to k(t, \bar{u}(t))$ is integrable and, for a.e. $t \in [S, T]$,

$$|f(t, x, u) - f(t, x', u)| \leq k(t, u)|x - x'|$$

for all $x, x' \in \bar{x}(t) + \delta B$ and $u \in U(t)$;

$(H2)$ GrU is an $\mathcal{L} \times \mathcal{B}^m$ mesurable set;

$(H3)$ g is locally lipschitz continuous.
Then there exist $p \in W^{1,1}([S, T]; \mathbb{R}^n)$ and $\lambda \geq 0$ such that

(i) $(p, \lambda) \neq (0, 0)$;

(ii) $-\dot{p}(t) \in co\partial_x \mathcal{H}(t, \bar{x}(t), p(t), \bar{u}(t))$ $a.e.$;

(iii) $\mathcal{H}(t, \bar{x}(t), p(t), \bar{u}(t)) = max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), p(t), u)$ a.e.;

(iv) $(p(S), -p(T)) \in \partial g(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T))$.

Now assume, also, that $f(t, x, u)$ and $U(t)$ are independant of t.
Then, in addition to the above conditions, there exist a constant r such that:

(v) $\mathcal{H}(t, \bar{x}(t), p(t) = r$
($\partial_x \mathcal{H}$ denotes the limiting subdifferential of $\mathcal{H}(t, ., p, u)$ for fixed (t, p, u).)

Elements (λ, p) whose existence is asserted in the Maximum Principle are called multipliers for (P).The componentes λ and p are refered to as the cost multiplier and adjoint arc, repectively.

Remark :
The adjoint inclusion (Condition (ii) in the Theorem statement often stated in terms of the Clarke's generalized jacobian:

3.Definition Take a point $y \in \mathbb{R}$ and a function $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ that is Lipschitz continuous on a neighborhood of $y\cdot$ Then the Generalized Jacobian $DL(y)$ of L at y is the set of $m \times n$ matrices:

$$DL(y) := co\{\eta : \exists y_i \longrightarrow y \text{ such that } \nabla L(y_i) \text{ exist } \forall i \text{ and } \nabla L(y_i) \longrightarrow \eta\}.$$

A noteworthy property of the generalized Jacobian $DL(y)$ of a function $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at a point y is that, for any row vector $r \in \mathbb{R}^m$,

$$rDL(y) = co\partial(rL)(y)$$

Here, $\partial(rL)(y)$ is the limiting subdifferential of the function $y \longrightarrow rL(y)$.It follows immediately that the adjoint inclusion can be equivalently written

$$-\dot{p}(t) \in pD_x f(t, \bar{x}(t), \bar{u}(t)),$$

in which $D_x f(t, x, u)$ denote the generalized jacobian with respect to the x variable.

4.A SMOOTH MAXIMUM PRINCIPLE .

This part , provides a self-contained proof of Conditions (i) throught (iv) of the Maximum Principle, Theorem 2.1, in the case when the dynamics constrait is "smooth" with respect to the state variable. The problem of interst remains:

$$
(P) \qquad \begin{cases} \text{Minimize g(x(S),x(T))} \\ \text{over} x \in W^{1,1}([S,T];\mathbb{R}^n) \\ \text{ and mesurable function } u : [S,T] \longrightarrow \mathbb{R}^m \text{ satisfying} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.,} \\ u(t) \in U(t) \quad \text{a.e.,} \\ (x(S), x(T)) \in C \end{cases}
$$

with data an interval $[S,T]$, functions $g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and $f : [S,T] \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, with a nonempty multifunction $U : [S,T] \rightsquigarrow \mathbb{R}^m$, and a closed set $C \subset \mathbb{R}^n \times \mathbb{R}^n$.

theorem 4.1 (A smooth Maximum Principle) Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer for (P). Assume that in addition to hypotheses of Theorem 6.2.1, nmely, there exist $\delta > 0$ sych that

$(H1)$: $f(.,x,.)$ is $\mathcal{L} \times \mathcal{B}^m$ mesurable for a fixed x. There exist a Borel measurable function $k : [S,T] \times \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $t \longrightarrow k(t, \bar{u}(t))$ is integrable and, for a.e. $t \in [S,T]$,

$$|f(t,x,u) - f(t,x',u)| \leq k(t,u)|x - x'|$$

for all $x, x' \in \bar{x}(t) + \delta B$, $u \in U(t)$,

$(H2)$: GrU is $\mathcal{L} \times \mathcal{B}^m$ mesurable,

$(H3)$: g is locally Lipschitz continuous,

and the following hypothesis satisfied.

$(S1):\ f(t,.,u)$ is a continously differentiable on $\bar{x}(t)+\delta$ int B for all $u \in U(t)$ a.e. $t \in [S,T]$.

Then there exist $p \in W^{1,1}([S,,T];\mathbb{R}^n)$ and $\lambda \geq 0$ such that

(i) $(p,\lambda) \neq (0,0)$,

(ii) $-\dot{p}(t) = \mathcal{H}_x(t,\bar{x}(t),p(t),\bar{u}(t))$ a.e,

(iii) $(p(S),-p(T)) \in \partial g(\bar{x}(S),\bar{x}(T)) + N_C(\bar{x}(S),\bar{x}(T)$,

(iv) $\mathcal{H}(t,\bar{x}(t),p(t),\bar{u}(t)) = \max_{u\in U(t)} \mathcal{H}(t,\bar{x}(t),p(t),u)$ a.e.

The smooth Maximum Principle is built up in stages, in which the optimality conditions are proved under hypotheses that are progressively less restrictive. The firt case treated is when the velocity set is compact and convex, the cost function is smooth, there is no end point constraints, and, finally, $(\bar{x},\bar{u})$ is a strong local minimizer.

Proposition 4.2 Let $(\bar{x},\bar{u})$ eb a strong local minimizer of Theorem 4.1 Then the assertions of Theorem 4.1 are valid with $\lambda = 1$ when we assumed that, in addition to $(H1)$ to $(H3)$ and $(S1)$, the following hypothese are satisfied:

$(S2)$ There exist $k_f \in L^1$ such that, for a.e $t \in [S,T]$,

$$|f(t,x,u) - f(t,x',u)| \leq k_f(t)|x-x'| \text{ and } |f(t,x,u)| \leq c_f(t)$$

for all $x,x' \in \bar{x}(t)+\delta B,\ u \in U(t)$;

$(S3):\ f(t,x,U(t))$ is a compact set for all $x \in \bar{x}(t)+\delta B,\ a.e.\ t \in [S,T]$;

$(S4) f(t,x,U(t))$ is a convex set for all $x \in \bar{x}(t)+\delta B,\ a.e.\ t \in [S,T]$;

$(S5)$ g is continously differentiable;

$(S6)$ $C = \mathbb{R}^n \times \mathbb{R}^n$.

Remark: Proofs of all theorems in this section are similar and use the same techniques, therefore it is sufficient to prove a single theorem which will be the following.

Proposition 4.3 The assertions of Proposition 4.2 are valid when, in addition to $(H1)$ throught $(H3)$ and $(S1)$, we impose merely $(S2)$ throught $(S5)$.

Proof .
Fix $\epsilon \in (0,1]$. Reduce the sixe of $\delta > 0$, if neccesary, to ensure that $(\bar{x}, \bar{u})$ is minimizing with respect to all feasible process (x,u) for (P) satisfying $||x - \bar{x}||_{L^\infty} \leq \delta$. Embed (P) (augmented by the constraint $||x - \bar{x}||_{L^\infty} \leq \delta$) in family of problems $\{P(a) : a \in \mathbb{R}^n \times \mathbb{R}^n\}$

$$\text{P(a)} \qquad \begin{cases} \text{Minimize g(x(S),x(T))} \\ \text{over} x \in W^{1,1}([S,T];\mathbb{R}^n) \\ \text{ and mesurable function } u : [S,T] \longrightarrow \mathbb{R}^m \text{ satisfying} \\ \dot{x}(t) = (1-\epsilon) f(t, x(t), \bar{u}(t)) + \epsilon f(t, x(t), u(t)), \text{ a.e.,} \\ u(t) \in U(t) \text{ a.e.,} \\ (x(S), x(T)) \in C + a \\ ||x - \bar{x}||_{L^\infty} \leq \delta \end{cases}$$

Since $f(t, x, U(t)$ is convex and in view of the Generalized Filipov Selection Theorem (Theorem 2.3.13 R.Vinter "optimal Control"), $(\bar{x}, \bar{u})$ is a minimizer for $P(0)$.
We impose an interim hypothesis,
(HS): If (x,u) is a minimizer fo $P(0)$ then $x = \bar{x}$.

(It is discrated later in the proof.)

Denote by V(a) the infimum cost of P(a).(Set $V(a) = +\infty$ if there exist no (x, u)s satisfying the cinstraints of P(a).)
Note the following properties of V.

(i) $V(a) > -\infty$ for all $a \in \mathbb{R}^n \times \mathbb{R}^n$ and if $V(a) < +\infty$ then P(a) has a minimizer

(ii)V is a lower semicontinous function on $\mathbb{R}^n \times \mathbb{R}^n$.

(iii) if $a_i \longrightarrow 0$ and $V(a_i) \longrightarrow V(0)$ and if (x_i, u_i) is a minimizer for $P(a_i)$ for each i, then $x_i \longrightarrow \bar{x}$ uniformly and $\dot{x}_i \longrightarrow \dot{\bar{x}}$ weakly in L^1 as $i \longrightarrow \infty$.

These are straintforward consequences of the Compactness of Trajectories Thoerem(theorem 2.5.3R.V."Optimalcontrol"), result of section 2.6 applied to the multifunction

$$F(t, x) := \{(1 - \epsilon) f(t, x(t), \bar{u}(t)) + \epsilon f(t, x(t), u(t)) : u \in U(t)\},$$

and the Generalied Filippov Selection Theorem, which tells that if $x \in W^{1,1}$, satisfies the differential inclusion

$$\dot{x} \in F(t, x(t)) \quad a.e,$$

then there is a $\bar{u} \in \mathcal{U}$ such that

$$\dot{x}(t) = (1 - \epsilon) f(t, x(t), \bar{u}(t)) + \epsilon f(t, x(t), \tilde{u}(t)) \quad a.e.$$

Since V is lower semicontinous and $V(0) < +\infty$, there exist a sequence $a_i \longrightarrow 0$ such that $V(a_i) \longrightarrow V(0)$ as $i \longrightarrow \infty$ and V has a proximal subdifferential ξ_i at a_i for each i. This means that, for each i, there exist $\alpha_i > 0$ and M_i such that

$$V(a) - V(a_i) \geq \xi_i.(a - a_i) - M_i |a - a_i|^2 \qquad (11)$$

for all $a \in \{a_i\} + \alpha_i B$.

In view of the above properties of V, $P(a_i)$ has a minimizer (x_i, u_i) for each i and $x_i \longrightarrow \bar{x}$ uniformaly. By eliminating initial terms in the sequence we may arrange that

$$||x_i - \bar{x}||_{L^\infty} < \frac{\delta}{4}$$

for all i.
Fix i.Take any (x, u) such that $u \in \mathcal{U}$, x satisfies the differential equation constraint of P(a), and also

$$||x - x_i||_{L^\infty} < \frac{\delta}{2}$$

choose an arbitrary point $c \in C$. Notice that

$$x(S), x(T)) \in C + (x(S), x(T)) - c).$$

This means that (x, u) is feasble process for $P(x(S), x(T)) - c)$. The cost of (x, u) cannot be smaller than the infimum cost $V(x(S), x(T)) - c)$. It follows that

$$g(x(S), x(T)) \geq V(x(S), x(T)) - c) \qquad (12)$$

Define

$$c_i := (x_i(S), x_i(T)) - a_i$$

Since (x_i, u_i) solves $P(a_i) = P((x_i(S), x_i(T)) - c_i))$, we have

$$g(x_i(S), x_i(T)) = V(x_i(S), x_i(T)) - c_i) \qquad (13)$$

Now define the function

$$J_i((x, u), c) := g(x(S), x(T)) - \xi.(x(S), x(T)) - c) + M_i(|(x(S), x(T)) - (x_i(S), x_i(T)) - (c - c_i)|^2).$$

From (11) throught (13), we deduce that

$$J_i((x, u), c) \geq J_i((x_i, u_i), c_i) \qquad (14)$$

for all $c \in C$ and all (x, u)s satisfying

$\dot{x}(t) = (1 - \epsilon)f(t, x(t), \bar{u}(t)) + \epsilon f(t, x(t), u(t)) \quad a.e$
$u(t) \in U(t) \quad a.e,$
$||x - \bar{x}||_{L^\infty} \leq \delta$

