UNIVERSITE DE NANTES

NORMAL BIRKHOFF FORMS IN THE ENERGY SPACE

by

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As a Mathematics student, I am frequently asked about my choice of study and its practical applications in our daily lives. As with many questions, the answers to these depend on relating disguised Math problems to features of the real world. Once you study Mathematics, you encounter the real meaning of struggling with a hard problem followed by developing hope that you will actually solve it. Besides being known as the basis of all sciences, it is often a mind-blowing source of joy to the people who understand it, and as Bertrand Russel once wrote ,“Mathematics, rightly viewed, possesses not only truth, but supreme beauty”.

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An Abstract

Title: Normal Birkhoff Forms in the Energy Space

Given small solutions of semi-linear Hamiltonian partial differential equations, we are interested in their long time behavior in H^s with s small. In order to do so, we followed the work done by [Bernier and Grébert, 2021] where they proved the almost global preservation for very long times of the low super-actions of non-resonant systems. Furthermore, we try to simplify the results done by setting a suitable formalism and by applying the results for a specific equation, the Beam equation.

Key words: Normal Forms in low regularity, non-resonanct condition, energy preservation, Beam equation

Contents

Chapter 1

Introduction

For half a century, the theory of partial differential equations has mainly focused on the study of the local or global existence of solutions, in well-chosen functional spaces. Nevertheless, the advances of this theory made it possible to consider other types of questions, in particular that of the qualitative behavior of solutions once their existence has been established.

Given a non-resonant[1] Hamiltonian partial differential equation and a small smooth initial datum, what can be said about the solution in H^s? In their paper, [Bambusi and Grébert, 2006] answered this question by proving that the super-actions are almost preserved in H^s for s large enough, leading to the stability of the solution. i.e. the solution remains small in H^s. Unfortunately, so far this theory of normal forms for Hamiltonian partial differential equations has only been developed for solutions of high regularity, the assumption that seems to be irrelevant.

Our goal here is to study the behavior of small solutions for such equations, in patricular the Beam Equation, over very long time $|t| \le \varepsilon^{-r}$ (r is very large) imposing less

[1]The eigenvalues of the linearized vector field enjoy a diophantine condition, in particular rational independency.

regularity by only setting them in the energy space with s small. The work done is inspired by the paper [Bernier and Grébert, 2021], however in this report we will be working with a simpler framework suitable for the Beam Equation defined on the 1-dimensional torus as follows:

$$\begin{cases} \partial_{tt}\psi + \partial_{xxxx}\psi + m\psi + p\psi^{p-1