Set $(x, u) = (x_i, u_i)$ in (14). The inequality implies

$$-\xi.(c - c_i) \leq M_i|c - c_i|^2 forallc \in C.$$

We conclude that

$$-\xi_i \in N_C^P((x_i(S), x_i(T)) - a_i),$$

Next set $c = c_i$. We see that (x_i, u_i) is a strong local minimizer for

$$\begin{cases} \text{Minimize } g(x(S), x(T)) + M_i(|(x(S), x(T)) - (x_i(S), x_i(T))|^2 - \xi_i((x(S), x(T)) \\ \text{over } x \in W^{1,1} \\ \text{and mesurable function u satisfying} \\ \dot{x}(t) = (1-\epsilon)f(t, x(t), \bar{u}(t)) + \epsilon f(t, x(t), u(t)), \text{ a.e.,} \\ u(t) \in U(t) \text{ a.e.,} \end{cases}$$

This is an "endpoint constraint-free" optimal control problem to which the necessary conditions of the special case of the Maximum Principle (4.2), are applicable.We deduce the existence of an adjoint arc $p_i \in W^{1,1}$ such that

$$-\dot{p}_i(t) = p_i(t)((1-\epsilon)f_x(t, x_i(t), \bar{u}(t)) + \epsilon f_x(t, x_i(t), u_i(t))), \tag{15}$$

$$p_i(t).\dot{x}_i(t) \geq p_i(t).((1-\epsilon)f(t, x_i(t), \bar{u}(t)) + \epsilon f(t, x_i(t), u)) \text{ for all } u \in U(t), \tag{16}$$

$$\begin{aligned}(p_i(S), -p_i(T))(&= \lambda_i \nabla g(x_i(S), x_i(T)) - \lambda_i \xi_i) \\ &\in \lambda_i \nabla g(x_i(S), x_i(T)) + N_C(x_i(S), x_i(T)) - a_i),\end{aligned} \tag{17}$$

in which $\lambda_i = 1$. Now scale p_i and λ_i (we do not relabel) so that

$$|p_i(S)| + \lambda_i = 1 \tag{18}$$

Recall that

$$x_i \longrightarrow \bar{x} \quad uniformly.$$

Since $\{p_i(S)\}$ is a bounded sequence, we deduce from (15) that the $p_i s$ are uniformly bounded and $\dot{p}_i s$ are uniformly integrably bounded. Along a subsequence then $p_i \longrightarrow p$ uniformly for some $p \in W^{1,1}$. Since $\{\lambda_i\}$ is a bounded sequence, we may arrange by yet another subsequence extraction that $\lambda_i \longrightarrow \lambda$. We deduce from (15) with the help of compactness of trajectories Theorem that p satifies

$$-\dot{p}(t) \in p(t)f_x(t, \bar{x}(t), \bar{u}(t)) + 2\epsilon |p(t)| k_f(t) B.$$

From (16) we see that, for arbitrary $u \in \mathcal{U}$,

$$\int_S^T p_i(t).\dot{x}_i(t)\, dt \geq \int_S^T p_i(t).((1-\epsilon)f(t, x_i(t), \bar{u}(t)) + \epsilon f(t, x_i(t), u(t)))\, dt.$$

Now $\dot{x}_i \longrightarrow \dot{\bar{x}}$ weakly in L^1 and $p_i \longrightarrow p$ and $x_i \longrightarrow \bar{x}$ uniformly. Passing to the limit (with the help of the dominated convergence theorem), noting that $\dot{\bar{x}} = f(t, \bar{x}(t), \bar{u}(t))$ and deviding across the resulting inequality by ϵ yields

$$\int_S^T p(t).(f(t, \bar{x}(t), u(t)) - f(t, \bar{x}(t), \bar{u}(t)))\, dt \leq 0 \quad (20)$$

From (17) and the closure properties of the limiting normal cone we deduce that

$$(p(S), -p(T)) \in \lambda \nabla g(\bar{x}(S), \bar{x}(T)), N_C(\bar{x}(S), \bar{x}(T)). \quad (21)$$

It follows from (18) that

$$|p(S)| + \lambda = 1 \quad (22)$$

All the assertions of the proposition have been verified except that condition (20) is an "integral" form of the weierstrass condition and a perturbation term $2\epsilon k_f |p| B$ currently appears in the adjoint inclusion (19).Notice however that p and λ have been constructed for a particular $\epsilon > 0$. Take a sequence $\epsilon_j \downarrow 0$. For each j, there are elements p_j and λ_j satisfying (18) to (22) (when p_j, λ_j, and ϵ_j replace p, λ, and ϵ). A by now familiar convergence analysis yields limits p and λ satisfying (18) to (22), but with the perturabtion absent.

We next allow a possibility nonconvex velocity set and a general Lipschitz continuous coast function, provided that a constraint is imposed only on the left endpoint of state trajectories.

Proposition 4.4 Consider the special cas of (P) in wich the endpoint constraint set C can be expressed

$$C = C_0 \times \mathbb{R}^n$$

for some closed set $C_0 \subset \mathbb{R}^n$. Let $(\bar{x}, \bar{u})$ be a strong local minimizer. Then the assertions of Theorem 4.1 are valid with $\lambda = 1$ when, in addition to $(H1)$ throught $(H3)$ and $(S1)$, we impose merely the hypotheses $(S2)$ and $(S3)$.

The final step is to allow a general endpoint constraint.

Proposition 4.5 The assertions of theorem 4.1 are valid when $(\bar{x}, \bar{u})$ is assumed to be a strong local minimizer and when, in addition to $(H1)$ throuht $(H3)$ and $(S1)$ of theorem 4.1, Hypotheses $(S2)$ and $(S3)$ of Proposition 4.2 are imposed.

Part III
A GENERAL VERSION OF EULER-LAGRANGE CONDITIONS

Till now we derived necessary conditions of optimality for different optimisation problems over standard hypotheses, most of times. Now we will introduce a proof for an optimisation problem with more general hypotheses. To be more precise, the bound hypothese on the velocity sets or the lipschitz continuity are not always valid.

For exemple let's take the velocity set F dependent only on x and expressed by

$$F(x = (x_1, x_2)) := \{v = v_1, v_2)/v_1 \leq x_1 v_2\}$$

we notive that this set is not bounded nor verify a lipschitz inequality so we have an unvalid hypothese... To make things work, and so derive some necessary conditions of optimality, an hypothese of pseudo-lipschitz or a bounded slope conditions comes... Let's begin first by introducing ower problem.

Minimize $J(x) := l(x(a), x(b))$ over an arc x satisfying the differential inclusion and boundary condition

$$\dot{x}(t) \in F(t, x(t)) \;\; a.e, \quad (x(a), x(b)) \in E \tag{1}$$

An arc x refers to an absolutely continous function, $x : [a, b] \longrightarrow \mathbb{R}^n$; is said to be admissible fo the problem if it satisfies (1)

F here is a mulifunction mapping $[a, b] \times \mathbb{R}^n$ to the subsets of $\mathbb{R}^n$

For each $t \in [a, b]$ the graph of the multifunction $F_t(.)$ is the set

$$G_t := \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n : v \in F(t, x)\}$$

Local minimizer let x_* be an admissible trajectory for the problem, and let R be multifunction from $[a, b]$ to $\mathbb{R}^n$ such that $x'_* \in R(t)$ a.e. We say that x_* is a local minimum for the problem in the following sense: for some $\epsilon_* > 0$, for all admissible arc x satisfying

$$|x(t) - x_*| \leq \epsilon_*, \int_b^a |\dot{x}(t) - \dot{x}'_*(t)|\, dt \leq \epsilon_*, \dot{x}(t) \in F(t, x(t)) \cap R(t) \; a.e,$$

we have $J(x_*) \leq J(x)$. The multifunction R will be called a radius.

We make the hypothesis that all functions and multifunctions that appear in the formulations of problems and theorems are mesurable, in the sens of Lebesgue if they depend only on t, or else in the $\mathcal{L} \times \mathcal{B}$ sens if they depend on t and x.

Hypotheses of Theorem 1.1 .

(H1) The function l is locally lipschitz; the set E is closed; for almost every t, the set G_t is locally closed in the following sens:

$$|x - x_*(t)| < \epsilon_*, v \in F(t,x) \cap R(t), C compact \Rightarrow G_t \cap C$$

(H2) (bounded slope condition) There exist a summable function k such that, for almost every t, we have

$$|x - x_*(t)| < \epsilon_*, v \in F(t,x) \cap R(t), (\alpha, \beta) \in N^P_{G_t} \Rightarrow |\alpha| \leq K_t|\beta|.$$

(H3) For some $\eta > 0$, for almost every t, we have: $R(t)$ is an open convex set satisfying

$$R(t) \supset B(x'_*(t), \eta k_t).$$

Theorem 1.1 Under the hypotheses $(H1),(H2),(H3)$ above there exist an arc p and $\lambda_0 \in \{0,1\}$ with $(\lambda_0, p(t)) \neq 0 \;\; \forall t$ satisfying the Euler inclusion

$$p'(t) \in co\{w : (w, p(t)) \in N^L_{G_t}(x_*(t), x'_*(t))\} a.e \tag{2}$$

together with the Weierstrass condition of radius R: for almost every t we have

$$p(t).v \leq p(t).x'_*(t) \;\forall v \in F(t, x_*(t)) \cap R(t), \tag{3}$$

and the transversality condition

$$(p(a), -p(b)) \in \partial_L \lambda_0 l(x_*(a), x_*(b)) + N_E^L(x_*(a), x_*(b)). \qquad (4)$$

Theorem 2.1 Let Y be a compact, convex of $\mathbb{R}^n$, and $\phi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ a lower semicontinous function. Let any real number m no greater than the quantity $\min_Y \phi - \phi(x_0)$. Then there exist a point z in the δ-neighberhood of the interval

$$[x_0, Y] := co[Y \cup \{x_0\}]$$

together with an element $\zeta \in \partial_P \phi(z)$ such that

$$m < \zeta.(y - x_0) + \delta \ \forall u \in Y, \quad \phi(z) < \min_{[x_0,Y]} \phi + |m| + \delta$$

Lipschitz continous function Let Γ be a multifunction from $\mathbb{R}^n$ to $\mathbb{R}^n$, with closed graph, and let d_G denote the euclidean distance function.

Proposition 1 Suppose that Γ satisfies the Lipschitz condition

$$\Gamma(y) \subset \Gamma(z) + B(0, k|y - z|) \qquad (5)$$

for all $y, z \in B(x_0, r)$, where $x_0 \in \mathbb{R}^n, r > 0$. Let $v_0 \in \Gamma(x_0)$. Then

$$(\alpha, \beta) \in N_G^L(x_0, v_0) \Rightarrow |\alpha| < k|\beta| \qquad (6)$$

If (5) holds for all $y, z \in \mathbb{R}^n$ then for any $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$(\alpha, \beta) \in \partial_L d_G(x, v) \Rightarrow |\alpha| < k|\beta| \qquad (7)$$

and

$$d_G(x, v) > 0, (\alpha, \beta) \in \partial_L d_G(x, v) \Rightarrow |\beta| \geq (1 + k^2)^{-1/2} \qquad (8)$$

Proof It suffices to prove (6) for $(\alpha,\beta)\in N_G^P(x_0,v_0)$ since $N_G^P generate N_G^L$ via Limits.
Sinxe $(x_0,v_0)\in G$, it suffices to consider $(\alpha,\beta)\in\partial_P d_G(x_0,v_0)$
The proximal inequality asserts that locally for some $\sigma\geq 0$, we have

$$d_G(x,v)+\sigma|(x-x_0,v-v_0|^2\geq(\alpha,\beta).(x-x_0,v-v_0).$$

for all x near x_0 . By (5) there exist $v\in\Gamma(x)$ such that $|v-v_0|\leq k|x-x_0|$.
Since $d_G(x,v)=0$ the proximal inequality leads to

$$\alpha.(x-x_0)\leq\beta.(v-v_0)+\sigma(|x-x_0|^2+|v-v_0|^2)\leq|\beta|k|x-x_0|+\sigma(1+k^2)|x-x_0|^2$$

for all x near x_0. Suppose that $|x-x_0|\leq\epsilon$ and Let$\epsilon\downarrow 0$ then $|\alpha|\leq|\beta|$ as we want.
Now consider (7)(8), for wich only the case $d_G(x,v)>0$ need to be considered. We know that (α,β) takes the form $\dfrac{(x-\bar{x}),(v-\bar{v})}{|(x-\bar{x}),(v-\bar{v})|}$ where $(\bar{x},\bar{v})$ is the closest point in G to (x,v). We have $(\alpha,\beta)\in N_G^L(\bar{x},\bar{v})$, then by (6) we have $|\alpha|\leq|\beta|$. Note also that $|(\alpha,\beta)|=1\Rightarrow|\alpha|^2+|\beta|^2=1$, then
$|\beta|^2=1-|\alpha|^2|$ or$|\beta|^2\geq\dfrac{|\alpha|^2}{k^2}$ then $|\beta|^2\geq\dfrac{1}{k^2}-\dfrac{|\beta|^2}{k^2}$ and so $|\beta|\geq(1+k^2)^{-1/2}$.

Theorem 2.2 Let $\Gamma:[a,b]\times\mathbb{R}^n\to\mathbb{R}^m$ be a measurable multifunction, f_i,x_i,r_i measurable multifunctions, and Ω_i measurable subset of $[a,b](i=1,2,....)$ such that for each i we have

$$f_i(t)\in\Gamma_t(x_i(t))+B(0,r_i(t))t\in\Omega_i.$$

We suppose more that:

1. $\lim_{i\longrightarrow\infty}$ meas $\Omega_i=b-a$ and r_i converges weakly to 0 in $L^1(a,b)$.
2.$\Gamma_t(.)$ has closed graph for each t, and Γ has convex values.
3. There is a function x such that $x_i(t)\longrightarrow x(t)$ a.e. in $[a,b]$.
4. There is a summable function ϕ such that, for each i, $|f_i(t)|\leq\phi(t)$ a.e. in $[a,b]$.
5.f_i converges weakly in $L^1(a,b)$ to a limit f.
Then $f(t)\in\Gamma_t(x(t))$ a.e in $[a,b]$.

The lipschitz problem of Bolza We now consider the problem of minimizing for thr bolza functional

$$J(x) := l_0(x(a)) + l_1(x(b)) + \int_a^b \Lambda_t(x(t), \dot{x}(t))\, dt \qquad (9)$$

over all arcs $x : [a, b] \longrightarrow \mathbb{R}^n$ satisfying the constraints

$$x(a) \in C_0, x(b) \in C_1, \dot{x}(t) \in V(t) \quad a.e. \quad (10)$$

Where $[a, b]$ is a given fixed interval in $\mathbb{R}, C_0, C_1$ are closed subset of $\mathbb{R}^n, l_0, l_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$ are locally lipschitz functions, and V is a measurable mapping from $[a, b]$ to the closed convex subsets of $\mathbb{R}^n$

x_* is said to be a local minimizer if for some ϵ_*, for any arc x admissible satisfying

$$|x(t) - x_*(t)| \leq \epsilon_*, \forall t \in [a, b], \int_a^b |\dot{x}(t) - x'_*(t)|\, dt \leq \epsilon_*$$

We have $J(x_*) \leq J(x)$.

Λ is a mapping from $[a, b] \times \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ with Λ_t locally lipschitz. There exist a summable function $k : [a, b] \longrightarrow \mathbb{R}$ such that for almost all t, for all $x, y \in B(x_*(t), \epsilon_*)$ and $v, w \in V_t$,

$$|\Lambda_t(x, v) - \Lambda_t(y, w)| \leq k_t\{|x - y| + |v - w|\} \qquad (11)$$

We suppose more That there is a positive δ such that $V_t \supset B(x'_*(t), \delta) a.e.$

Theorem 2.3 Under these hypotheses, there exist an arc p wich satisfies the Euler inclusion

$$p'(t) \in co\{w : (w, p(t)) \in \partial_L \Lambda_t(x_*(t), x'_*(t)) a.e.\ t \in [a, b] \quad (12)$$

together with the Weierstrass condition, for almost every t we have

$$p(t).v - \Lambda_t(x_*(t), v) \leq p(t).x'_*(t)) \quad \forall v \in V_t \quad (13)$$

and the transversality condition

$$p(a) \in \partial_L l_0(x_*(a)) + N^L_{C_0}(x_*(a)), \quad -p(b) \in \partial_L l_1(x_*(b)) + N^L_{C_1}(x_*(b)). \quad (14)$$

We will focus on doing a full long proof of one of the theorems then as ususal the rest will be similar or a spetial case. F now is taken lipschitz, hypothese $(H3)$ holds since the radius $R(t)$ is taken to be $\mathbb{R}^n$

We consider the following minimization problem

$$J(x) := l(x(a), x(b)) + \int_a^b \Gamma_t(x(t), \dot{x}(t))\, dt \quad (15)$$

over the arc x satisfying the differential inclusion and boundary condition

$$\dot{x}(t) \in F(t, x(t))a.e., (x(a), x(b)) \in E \quad (16)$$

an arc is said to be admissible if it satisfes (16), and if the integral (15) is well defined and finite.

An arc x_* is said to bed local minimizer for our problem if for some ϵ_*, for any arc x admissible satisfying

$$|x(t) - x_*(t)| \leq \epsilon_*, \forall t \in [a, b], \int_a^b |\dot{x}(t) - \dot{x}_*(t)|\, dt \leq \epsilon_*$$

We have $J(x_*) \leq J(x)$.

3.Basic theorem :

Hypotheses of the theorem 3.1 .

(HP1) The function l is locally lipschitz, the set E is colsed, the following set is closed for almost every t

$$\{(x, v) \in G_t : |x - x_*(t)| \leq \epsilon\}$$

(HP2) There exist a constant k such that, for almost every t,

$$x, y \in B(x_*(t), \epsilon_*) \Rightarrow F(t, y) \subset F(t, (x) + B(0, k|x - y|).$$

(HP3) for almost every t, Λ_t is locally lipschitz function, there exist a summable functions k^x, k^v such that, for almost every t, for all $y, z \in B(x_*(t), \epsilon_*)$, for all $(u, w) \in F(t, y) \times F(t, z)$, we have

$$|\Lambda_t(y, u) - \Lambda t(z, w)| \leq k_t^x |y - z| + k_t^v |u - w|.$$

Theorem 3.1. Under these hypothese, there exist an arc p and $\lambda_0 \in \{0, 1\}$ with $(\lambda_0, p(t)) \neq 0 \forall t$ satisfying the Euler inclusion

$$p'(t) \in co\{w : (w, p(t)) \in \partial_L \lambda_0 \Lambda_t(x_*(t), x_*'(t)) + N_{G_t}^L(x_*(t), x_*'(t))\} a.e. \quad (17)$$

together with the Weierstrass condition, for almost every t we have

$$p(t).v - \lambda_0 \Lambda_t(x_*(t), v) \leq p(t).x_*'(t)) - \lambda_0 \Lambda_t(x_*(t), x_*'(t)) \quad \forall v \in F_t(x_*(t). \quad (18)$$

and the transversality condition

$$(p(a) - p(b)) \in \partial_L \lambda_0 l(x_*(a), x_*(b)) + N_E^L(x_*(a), x_*(b)). \quad (19)$$

Proof we will prove the thoerem first in the presence of two additional hypotheses, then we will remove them

$(TH1)$ $l(x_1, x_2)$ in on the form $l(x_2)$ and E is of the form $C_0 \times C_1$

$(TH2)$ Λ is identically zero

For a positive sequence ϵ_i decreasing to 0, we consider the problem of minimizing

$$J_i(x) := l_i(x(b)) + (\frac{1}{\epsilon_i}) \int_a^b d_{G_t}(x_t, \dot{x}(t))\, dt$$

over the set A of arcs x satisfying

$$x(a) \in C_0, x(b) \in C_1, |x(t), x_*(t)| \leq \epsilon_* \forall t, \int_a^b |\dot{x}(t)$$

where

$$l_i(x) := [l(x) - l(x_*(b) + \epsilon_i^2]_+$$

avec $[a]_+ := max\{0, a\}$.

The set A is a complete metric space when equipped with the norm

$$d(x, y) := |x(a) - y(a)| + \int_a^b |\dot{x}(t) - y'(t)|\, dt$$

Note that the infimum of this problem is nonnegative, we remark also that Ekeland's Theorem is applcable since x_* is an $\epsilon_i^2 - minimum(J_i(x_*) = l(x_*(b) - l(x_*(b) + \epsilon_i^2 + 0 = \epsilon_i^2)$. So we now now that there exist an arc x_i minimizer for the perturbed new cost

$$l_i(x(b)) + \epsilon_i |x(a) - x_i(a)| + (\frac{1}{\epsilon_i}) \int_a^b d_{G_t}(x_t, \dot{x}(t))\, dt + \epsilon_i \int_a^b |\dot{x}(t) - \dot{x}_i(t)|\, dt.$$

and we have

$$||x_i - x_*||_\infty + ||\dot{x}_i - \dot{x}_*||_1 < \epsilon_*$$

for i sufficiently large. By theorem 2.3 x_i is a local minimum(with $V_t \equiv \mathbb{R}^n$)). Then there exist an arc p_i such that

$$-p_i(b) \in \partial_L l_i(x_i(b)) + N_{C1}^L(x_i(b)), p_i(a) \in \partial_L l_i(x_i(a)) + N_{C0}^L(x_i(a)) + \epsilon_i B \qquad (20)$$

$$p_i'(t) \in co\{w : (w, p_i(t) \in (\frac{1}{\epsilon_i})\partial_L d_{G_t}(x_i(t), x_I'(t)) + \{0\} \times \epsilon_i B\}\ a.e \qquad (21)$$

$$p_i(t).v - (\frac{1}{\epsilon_i}) d_{G_t}(x_i(t), v) - \epsilon_i |v - x_i'(t) \leq p_i(t).x_i'(t) - (\frac{1}{\epsilon_i}) d_{G_t}(x_i(t), x_i'(t)) \forall\, v\ a.e \quad (22).$$

Note that G_t is locally closed a.e. at $(x_i(t), x_i'(t))$ by hypotheses $(HP1)$ and since $||x_i - x_*||_\infty + ||x_i' - x_*'||_1 < \epsilon_*$. Applying proposition 1, with the lipschitz hypothese $(HP2)$ we deduce that

$$|p_i'(t)| \leq k(|p_i(t)| + \epsilon_i)\ a.e \quad (23)$$

and that's because $p_i'(t) \in co\{w : (w, p_i(t) :\in (\frac{1}{\epsilon_i})\partial_L d_{G_t}(x_i(t), x_I'(t)) + \{0\} \times \epsilon_i B\}$, so $p_i'(t) \in co\{w : \epsilon_i(w, p_i(t) - \epsilon_i B) \in \partial d_{G_t}(x_i(t), x_i'(t))$. Proposition 1 implies that $\epsilon_i |w| \leq k\epsilon_i(p_i(t) + \epsilon_i)$ and by Carathéodory's theorem for convex hull we have the desire $|p_i'(t)| \leq k(|p_i(t)| + \epsilon_i)$.

Note also that (22) implies, for almost every t:

$$p_i(t).v - \epsilon_i |v - x_i'(t)| \leq p_i(t).x_i'(t)\ \forall v \in F_t(x_i(t). \quad (24)$$

Convergence: By taking a subsequence as necessary (without relabeling), we may arrenge that $\int_a^b d_{G_t}(x_i, x_i'(t))\, dt$ is strictly positive for all i or lese zero for every i. We also arrange to have x_i' converge almost everywhere to x_*'.

$Case 1 : \int_a^b d_{G_t}(x_i, x_i'(t))\, dt > 0 \ \forall i.$

In this case there exist a set S_i of positive measure on which $d_{G_t}(x_i, x_i') > 0$. Proposition 1 and (21) implies

$$\frac{1/\epsilon_i}{(1+k^2)^{1/2}} - \epsilon_i \leq |p_i(t)| \leq 1/\epsilon_i + \epsilon_i \ a.e.t \in S_i \quad (25)$$

We proceed to write $(20) - (22)$ with p_i replaced by $\epsilon_i p_i$, we obtain then

$$-p_i(b) \in \partial_L \epsilon l_i(x_i(b)) + N^L_{C1}(x_i(b)), p_i(a) \in \partial_L l_i(x_i(a)) + N^L_{C0}(x_i(a)) + \epsilon_i^2 B \qquad (26)$$

$$p_i'(t) \in co\{w : (w, p_i(t) :\in \partial_L d_{G_t}(x_i(t), x_I'(t)) + \{0\} \times \epsilon_i^2 B\} \ a.e \qquad (27)$$

$$p_i(t).v - d_{G_t}(x_i(t), v) - \epsilon_i^2 |v - x_i'(t) \leq p_i(t).x_i'(t) - (\frac{1}{\epsilon_i}) d_{G_t}(x_i(t), x_i'(t)) \forall \ v \ a.e \quad (28).$$

The inequality (23) becomes $|p_i'(t)| \leq k(|p_i(t)| + \epsilon_i^2)$, and (25) yields

$$\frac{1/\epsilon_i}{(1+k^2)^{1/2}} - \epsilon_i^2 \leq |p_i(t)| \leq 1 + \epsilon_i^2 \ \forall t \in S_i.$$

All these fact allow us to deduce with the aid of Gronwall's Lemma and Ascoli's Thoerem that for a subsequence, p_i converge uniformaly to an arc p : and p_i' converge weakly in L^1 to p'. Note that we have $||p||_\infty \geq (1+k^2)^{-1/2}$. We want now passing to limit in (27) but it's not simple like that so we proceed by defining a multi function Γ

$$\Gamma_t(x, v, p, a^0, a^1, ..., a^n) := co \cup_{j=0}^n \{w : (w, p) \in \partial_L d_{G_t}(x, v) + (0, a^j)\},$$

Where a_j belongs to $\mathbb{R}^n$. By closure properties of ∂_L we deduce that the set $\{w : (w, p) \in \partial_L d_{G_t}(x, v) + (0, a)\}$ is closed and by Proposition1 we deduce that it's uniformaly bounded if $x \in B(x_*(t), \epsilon_*)$, and (a, p) is restricted to a bouded set,then it follows that Γ_t has closed graph.
By caratheodory's theorem, and (27), there exist a convex combination λ^j, points $a^j \in B(0, \epsilon_i^2)$, and w^j such that

$$(w^j, p_i) \in \partial_L d_{G_t}(x_i, x_i') + (0, a^j), p_i' = \sum_{j=0}^n \lambda^j w^j.$$

It follows that

$$p_i' \in \Gamma_t(x_i, x_i', p_i, a^0, a^1, ..., a^n).$$

Now using Compactness theorem 2.2 and passing to the limit we deduce that

$$p'(t) \in \Gamma_t(x_*(t), x_*'(t), p(t), a^0, a^1, 0, ..., 0) = co\{w : (w, p(t) \in \partial_L d_{G_t}(x_*(t), x_*'(t))\} a.e.$$

This implies (17) since $(x_*(t), x_*'(t)) \in G_t$ and $\partial_L d_{G_t}(x_*(t), x_*'(t)) = N_{G_t}(x_*(t), x_*'(t))$ and $\Gamma = 0 (TH2)$ A further consequence is that $|p'(t)| \leq k|p(t)|$ a.e., or else it would be identically zero by Gronwall's lemma and that's a contradiction since $|p|$ is bigger that $(1+k^2)^{-1/2}$ Finally it's clear that (26) leads to (19), with $\lambda_0 = 0$ amd (28) gives (18) in limits, so all conditions are now verified in case 1.

$Case2 : \int_a^b d_{G_t}(x_i, x_i')\, dt = 0\ \forall i.$

It follows in this case taht x_i verifies the differential inclusion and it's a F trajectory . Then $l_i(x(b)) > 0 \forall i$ since x_* is a local minimizer. Now observe that (21) implies

$$p_i'(t) \in co\{w : (w, p(t)) \in N^L_{G_t}(x_i(t), x_i'(t)) + \{0\} \times \epsilon_i B\} a.e \quad (30)$$

Now seperate the two cases $||P||_\infty$ bouded or $||P||_\infty \longrightarrow 0$

In the first case, Since p_i bounded(Gronwall's lemma) together with (23) implies that for a subsequence

$$p_i \rightrightarrows p \quad and \quad p_i' \rightharpoonup p' \text{ in} L^1$$

So we pass to the limits in (30) as expalined above to deduce (17) with $\Gamma \equiv 0$, and it's clear that (20) leads to (19), with $\lambda_0 = 1$. Now it's remains (18

Fix any t for wich $x_i'(t) \longrightarrow x_*'(t)$ as well as $x_i'(t) \in f_t(x_i(t)) \forall i$, for which $(HP2)$ and (24) holds. choose now any $v \in F_t(x_*(t))$, for each i, and by$(HP2)$ there exist $v_i \in F_t(x_i(t))$ such that $|v - v_i| \leq k|x_i(t) - x_*(t)|$. Then (24) holds for v_i and passing to the limit we deduce that $p(t).v \leq p(t).x_*'(t)$ as required.

In the second case, when $||p||_\infty \longrightarrow \infty$, we devide by $||p_i||_\infty$ in (20)(24)(30) then buy the same convergence argument give the existence of an arc p with $||p||_\infty = 1$ which satisfies the conditions needed with $\lambda_0 = 0$

Removal of temporary hypotheses .

Suppose now that $(TH2)$ is not satisfied and that only $(TH1)$ is given. We extend the state by a new coordinate y,and a new multifunction

$$\hat{F}(x,y) := \{(v, \Gamma_t(x,v)) : v \in F_t(x)\}.$$

We also define the new $\hat{l}$ by

$$\hat{l}(x,y) := l(x)+y, \hat{\Gamma} \equiv 0, \hat{C}_0 := C_0 \times \{0\}, \hat{C}_1 := C_1 \times \mathbb{R}, y_*(t) := \int_a^t \Gamma_s(x_*(s), x'_*(s))\, ds.$$

It follows that the arc (x_*, y_*) is a solution to the new extended problem (in the saame local sens). Since $\hat{F}$ and $\hat{l}$ satisfies the hypothese of the theorem as we as $(TH1)(TH2)$ and then buy the same steps above we prove the existence of all conditions without $(TH2)$

Finally consider the theorem with absence of any temporary hypothesis, we define a new extended sate (x,y) a new multifunction F_t^+, new l^+ and Γ^+ such taht

$$F_t^+(x,y) := \{(v,0) : v \in f_t(x)\}, l^+(x,y) := l(y,x), \Gamma_t^+(x,y,v,w) := \Gamma_t(x,v)$$

and the boundary constraint

$$C_0^+ := \{(x,y) : x = y\}, C_1^+ := \{(x,y) : (y,x) \in E\}$$

Then the arc $(x_*, x_*(a))$ is a solution to the problem corresponding to the cost $l^+(x(b), y(b))$. Sice $(HP1) to (HP3)$ are verified as well as $(TH1)$ and by the same way as above we have ower results.

Part IV
OPTIMAL MULTIPROCESSES

1.Introduction We consider now optimisation problems in which the process is replaced by a multiple processes, this is a modification to the old problems at a finite number of times and change of dynamics in each intervale. An exemple to this type of study is the refraction of light or the problems studied in different environments which imposes a change in the dynamics. Let's begin first buy some essential definitions to our new study.

2. Essential values Let $S \subset \mathbb{R}$ be an open subset subset , T a point in S, and $\psi : S \longrightarrow \mathbb{R}^k$ a mesurable function. The set of essential values of ψ at T, denoted $ess_{t \longrightarrow T}\Psi(t)$, is defined as follows.$\zeta$ belongs to this set if and only if, for any positive number $\epsilon > 0$, the following set has positive Lebesgue measure:

$$\{t : T - \epsilon < t < T + \epsilon, |\zeta - \psi(t)| < \epsilon\}.$$

If a point lies in co $ess_{t \longrightarrow T}\xi(t)$ we say it is a convex essential value of ψ at T.
It's clear that if ψ is continous at T then:

$$ess_{t \longrightarrow T}\psi(t) = \{\psi(T)\}$$

Closed multifunction Given a set $D \subset \mathbb{R}^l$ and a multifunction $A : D \rightsquigarrow \mathbb{R}^k$, we say that A is colsed if , for any convergence subsequences $\{y_i\} \subset D$ and $\{a_i\} \subset \mathbb{R}^k$ sunch that $a_i \in Ay_i$ and $y \in D$ we have $a \in Ay(y, a$ limits of $\{y_i\}$ and a_i respectively).

Lemma.2.1 Let P, Q be open subsets of $\mathbb{R}, \mathbb{R}^n$, respectively, and let $h : P \times Q \longrightarrow \mathbb{R}^k$ be a given finction. Suppose $x \longrightarrow h(t, x)$ is continuous, umifornaly in t, and
$t \longrightarrow h(t, x)$ is measurable for every $x \in Q$.

Then the multifunction $G : P \times Q \rightsquigarrow \mathbb{R}^k$ defined by $G(t, x) = ess_{s \longrightarrow t}h(s, x)$ is closed. If in addition we have

$$\sup_{x \in P} \; ess_{t \longrightarrow t}|h(s, x)| < \infty,$$

Then $(t,x) \longrightarrow co\ G(t,x)$ is also a closed multifunction.

Proof Consider (t_i, x_i) in $P \times Q$ such that $(t_i, x_i) \longrightarrow (t,x)$ where $t \in P$ and $x \in Q$. Consider also $r_i \in ess_{s \longrightarrow t_i} h(s, x_i)$ for each $i, t \in P$, and $x \in Q$. We must show that $r \in ess_{s \longrightarrow t} h(s,x)$.

choose $\epsilon > 0$ and define

$$S_i^\epsilon := \{s \in (t_i - \epsilon/2, t_i + \epsilon/2) \cap P / |h(s, x_i) - r_i| < \epsilon/2\}.$$

By definition of essential values, this set has positive measure, since $r_i \in ess_{s \longrightarrow t_i} h(s, x_i)$
for i sufficiently large $|t_i - t| < \epsilon/2$ and $|h(s, x_i - h(s,x)| + |r - r_i| < \epsilon/2$ for all $s \in P$ since h is continue in x, and r_i goes for r. It follows that

$$S_i^\epsilon \subset S$$

where

$$S^\epsilon = \{s \in (t - \epsilon, t + \epsilon) \cap P / |h(s,x) - r| < \epsilon\}.$$

The set S^ϵ then has positive measure. Since ϵ is arbitrary,$r \in ess_{s \longrightarrow t} h(s,t)$.

it remains the second assertion wich is simple by a compactness argument. Suppose that

$$\sup_{x \in P} ess_{t \longrightarrow t} |h(s,x)| < \infty,$$

, consider $(t,x) \longrightarrow co\ G(t,x)$. By caratheodoty's theorem, $\xi \in coG(t,x)$ can be writenn as the combination $\lambda_i \xi_i$ where $\xi \in G(t,x)$, and $\sum \lambda_i = 1$. Consider $(t_i, x_i, r_i) \in P \times Q \times co\ G(t, x_i) \longrightarrow (t, x, r)$ since the convexe hull of a compact is also compact we have $(t, x, r) \in P \times Q \times co\ G(t,x)$ and so $(t,x) \longrightarrow co\ G(t,x)$ is closed.

3. A maximum principle for optimal multiprocesses .

To simplify we denote a point $((a_1, b_1, ...), (a_2, b_2, ...), ..., (a_k, b_k, ...))$ by $\{a_i, b_i, ...\}_{i=1}^k$

or, $\{a_i, b_i, ...\}$.

The following data are given:

positive integeres k, and n_i, m_i, $\quad i = 1, ..., k$,
functions $\phi_i : \mathbb{R} \times \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \longrightarrow \mathbb{R}^{n_i}, \quad i = 1, ..., k$,
subsets U^i of $\mathbb{R} \times \mathbb{R}^{m_i}, \quad i = 1, ..., k$,
subsets X^i of $\mathbb{R} \times \mathbb{R}^{n_i}, \quad i = 1, ..., k$,

A multiprocess is a point $\{\tau_0^i, \tau_1^i, x_i(.), w_i(.)\}$ comprising left and right end-points τ_0^i and τ_1^i of a closed interval $[\tau_0^i, \tau_1^i]$ of $\mathbb{R}$, absolutely continous functions $x_i(.) : [\tau_0^i, \tau_1^i] \longrightarrow \mathbb{R}^{n_i}$ and mesurable functions $w_i(.) : [\tau_0^i, \tau_1^i] \longrightarrow \mathbb{R}^{m_i}$ such that

$$
\begin{aligned}
&x_i(t) = \phi_i(t, x_i(t), w_i(t)) && a.e.\ t \in [\tau_0^i, \tau_1^i],\\
&w_i(t) \in U_t^i, && a.e.\ t \in [\tau_0^i, \tau_1^i],\\
&x_i(t) \in X_t^i, && \text{for all } t \in [\tau_0^i, \tau_1^i],
\end{aligned}
$$

for $i = 1, ..., k$. Here U_t^i is the set $\{u | (t, u) \in U^i\}$, and $X_i^t := \{x | (t, x) \in X^i\}$.

To generate our neccesary conditions we assume that th data defined satisfies the following hypotheses.

$(H1)$ For each $x \in \mathbb{R}^{n_i}, \phi_i(., x, .)$ is $\mathcal{L} \times \mathcal{B}$ mesurable

$(H2)$ U^i is a Borel mesurable set for $i = 1, ..., k$.

$(H3)$ $|\phi(t, y, w)| \leq K$ whenever $(t, y, w) \in \mathbb{R} \times X_t^i \times U_t^i$.

$(H4)$ $|\phi_i(t, y, w) - \phi_i(t, y', w)| \leq K|y - y'|$ whenever $(t, y, w), (t, y', w) \in \mathbb{R} \times X_t^i \times U_t^i$.

Reachable set Let C be a given set in

$$\prod_i \{(\tau_0^i, \tau_1^i, a_0^i) | a_0^i \in \mathbb{R}^{n_i}, \tau_0^i, \tau_1^i \in \mathbb{R}, \tau_0^i \leq \tau_1^i\}$$

and let $\psi : \mathbb{R}^{n_i} \times ... \times \mathbb{R}^{n_k} \longrightarrow \mathbb{R}^d$ be a given Lipschitz continous function. We define the reachable set (with respect to C and ψ), written $\mathcal{R}_{\psi,C}$, to be

$$\mathcal{R}_{\psi,C} := \{\psi(\{y_i(\tau_1^i)\}) | \{\tau_0^i, \tau_1^i, y_i(.), w_i(.)|\} \textit{ is a multiprocess such that } \{\tau_0^i, \tau_1^i, y_i(\tau_0^i) \in C\}.$$

We say that a multiprocess $\{\tau_0^i, \tau_1^i, y_i(.), w_i(.)\}$ is a boundary multiprocess relative to ψ and C if

$$\{\tau_0^i, \tau_1^i, y_i(\tau_0^i) \in C \quad and \quad \psi(\{y_i(\tau_1^i)\}) \in \partial\mathcal{R}_{\psi,C}$$

(∂ denote the boundary).

Unmaximized Hamiltonian Define the unmaximized Hamiltonian to be the function H_i such that

$$H_i(t, x, u, p) := p.\phi(t, x, u), \quad i = 1, ..., k.$$

Theorem 3.1. Let $\{T_0^i, T_1^i, x_i(.), u_i(.)\}$ be a boundary multiprocess with respect to C and ψ. Assume that

$$graph\{x_i(.)\} \subset \{X^i\}$$

for $i = 1, ..., k$ and that $(H1) - (H4)$ are satisfied. Then there exist a vector v of unit length, numbers h_0^i, h_1^i and absolutely continous functions $p_i(.) : [T_0^i, T_1^i] \longrightarrow \mathbb{R}^{n_i}$ for $i = 1, ..., k$, and a number c (whose magnitude is governed by the constant K in the Hypotheses $(H3)$ and $(H4)$ together with the Lipschitz rank of ψ restricted to some neighbourhood of $\{x_i(T_1^i)\}$, with the following properties:

$$-\dot{p}_i(t) \in \partial_x H_i(t, x_i(t), u_i(t), p_i(t)) \quad a.e.\ t \in [T_0^i, T_1^i],$$

$$H_i(t, x_i(t), u_i(t), p_i(t)) = \max_{w \in U_t^i} H_i(t, x_i(t), w, p_i(t)) \ a.e. t \in [T_0^i, T_1^i],$$

$h_0^i \in co\ ess_{t\longrightarrow T_0^i}[\sup_{w\in U_i^t}[H_i(t,x_iT_0^i,w,p_i(T_0^i))]$,

$h_1^i \in co\ ess_{t\longrightarrow T_o^i}[\sup_{w\in U_i^t} H_i(t,x_iT_1^i,w,p_i(T_0^i))]$,

for $i=1,...,k$

$$\{p_i(T_1^i)\} \in \partial\psi^*(\{x_i(t_1^i\})v$$

and

$$\{-h_0^i,h_1^i,p_i(T_0^i)\} \in c\ \partial d_C(\{T_0^i,T_1^i,x_i(T_0^i)\}).$$

Here $\partial_x H_i$ denote the partial generalized gradient in the second variable and $\partial\psi^*$ is the transpose of the generalized jacobian of ψ.(theorem will be proved after as a spetial case). Let's define now some preparation theorems. Let

$$f:\prod_i(\mathbb{R}\times\mathbb{R}^{n_i}\times\mathbb{R}^{n_i})\longrightarrow\mathbb{R}$$

be a given locally Lipschitz continous function and let

$$\Lambda\subset\prod_i\{(\tau_0^i,\tau_1^i,a_0^i,a_1^i)|\tau_0^i,\tau_1^i\in\mathbb{R},a_0^i,a_1^i\in\mathbb{R}^{n_i},\tau_0^i\leq\tau_1^i\}$$

be a given closed set.

We pose the optimal multiprocess problem:

$$\text{(P)}\qquad\begin{cases}\text{Minimize} f(\{\tau_0^i,\tau_1^i,y_i(\tau_0^i,y_i\tau_1^i)\})\\ \text{over multiprocesses } \{\tau_0^i,\tau_1^i,y_i(.)\}\\ \text{satisfying}\\ \{\tau_0^i,\tau_1^i,y_i(\tau_0^i,y_i(\tau_1^i)\subset\Lambda\end{cases}$$

Theorem 3.2 Let $\{T_0^i,T_1^i,x_i(.),u_i(.)\}$ be a solution to (P). Assume that

$$graph\{x_i(.)\}\subset interior\{X^i\}$$

for $i=1,...,k$ and that hypotheses $(H1)-(H4)$ are satisfied. Then there exist a real number $\lambda\geq 0$, real numbers h_0^i,h_1^i, and absolutely continous functions $p_i(.):[T_0^i,T_1^i]\longrightarrow\mathbb{R}^{n_i}$ for $i=1,...,k$ and a constant c(whose magnitude is determined by the constant K of the hypotheses $(H3)$ and $(H4)$ together with the Lipschitz rank of f in the neighbourhood of $\{T_0^i, T_1^i, x_i(T_0^i), x_i(T_1^i)\}$ such that $\lambda + \sum_i |p_i(t_1^i)| = 1$ and we have

(3.1) $\quad -\dot{p}_i(t) \in \partial_x H_i(t, x_i(t), u_i(t), p_i(t)) \quad a.e.\ t \in [T_0^i, T_1^i],$

$H_i(t, x_i(t), u_i(t), p_i(t)) = \max_{w \in U_t^i} H_i(t, x_i(t), w, p_i(t))\ a.e.t \in [T_0^i, T_1^i],$

(3.2) $\quad h_0^i \in co\ ess_{t \longrightarrow T_0^i}[\sup_{w \in U_i^t} H_i(t, x_i T_0^i, w, p_i(T_0^i))],$

$h_1^i \in co\ ess_{t \longrightarrow T_o^i}[\sup_{w \in U_i^t} H_i(t, x_i T_1^i, w, p_i(T_0^i))],$

for $i = 1, ..., k$, and

(3.3) $\ \{-h_0^i, h_1^i, p_i(T_0^i)\} \in c\ \partial d_\Lambda + \lambda \partial f$

where the generalized gradient ∂d_Λ and ∂f are evaluated at $\{T_0^i, T_1^i, x_i(T_0^i), x_i(T_1^i)\}$

4.Coupled dynamic optimisation problems: a differential inclusion formulation . It is well known that we may choose a variety of starting points for derivation of conditions on solutions to dynamic optimisation problems over a single time interval. Two notable instance are, first, taht the dynamics are nodeled by a differential equation with control and , second, taht involving a dofferential inclusion. We will show now a second preparation theorem in which the velocity verify a differential inclusion with the control. The following data are given:

posistive integers k, and n_i, $\quad i = 1, ..., k$

a function $g : \prod_{i=1}^k (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_i}) \longrightarrow \mathbb{R}$,

multifunctions $F_i : \mathbb{R} \times \mathbb{R}^{n_i} \rightsquigarrow \mathbb{R}^{n_i}, \quad i = 1, ..., k,$

sets $\Gamma^i \subset \mathbb{R} \times \mathbb{R}^{n_i}, \quad i = 1, ..., k,$

and a subset M of

$\prod_{i=1}^k \{(\tau_0^i, \tau_1^i, a_0^i, a_1^i) | \tau_0^i, \tau_1^i \in \mathbb{R}^{n_i} \text{ and } \tau_0^i \leq \tau_1^i\}.$

Consider the following problem:

$$
(Q) \qquad \begin{cases} \text{Minimize} g(\{\tau_0^i, \tau_1^i, y_i(\tau_0^i), y_i \tau_1^i)\}) \\ \text{over multiprocesses } \{\tau_0^i, \tau_1^i, y_i(.)\} \\ \text{satisfying} \\ \dot{y}_i \in F_i(t, y_i(t)) \quad a.e.t \in [\tau_0^i, \tau_1^i], \\ y_i(t) \in \Gamma_t^i \quad a.e.t \in [\tau_0^i, \tau_1^i], \\ \{\tau_0^i, \tau_1^i, y_i(\tau_0^i), y_i(\tau_1^i) \subset M \end{cases}
$$

Hypotheses of the theorem .

$(I1)$ g is locally lipcshtz continous

$(I2)$ M is closed

$(I3)$ For each i, F_i takes values closed convex sets, and given any point $x \in \mathbb{R}^{n_i}$ and closed set $D \subset \mathbb{R}^{n_i}$, the set $\{t | D \cap F_i(t, x) \neq \emptyset\}$ is Lebesgue mesurable.
There exist a constant K such that we have the following:

$(I4)$ $|v| \leq K$ whenever $v \in F_i(t, x), (t, x) \in \Gamma^i, i = 1, ..., k$.

$(I5)$ $dist\{F_i(t, x), F_i(t, y)\} \leq K|x - y|$, whenever $(t, x), (t, y) \in \Gamma^i, i = 1, ..., k$

(dist here is the hosdorff distance).

We define the Hamiltionian functions $\mathcal{H}_i : \Gamma^i \times \mathbb{R}^{n_i} \longrightarrow \mathbb{R}$ to be

$$\mathcal{H}_i(t, x, p) := \sup_{e \in F_i(t,x)} p.e, \quad i = 1, ..., k.$$

Theorem 4.1. Let $\{T_0^i, T_1^i, x_i(.)\}$ solve the problem (Q). Assume that

$$graph\{x_i(.)\} \subset interior\{\Gamma^i\}$$

for $i = 1, ..., k$, and that hypotheses $(I1) - (I5)$ are satisfied. Then there exist a real number $\lambda \geq 0$, real numbers h_0^i, h_1^i, absolutely continous functions

$p_i(.) : [T_0^i, T_1^i] \longrightarrow \mathbb{R}^{n_i}, i = 1, ..., k$, and a constant c(whose magnitude is determined by the constant K of the hypotheses $(I4)$ and $(I5)$ together with the Lipschitz rank of g in the neighbourhood of $\{T_0^i, T_1^i, x_i(T_0^i), x_i(T_1^i)\}$) such that $\lambda + \sum_i |p_i(t_1^i)| = 1$ and we have

$(-\dot{p}_i(t), \dot{x}_i(t)) \in \partial_{x,p} H_i(t, x_i(t), p_i(t)) \quad a.e.\ t \in [T_0^i, T_1^i],$

$h_0^i \in co\ ess_{t \longrightarrow T_0^i}[\sup_{w \in U_i^t} H_i(t, x_i(T_0^i), p_i(T_0^i))],$

$h_1^i \in co\ ess_{t \longrightarrow T_o^i}[\sup_{w \in U_i^t} H_i(t, x_i(T_1^i), p_i(T_0^i))],$

for $i = 1, ..., k$, and

$\{-h_0^i, h_1^i, p_i(T_0^i), (p_i(T_0^i)\} \in c\ \partial d_M + \lambda \partial g$

where the generalized gradient ∂d_C and ∂g are evaluated at $\{T_0^i, T_1^i, x_i(T_0^i), x_i(T_1^i)\}$

Proof of Theorem 4.1 .

The theorem will be prooved first in a special case then as usual we will remove theese temporarly hypotheses. The proof of this theoremm will lead to all the others mentioned before in this part. So we imposed now the following hypotheses:

(IU) $\{T_0^i, T_1^i, x_i(.)\}$ is the unique solution to (Q)

(IL) g is linear function of the form $g(\{\tau_0^i, \tau_1^i, y_0^i, y_1^i\}) = \sum_{i=1}^k g_i.y_1^i$ in wich g_i is a given vector in $\mathbb{R}^{n_i}, i = 1, ..., k$. We introduce a family of problems $Q(\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\})$ generated by perturbation to the constraint set M. choose $\epsilon > 0$ such that

$$graph\{x_i(.)\} + 2\epsilon B \subset \Gamma^i, \quad i = 1, ..., k,$$

and define the closed set $\tilde{\Gamma}^i, i = 1, ..., k$ to be

$$\tilde{\Gamma}^i := graph\{x_i(.)\} + \epsilon \bar{B}.$$

for each vector $\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\} \in \prod_i (\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n_i}\times\mathbb{R}^{n_i})$ problem $Q(\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\})$ is taken to be the following:

$$\begin{cases} \text{Minimize} g(\{\tau_0^i, \tau_1^i, y_i(\tau_0^i), y_i(\tau_1^i)\}) \\ \text{over multiprocesses } \{\tau_0^i, \tau_1^i, y_i(.)\} \\ \text{satisfying} \\ \dot{y}_i \in F_i(t, y_i(t)) \quad a.e. t \in [\tau_0^i, \tau_1^i], \\ \dot{y}_i(t) = 0 \quad a.e. t \in I_i / [\tau_0^i, \tau_1^i], \\ graph\{y_i(.)\} \subset \tilde{\Gamma}^i \text{for} i = 1, ..., k \ and \\ (5.1) \qquad \{\tau_0^i, \tau_1^i, y_i(\tau_0^i, y_i(\tau_1^i) \subset M + \{\rho_0^i, \rho_1^i, y_i(\tau_0^i), y_i(\tau_1^i)\}. \end{cases}$$

where I_i is take here to be the fixed time interval

$$I_i = [T_0^i - \epsilon, T_1^i - \epsilon],$$

$i = 1, ...k..$ The problem $Q(\{0,0,0,0\})$ will be called a refinement of initial problem (Q). Clearly the point $\{T_0^i, T_1^i, x_i(.)\}$ remain solution to $Q(\{0,0,0,0\})$.

We donoted but V the value fuction associated to the perturbation pf the problem (Q) (the infimum cost of the cost function).

Lemma 5.1. (i) Let $\{\bar{\rho_0^i}, \bar{\rho_1^i}, \bar{\sigma_0^i}, \bar{\sigma_1^i}\}$ a sequence converging to $\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\}$ and let $\{\bar{\tau_0^i}, \bar{\tau_1^i}, \bar{y}_i(.)\}$ be a solution to $Q(\{\bar{\rho_0^i}, \bar{\rho_1^i}, \bar{\sigma_0^i}, \bar{\sigma_1^i}\})$. Then we have $\bar{\tau_0^i} \longrightarrow \tau_0^i, \bar{\tau_1^i} \longrightarrow \tau_1^i$ for each i, and $\bar{y}_i(.) \longrightarrow y_i(.)$ uniformaly where $\{\tau_0^i, \tau_1^i, y_i(.)\}$ is an admissible trajectory for $Q(\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\})$ (by theorem of compactness of trajectories, the limiting trajectory y_i still verify the differential inclusion).

(ii) if in part $(i)\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\} = \{0,0,0,0\}$ and also $V(\{\{\bar{\rho_0^i}, \bar{\rho_1^i}, \bar{\sigma_0^i}, \bar{\sigma_1^i}\} \longrightarrow V(\{0,0,0,0\}$ then $\{\bar{\tau_0^i}, \bar{\tau_1^i}, \bar{y}_i(.)\} = \{T_0^i, T_1^i, x_i(.)\}$. In fact if $\{\tau_0^i, \tau_1^i, y_i(.)\}$ solve the problem $Q(\{\{\bar{\rho_0^i}, \bar{\rho_1^i}, \bar{\sigma_0^i}, \bar{\sigma_1^i}\})$ we know that this trajectory as a subsequence if necessary converge uniformaly to an admissible trajectory, and since $V(\{\{\bar{\rho_0^i}, \bar{\rho_1^i}, \bar{\sigma_0^i}, \bar{\sigma_1^i}\} \longrightarrow V(\{0,0,0,0\}$, with the help of hypothese (IU), we conclude that this limit is nothing then $\{T_0^i, T_1^i, x_i(.)\}$

(iii) The epigraph of V is closed, which is equivaleny to say that v is lower semicontinous fucntion. In fact suppose that a vector $a_i \longrightarrow a$ and y_i is solution to the problem $Q(a_i)$ we know that y_i goes for an admissible trajectory for $V(a)$ and by minimum properties we have that $\liminf g(y_i) \geq g(a)$ and then V is lower semi continous

Lemma 5.2 Let $[\{h_0^i, -h-1^i, -s_0^i, s_1^i\} - \lambda]$ be a proximal normal to epi V at the point $[\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\}, V(\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\}) + \delta]$. (with $\delta \geq 0$). Let $\{\tau_0^i, \tau_1^i, z_i(.)\}$ solve $Q(\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\})$ and supppose that graph $\{z_i(.) : [\tau_0^i, \tau_1^i] \longrightarrow \mathbb{R}^{n_i}\}$ is interior to $\tilde{\Gamma}^i$ for $i = 1, ..., k$. Let $\{\alpha_0^i, \alpha_1^i, \gamma_0^i, \gamma_1^i\}$ be the point in M sych that $\{\tau_0^i, \tau_1^i, z_i(\tau_0^i), z_i(\tau_1^i)\} = \{\alpha_0^i, \alpha_1^i, \gamma_0^i, \gamma_1^i\} + \{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\}$. Then for $i = 1, ..., k$ there exist an absolutely continous function $p_i(.) : I_i \longrightarrow \mathbb{R}^{n_i}$ such that

$$(5.3)\quad (-\dot{p}_i(t), \dot{z}_i(t)) \in \begin{cases} \partial\mathcal{H}_i(t, z_i(t), p_i(t)) & \text{a.e. } t \in [\tau_0^i, \tau_1^i], \\ \{0, 0\} & \text{a.e. } \quad I_i/[\tau_0^i, \tau_1^i], \end{cases}$$

$$(5.4)\quad p_i(\tau_0^i) = s_0^i,$$

$$(5.5)\quad p_i(\tau_1^i) = s_1^i - \lambda g_i,$$

$$(5.6)\quad h_0^i \in co\ ess_{t \longrightarrow \tau_0^i} \mathcal{H}_i(t, z_i(\tau_0^i, p_i(\tau_0^i)),$$

$$(5.7)\quad h_1^i \in co\ ess_{t \longrightarrow \tau_1^i} \mathcal{H}_i(t, z_i(\tau_1^i, p_i(\tau_1^i)).$$

Furtheremore,

$$(5.8)\quad \{h_0^i, -h_1^i, -s_0^i, s_1^i\} \in |\{h_0^i, -h_1^i, -s_0^i, s_1^i\}| \partial d_M(\{\alpha_0^i, \alpha_1^i, \gamma_0^i, \gamma_1^i\}).$$

Proof. Let $\{t_0^i, t_1^i, y_i(.)\}$ be an arbitrary admissible trajectory. Let $\{\bar{\alpha}_0^i, \bar{\alpha}_1^i, \bar{\gamma}_0^i, \bar{\gamma}_1^i\}$ be any point in M and $\bar{\delta}$ any nonnegative number. we have that $[\{t_0^i - \bar{\alpha}_0^i, t_1^i - \bar{\alpha}_1^i, y_i(t_0^i) - \bar{\gamma}_0^i, y_i(t_1^i) - \bar{\gamma}_1^i\}, \sum_i g_i.y_i(t_1^i) + \bar{\delta}] \in epi\ V$. we shall use this point in a proximal inequality, but let's first define that.

Proximal normal vector We say that a vector ζ is proximal normal to a closed set $S \subset \mathbb{R}^q$ at $s \in S$ if there exist $m \geq 0$ such that

$$-\zeta.s' + m|s' - s| \geq -\zeta.s \quad \forall s' \in S$$

So $[\{t_0^i - \bar{\alpha_0^i}, t_1^i - \bar{\alpha_1^i}, y_i(t_0^i) - \bar{\gamma_0^i}, y_i(t_1^i) - \bar{\gamma_1^i}\}, \sum_i g_i.y_i(t_1^i) + \bar{\delta}] \in epi\ V$ will be used in this proximal inequality at the point

$$\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\}, \sum_i g_i.z_i(\tau_1^i) + \delta$$

By hypothese the last point can be written as

$$[\{\tau_0^i - \alpha_0^i, \tau_1^i - \alpha_1^i, z_i(\tau_0^i) - \gamma_0^i, z_i(\tau_1^i) - \gamma_1^i\}, \sum_i g_i.z_i(\tau_1^i) + \delta]$$

Taking ζ as in the hypothese and putting all these points in our prozimal inequality we have

(5.9) $\sum_i[-h_0^i(t_0^i - \bar{\alpha_0^i} - \tau_0^i + \alpha_0^i) + -h_1^i(t_1^i - \bar{\alpha_1^i} - \tau_1^i + \alpha_1^i) + s_0^i.(y_i(t_0^i) - \bar{\gamma}_0^i - z_i(\tau_0^i + \gamma_0^i) - s_1^i.(y_i(t_1^i) - \bar{\gamma}_1^i - z_i(\tau_1^i + \gamma_1^i) + \lambda(\lambda(\sum_i g_i.y_i(t_1^i + \bar{\delta} - \sum_i g_i.z_i(\tau_1^i - \delta)] + m\Delta \geq 0$

With $\Delta = |\sum_i g_i.y_i(t_1^i + \bar{\delta} - \sum_i g_i.z_i(\tau_1^i - \delta|^2 + \sum_i(|t_0^i - \bar{\alpha}_0^i - \tau_0^i + \alpha_0^i|^2 + |t_1^i - \bar{\alpha}_1^i - \bar{\tau_1^i} + \alpha_1^i|^2) + \sum_i(|y_i(t_0^i - \bar{\gamma_0^i} - z_i(\tau_0^i + \gamma_0^i|^2 + |y_i(t_1^i - \bar{\gamma_1^i} - z_i(\tau_1^i + \gamma_1^i|^2)$.

Remember that $\{t_0^i, t_1^i, y_i(.)\}$ is taken arbitrary we replace it by $\{\tau_0^i, \tau_1^i, z_i(.)\}$ in our proximal inequality to obtain

$\sum_i(-h_0^i(\alpha_0^i - \bar{\alpha}_0^i) + h_1^i(\alpha_1^i - \bar{\alpha}_1^i) + s_0^i(\gamma_0^i - \bar{\gamma}_0^i) - s_1^i(\gamma_1^i - \bar{\gamma}_1^i)) + \lambda(\delta - \bar{\delta}) + m(\sum_i(|\alpha_0^i - \bar{\alpha}_0^i|^2 + |\alpha_1^i - \bar{\alpha}_1^i|^2 + |\gamma_0^i - \bar{\gamma}_0^i|^2 + |\gamma_1^i - \bar{\gamma}_1^i|^2 + |\delta - \bar{\delta}|^2) \geq 0$, for all $\bar{\delta} \geq 0$ and $\{\bar{\alpha}_0^i, \bar{\alpha}_1^i, \bar{\gamma}_0^i, \bar{\gamma}_1^i\} \in M$. setting $\bar{\delta} = \delta$ and deviding by $|\{h_0^i, -h_1^i, -s_0^i, s_1^i\}|$ we conclude that

$$\frac{\{h_0^i, -h_1^i, -s_0^i, s_1^i\}}{|\{h_0^i, -h_1^i, -s_0^i, s_1^i\}|} \in N_M^L(\{\alpha_0^i, \alpha_1^i, \gamma_0^i, \gamma_1^i\})$$

Since at any point we have $N_M^L \cap B = \partial d_M$ we conclude that $\frac{\{h_0^i, -h_1^i, -s_0^i, s_1^i\}}{|\{h_0^i, -h_1^i, -s_0^i, s_1^i\}|} \in \partial d_M(\{\alpha_0^i, \alpha_1^i, \gamma_0^i, \gamma_1^i\})$ and so we obtain (5.8). We need now to proof (5.3) to (5.5) by taking another spetial case and it's all about the proximal inequality noted.

Set now $\{\bar{\alpha}_0^j, \bar{\alpha}_1^j, \bar{\gamma}_0^j, \bar{\gamma}_1^j\} = \{\bar{\alpha}_0^i, \bar{\alpha}_1^i, \bar{\gamma}_0^i, \bar{\gamma}_1^i\}, t_0^i = \tau_0^i$ and $t_1^i = \tau_1^i$ for all j, $1 \leq j \leq k$ and set $\delta = \bar{\delta}$. Select $i, 1 \leq i \leq k$, and set $y_j(.) = z_j(.)$ for all $j \neq i$. Since z_i solves the problem $Q(\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\})$ we see that z_i solves the following minimization problem

$\lambda g_i.y(\tau_1^i)+s_0^i.y(\tau_0^i)-s_1^i.y(\tau_1^i)+m[|g_i.y(\tau_1^i)-g_i.z_i(\tau_1^i)|^2+|y(\tau_0^i)-z_i(\tau_0^i)|^2+y(\tau_1^i-z_i(\tau_1^i)|^2]$ If $\tau_0^i \neq \tau_1^i$ Now since F is Lipschitz and takes a closed ,covexe value and g is locally Lipschitz continous, furthermore z_i solves the minimization problem problem above, we deduce the presence of five tuples $[p, \gamma, a, \zeta, b]$ with $a = 0$ and $b = 1$ such that

$$(-\dot{p_i}(t), \dot{z}_i(t)) \in \begin{cases} \partial\mathcal{H}_i(t, z_i(t), p_i(t)) & \text{a.e. } t \in [\tau_0^i, \tau_1^i], \\ \{0, 0\} & \text{a.e.} \quad I_i/[\tau_0^i, \tau_1^i], \end{cases}$$

$\zeta \in \partial g(y_i(\tau_1^i) = \nabla g(y_i(\tau_1^i) = \lambda g_i - s_1^i$ and since $p_i(\tau_1^i) = -1.\zeta$ we conclude that $p_i(\tau_1^i) = s_1^i - \lambda g_i$

$p_i(\tau_0^i = \nabla g(z_i(\tau_0^i) = s_0^i$

so (5.3)(5.4)(5.5) are verified. Suppose now that $\tau_0^i = \tau_1^i(:= \tau^i)$. Since z_i solves minimization problem $\lambda g_i.y(\tau_1^i) + s_0^i.y(\tau_0^i) - s_1^i.y(\tau_1^i) + m[|g_i.y(\tau_1^i) - g_i.z_i(\tau_1^i)|^2 + |y(\tau_0^i) - z_i(\tau_0^i)|^2 + y(\tau_1^i - z_i(\tau_1^i)|^2]$ we conclude that $\lambda g_i + s_0^i - s_1^i = 0$. Setting $p_i(\tau^i) := s_0^i)$ we deduce the existence of a functions such that $p_i(\tau^i) = s_0^i$ and $p_i(\tau^i = s_1^i - \lambda g_i$ which verifies (5.4)(5.4)(in this case (5.3) is trivial).

It remains (5.6) and (5.7). Since z_i is assumed in the interior of $\tilde{\Gamma}^i$, we may choose $t_1^i \in I_i$ such that $t_1^i > \tau_1^i$. We proceed to extend $z_i|_{[\tau_0^i, \tau_1^i]}$ to $[\tau_0^i, t_1^i]$ defining a new trajectory $y_i(.)$. By Aumann's selection theorem we conclude the existence of an absolute continous function $\bar{\xi} : [\tau_1^i, t_1^i] \longrightarrow \mathbb{R}^{n_i}$ such that $\bar{\xi}(\tau_1^i = z_i(\tau_1^i)$ and

$$\dot{\bar{\xi}}(t) \in F_i(t, z_i(t, z_i(\tau_1^i)) \cap E_i(t) \ a.e.$$

With

$$E_i(t) = \{e | p_i(\tau_1^i).e = \max[p_i(\tau_1^i).e' | e' \in F_i(t, z_i(\tau_1^i)]\}.$$

The hypotheses on the velocity set implies the existence of an absolute function $\xi(.) : [\tau_1^i, t_1^i] \longrightarrow \mathbb{R}^{n_i}$ such that $\dot{\xi}(t) \in F_i(t, \xi(t)) \;\; a.e.t \in [\tau_1^i, t_1^i]$.
$\xi(t_1^i) = z_i 9\tau_1^i)$,
$\dfrac{1}{t_1^i - \tau_1^i} \int_{\tau_1^i}^{t_1^i} |\dot{\xi}(s) - \dot{\bar{\xi}}(s)|\, ds \leq K^2 exp\{K(t_1^i - \tau_1^i)\}(t_1^i - \tau_1^i)$

With $t_1^i \downarrow \tau_1^i$. We return now as always to (5.9) in which the following special case will be taken. Set $\bar{\delta} = \delta$ and $\{\bar{\alpha}_0^j, \bar{\alpha}_1^j, \bar{\gamma}_0^j, \bar{\gamma}_1^j\} = \{\alpha_0^j, \alpha_0^j, \gamma_0^j, \gamma_1^j\}$ for all j, $1 < j \leq k$. For $j \neq i$ take $(t_0^i, t_1^i, y_j(.)) = (\tau_0^j, \tau_1^j, z_j(.))$. Tke also $t_0^i = \tau_0^i$ and define $y_i(.) : [\tau_0^i, t_1^i] \longrightarrow \mathbb{R}^{n_i}$ to be

$$y_i(t) = \begin{cases} z_i(t) & \text{for } t \in [\tau_0^i, \tau_1^i] \\ \xi(t) & \text{for } t \in [\tau_1^i, t_1^i] \end{cases}$$

Write $\epsilon' = t_1^i - \tau_1^i$ we obtain $h_1^i \epsilon' - s_1^i(y_i(t_1^i) - z_i(\tau_1^i) + \lambda g_i(y_i(t_1^i - z_i(\tau_1^i) + m\Delta \geq 0$ and since$y_i(t_1^i) - z_i(\tau_1^i) = \xi(t_1^i) - z_i(\tau_1^i) = \int_{\tau_1^i}^{t_1^i + \epsilon'} \dot{\xi}(s)\, ds$ and deviding across by ϵ' we obtain

$$h_1^i - (s_1^i - \lambda g_i)((\epsilon')^{-1} \int_{\tau_1^i}^{\tau_1^i + \epsilon'} \dot{\xi}(s)\, ds) + \epsilon'^{-1} m\Delta \geq 0.$$

Since $p_i(\tau_1^i) = s_1^i - \lambda g_i$ and by (5.10), we have

$$-h_1^i + (\epsilon')^{-1} \int_{\tau_1^i}^{\tau_1^i + \epsilon'} \mathcal{H}_i(t, z_i(\tau_1^i, p_i(\tau_1^i)\, dt \leq (\epsilon')^{-1}(p_i(\tau_1^i). \int_{\tau_1^i}^{\tau_1^i + \epsilon'} |\dot{\xi}(s) - \dot{\bar{\xi}}(s)|\, ds$$

Since the velocity set verify a lipschitz inequality we have that $|y_i(t_1^i - z_i(\tau_1^i)| \leq K|t_1^i - \tau_1^i|$, and so $\Delta/\epsilon' \longrightarrow 0$ as $\epsilon' \downarrow 0$. So in the limite and by (5.11) we obtain that $\limsup_{\epsilon' \downarrow 0} (\epsilon')^{-1} \int_{\tau_1^i}^{\tau_1^i + \epsilon'} [\mathcal{H}_i(t, z_i(\tau_1^i), p_i(\tau_1^i) - h_1^i]\, dt \leq 0$ which is equivalent to say that

$$(5.12) \quad h_1^i \in ess_{t \longrightarrow \tau_1^i} \mathcal{H}_i(t, z_i(\tau_1^i), p_i(\tau_1^i)) + [0, +\infty),$$

Similar reasoning but now by choosing $t_1^i < \tau_1^i$, gives

$$\liminf_{\epsilon' \downarrow 0} (\epsilon')^{-1} \int_{\tau_1^i - \epsilon'}^{\tau_1^i} [\mathcal{H}_i(t, z_i(\tau_1^i), p_i(\tau_1^i) - h_1^i]\, dt \geq 0$$

which imply that

$$(5.13) \quad h_1^i \in ess_{t \longrightarrow \tau_1^i} \mathcal{H}_i(t, z_i(\tau_1^i), p_i(\tau_1^i)) + (-\infty, 0].$$

(5.12)(5.13) imply that

$$h_1^i \in co\ ess_{t\longrightarrow \tau_1^i}\mathcal{H}_i(t, z_i(\tau_1^i), p_i(\tau_1^i))$$

Same argument applied to the left endtime τ_0^i gives

$$h_0^i \in co\ ess_{t\longrightarrow \tau_0^i}\mathcal{H}_i(t, z_i(\tau_0^i), p_i(\tau_0^i)).$$

Suppose now that $\tau_0^i = \tau_1^i = \tau^i$ and returning to (5.9), Setting $\bar{\delta} = \delta$ and $\{\bar{\alpha}_0^j, \bar{\alpha}_1^j, \bar{\gamma}_0^j, \bar{\gamma}_1^j\} = \{\alpha_0^j, \alpha_0^j, \gamma_0^j, \gamma_1^j\}$ for $j \neq i$ and $y_j(.) \equiv z_i(\tau_1^i)$,Passing to the limite we have $0 \leq -h_0^i(\epsilon') + h_1^i(\epsilon') \leq 0$, and then $h_0^i = h_1^i$

Let's go back now to the proximal normal vector $[\{h_0^i, h_1^i, -s_0^i, s_1^i\}, -\lambda]$ at epi V at the point $\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\}, V(\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\}) + \delta]$ to proof the last differential inclusion of theorem 4.1, arranging $\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\} \longrightarrow 0$ and $V(\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\}) + \delta \longrightarrow V(\{0, 0, 0, 0\})$. Let $\{\tau_0^i, tau_1^i, z_i(.)\}$ be a solution to the perturabte problem $Q(\{\rho_0^i, \rho_1^i, \sigma_0^i, \sigma_1^i\}$. We can arrange a subsequence such that $\tau_0^i \longrightarrow T_0^i, \tau_1^i \longrightarrow T_1^i$ and $z_i(.) \longrightarrow x_i(.)$ uniformaly. From (5.8) we have

$$\{h_0^i, -h_1^i, -s_0^i, s_1^i\} \in |\{h_0^i, -h_1^i, -s_0^i, s_1^i\}|\partial d_M(\{\alpha_0^i, \alpha_1^i, \gamma_0^i, \gamma_1^i\}).$$

Since $p_i(\tau_0^i) = s_0^i$ and $p_i(\tau_1^i) = s_1^i - \lambda g_i$ we have :

$$\{h_0^i, -h_1^i, -p_i(\tau_i^i), p_i(\tau_1^i)+\lambda g_i\} \in |\{h_0^i, -h_1^i, -p_i(\tau_i^i), p_i(\tau_1^i)+\lambda g_i\}|\partial d_M(\{\alpha_0^i, \alpha_1^i, \gamma_0^i, \gamma_1^i\})$$

and so

$$\{h_0^i, -h_1^i, -p_i(\tau_i^i), p_i(\tau_1^i)\} \in |\{h_0^i, -h_1^i, -p_i(\tau_i^i), p_i(\tau_1^i)+\lambda g_i\}|\partial d_M(\{\alpha_0^i, \alpha_1^i, \gamma_0^i, \gamma_1^i\}) - \{0, 0, 0, \lambda g_i\}$$

By Lemma (5.2), $\{\alpha_0^i, \alpha_1^i, \gamma_0^i, \gamma_1^i\} = \{\tau_0^i - \rho_0^i, \tau_1^i - \rho_1^i, z_i(\tau_0^i) - \sigma_0^i, z_i(\tau_1^i) - \sigma_1^i\}$.
Then:

$$\{-h_0^i, h_1^i, p_i(\tau_i^i), -p_i(\tau_1^i)\} \in |\{h_0^i, -h_1^i, -p_i(\tau_i^i), p_i(\tau_1^i)+\lambda g_i\}|\partial d_M(\{\tau_0^i - \rho_0^i, \tau_1^i - \rho_1^i, z_i(\tau_0^i) - \sigma_0^i, z_i(\tau_1^i) - \sigma_1^i\}) + \lambda \partial g(\{\tau_0^i - \rho_0^i, \tau_1^i - \rho_1^i, z_i(\tau_0^i) - \sigma_0^i, z_i(\tau_1^i) - \sigma_1^i\})$$

(The limiting subdifferential of the cost function here is nothing then the gradient). Now the final step is to pass threw limits in (5.3), (5.14) with the aid of the theorem of compactness of trajectories and so (5.6)(5.7), then we have all our assertions of theorem 4.1 but with the temporary hypotheses(IU)(IL). As usual the removal of the additional hypotheses will be by taking cases lettting us to go back the hypotheses (IU)(IL). Suppose now that (IL) is not verified, consider an additiona trajectorie $z_i(.)$ verifying the differential inclusion :

$$\dot{y_i} \in F_i(t, y_i) \quad a.e.on\ [\tau_0^i, \tau_1^i] \ \ for\ i = 1, ..., k$$

and

$$\dot{z} \in \{0\} \quad on\ [\sigma_0, \sigma_1].$$

With the constraint

$$(\{\tau_0^i, \tau_1^i, y_i(\tau_0^i), y_i(\tau_1^i)\}, \sigma_0, \sigma_1, z(\sigma_0), z(\sigma_1)) \in \tilde{M}.$$

Such that

$$\tilde{M} := \{a, 0, 1, z_0, z_1)|a \in M, z_0 \geq g_e(a.0, 1, z_1\}$$

and

$$g_e(a, (\sigma_0, \sigma_1, z_0, z_1)) = z_1.$$

The new optimisation problem with the modified cost $\tilde{g}$ defined as $\tilde{g}(a, (\sigma_0, \sigma_1, z_0, z_1)) = z_1$ verifie hypotheses of theorem 4.1 with (IU). (endtimes taken here for a trajectory y is is t=0, t=1). we see that$(\{T_0^i, T_1^i, x_i(.)\}, 0, 1, y(.) \equiv g(\{T_0^i, T_1^i, x_i(.)\}))$ is a solution to the new problem with the additional trajectory here as the cost value of x_i. In addition our modified problem satisfy (IL) and so applying Theorem 4.1 with the additiona hypothese leads to the existence of $\lambda \geq 0$, numbers α, β, q, a fucntion $p_i(.)$ verifying the conditions of theorem (4.1). The difficultie here is to proof that $\{-h_0^i, h_1^i, p_i(\tau_i^i), -p_i(\tau_1^i)\} \in c\partial d_M + \lambda \partial g$ Applying hypotheses of the theorem we have only

$$(5.15)\{-h_0^i, h_1^i, p_i(T_0^i), p_i(T_1^i), \alpha, \beta, -q, q) \in cd_{\tilde{M}} + \lambda[0, (0, ..., 0, 1)].$$

at the point $(\{T_0^i, T_1^i, x_i(.)\}, (0, 1, g(\{T_0^i, T_1^i, x_i(.)\})).$

Lemma 5.3 .

Let $S \subset \mathbb{R}^k$ be a closed set and take $\bar{s} \in S$. Suppose there is a constant

$\delta > 0$ and a fucntion $l : \bar{s} + \delta B \longrightarrow \mathbb{R}$ such that l is lipschitz continous of rank at most K_1 on $\bar{s} + \delta B$. Then for all $R \geq (1 + K_1^2)^{1/2}$ we have

$$\partial d_{epi(l+\xi_s)}(\bar{s}, l(\bar{s}) \subset \{(\zeta, -\epsilon) | \zeta \in \epsilon \partial l(\bar{s}) + R \partial d_s(\bar{s}), \epsilon \geq 0\}.$$

Now it's easy to see that applying this Lemma and the fact that

$$\partial d_{M \times \{0,\} \times \{1\} \times \{R\}} \subset \partial d_M \times B \times B \times \{0\}$$

That

$$\{-h_0^i, h_1^i, p_i(T_0^i, -p_i(T_1^i)\} \in \lambda \partial g + c(1 + \bar{K}^2)^{1/2} \partial d_M.$$

$*$ Removual of (IU) is by a similar way considering the additional trajectory as

$$z_i = (y_i - x_i(t))^2 \ a.e.t \in [\tau_0^i, \tau_1^i],$$

verdying the differential inclusion

$$\dot{y}_i \in F_i(t, y_i) \ a.e.t \in [\tau_0^i, \tau_1^i].$$

with the cost

$\tilde{g}(\tau_0^i, \tau_1^i, (z_0^i, y_0^i), (z_1^i, y_1^i)\}) = g(\{(\tau_0^i, \tau_1^i, y_0^i, y_1^i\}) + \sum_i (|z_i(\tau_1^i)|^2 + |\tau_0^i - T_0^i|^2 + |\tau_1^i - T_1^i|^2)$

and the constraint set

$\tilde{M} = \{\{\tau_0^i, \tau_1^i, (z_0^i, y_0^i), (z_1^i, y_1^i)\}) = g(\{(\tau_0^i, \tau_1^i, y_0^i, y_1^i\} | \{\tau_0^i, \tau_1^i, y_0^i, y_1^i\} \in M$ and $z_0^i = 0$ for $i = 1, ..., k\}$.(The solution here is $\{T_0^i, T_1^i, (z_i(.) \equiv 0, x_i(.))\}$)

Proof of theorem 3.1. Choose $\epsilon > 0$ such that

$$graph\{x_i(.)\} + 2\epsilon B \subset X^i, \quad i = 1, ..., k,$$

and define the set

$$\tilde{X}^i = graph\{x_i(.)\} + \epsilon \bar{B}, \quad i = 1, ..., k.$$

and let the perturbation interval

$$I_i := [T_0^i - \epsilon, T_1^i + \epsilon], i = 1, , , k.$$

We define W as the set of extended process. An extended process is symply a process with an additional trajectory $w_i(.)$ satisfiying:

$\{\tau_0^i, \tau_1^i, y_i(.), w_i(.)\}$ with $[\tau_0^i, \tau_1^i] \subset I_i$ and $graph\{y_i(.)\} \subset \tilde{X}_i$ Define now the metric $\Delta : W \times W \longrightarrow \mathbb{R}$ as

$\Delta(\{\tau_0^i, \tau_1^i, y_i(.), w_i(.)\}, \{\bar{\tau}_0^i, \bar{\tau}_1^i, \bar{y}_i(.), \bar{w}_i(.)\}) := \sum_i [|\tau_0^i - \bar{\tau_0^i}| + |\tau_1^i - \bar{\tau_1^i}|, |y_i(\tau_0^i) - \bar{y}_i(\tau_0^i)| + \mathcal{L} - meas\{t \in [\tau_0^i \vee \bar{\tau}_1^i, \tau_1^i \bigwedge \bar{\tau}_1^i] | w_i(t) \neq \bar{w}_i(t)\}]$

$*$ This remind us for considering a perturbation problem in which the solution x_i will be a solution of order ϵ^n for the perturbated cost and then apply Euklend's theorem to pass threw limits.

Lemma 6.1 .

(W, Δ) is a complete metric space. Let $\{\tau_0^i, \tau_1^i, y_i(.), w_i(.)\}$ the general term in a sequence of points in (W, Δ) converging to $\{\bar{\tau}_0^i, \bar{\tau}_1^i, \bar{y}_i(.), \bar{w}_i(.)\}$ then $\limsup_{t \in I_i} |y_i(t), \bar{y}_i(t)| = 0$, for $i = 1, ..., k$.

Let $n > 0$, $\zeta \in \xi(\{x_i(T_1^i\}) + n^{-2}B$ such that $\zeta \neq \mathcal{R}_{\xi,C}$ and define F : $(W, \Delta) \longrightarrow \mathbb{R}$ to be

$$F(\{T_0^i, T_1^i, x_i(.), u_i(.)\}) := |\zeta - \xi(\{y_i(\tau_1^i)\}).$$

By lemma 6.1, F is continous and we have

$$F(\{T_0^i, T_1^i, x_i(.), u_i(.)\}) < \inf_{e \in w} F(e) + n^{-2}.$$

We see that $\{T_0^i, T_1^i, x_i(.)\}$ is an n^{-2} minimizer for the modified optimisation problem $f + n^{-1}\Delta$ so by Eukland's theorem there exist $\bar{e} = \{\bar{T}_0^i, \bar{T}_1^i, \bar{x}_i(.), \bar{u}_i(.)\}$ in W such that

(6.1) $\Delta(e, \bar{e}) \leq n^{-1}$

(6.2) $F(\bar{e}) \leq F(e') + n^{-1}\Delta(e', \bar{e})$ for all $e' \in W$.

Lemma 6.2 . Let $\{\tau_0^i, \tau_1^i, y_i(.), w_i(.)\}$ eb a multiprocess such that

$$\{\tau_0^i, \tau_1^i, y_i(.)\} \in C$$

and

$$sup_{t\in I_i\}}|y_i(t) - \bar{x}_i(t)| \leq \frac{\epsilon}{2}$$

for $i = 1, ..., k$. Then:

$|\zeta - \xi(\{y_i(\tau_1^i)\})| + n^{-1}\sum_i([\tau_0^i, \bar{T}_0^i) \vee 0] + [(\bar{T}_1^i - \tau_1^i) \vee 0] + (y(\tau_0^i - \bar{x}(\tau_0^i)| + \int_{\tau_0^i}^{\tau_1^i} \mathcal{H}_i(t, w_i(t))\, dt) \geq |\zeta - \xi(\{\bar{x}(T_1^i)\})|$.

Here

$$\mathcal{H}_i(t, w) = \begin{cases} 1 & \text{if } t \notin [T_0^i, T_1^i] \ or \ w \neq \bar{u}_i(t), \\ 0 & \text{otherwise} \end{cases}$$

To derive our condition for the minimum we need to apply the prooved theorem 4.1. So consider the state trajectory, $Y_i = (z_i, y_i)$ with the velocity set defined by

$$F_i(t, Y_i) := \{[\mathcal{H}_i(t, w), \phi(t, y, w)] | w \in U_t^i\}.$$

With the endpoints constaints

$$\Lambda := \{\{\tau_0^i, \tau_1^i, (z_0^i, y_0^i), (z_1^i, y_1^i)\} | \{\tau_0^i, \tau_1^i, y_0^i \in C, z_0^i = 0, i = 1, ..., k\}$$

and the cost function

$g(\{\tau_0^i, \tau_1^i, Y_0^i, Y_1^i\}) := |\zeta - \xi(\{y_1^i\})| + n^{-1}\sum_i(z_1^i + (\tau_0^i - \bar{T}_0^i) \vee 0 + (\bar{T}_1^i - \tau_1^i) \vee 0 + |y_0^i - \bar{x}(\bar{T}_0^i)|)$

in which $Y_0^i = (z_0^i, z_0^i)$ and $Y_1^i = (z_1^i, z_1^i)$.

Conside the trajectory $(\bar{T}_0^i, \bar{T}_1^i, \int_0^t \mathcal{H}(s, \bar{u}(s))\, ds, \bar{x}(.)))$ and calculate it cost g$((\bar{T}_0^i, \bar{T}_1^i, \int_0^t \mathcal{H}(s, \bar{u}(s))\, ds, \bar{x}(.))) = |\zeta - \xi(\{\bar{x}_1^i\})|$. By lemma 6.2 it is a minimizer to the problem

$$\text{(P(n))} \qquad \begin{cases} \text{Minimize } g(\{\tau_0^i, \tau_1^i, Y_i(\tau_0^i), Y_i(\tau_1^i)) \\ \text{ over} \\ (6.3) \quad \dot{Y}_i(t) \in F_i(t, Y_i(t)) \ a.e.t \in [\tau_0^i, \tau_1^i], \\ \text{ graph } Y_i(t) \subset \mathbb{R} \times (graph\{\bar{x}(.)\} + (\epsilon/2)B), \ i = 1, ..., k,) \\ (6.4) \quad \{\tau_0^i, \tau_1^i, Y_i(\tau_0^i), Y_i(\tau_1^i)\} \in \Lambda, \end{cases}$$

Recall $\bar{a}$ is a solution to a problem (P) with a differential inclusion. Considering the modified problem co(P) in which the differential inclusion into the velocity set is replaced by the covex hull of this set, we know that $\bar{a}$ remains solution to co(P).

Now considering the Problem co(P(n)) we see that our problem verifie hypotheses of theorem 4.1. and by the Lemma $\{T_0^i, T_1^i, \bar{z}_i(.), \bar{x}_i(.)\}$ is a solution to co(P(n)). By theorem 4.1 we conclude the existence of $\{p_i(.) : I_i \longrightarrow \mathbb{R}^{n_i}\}$, v of unit lengh such that

(6.6) $p_i(\bar{T}_1^i) \in \partial\xi^*(\{\bar{X}_i(t)\}).v$

(6.7) $-\dot{p}_i(t) \in \partial_x H(t, \bar{x}_i(t), u_i(t), p_i(t))$

(6.8) $H_i(t, \bar{x}_i(t), u_i(t), p_i(t)) \geq \max_{w \in U_t^i}\{H_i(t, \bar{x}_i(t), w, p_i(t))\} - n^{-1}$,

(6.9) $\{-h_0^i, h_1^i, o_i(\bar{T}_0^i)\} \in \bar{K}\partial d_C(\{\bar{T}_0^i, \bar{T}_1^i, \bar{x}_i(\bar{T}_0^i) + n^{-1}\bar{K}B$

(6.10) $h_0^i \in coess_{t \longrightarrow T_1^i} h_i(t, \bar{x}_i(\bar{T}_0^i), p_i(\bar{T}_1^i)) + n^{-1}\bar{K}B$,

(6.11) $h_1^i \in coess_{t \longrightarrow T_1^i} h_i(t, \bar{x}_i(\bar{T}_1^i), p_i(\bar{T}_1^i)) + n^{-1}\bar{K}B$

With

$$h_i(t, x, p) := \max_{w \in U_t^i} H_i(t, x, w, p).$$

Now passing threw limits we have all assumptions of the theorem.

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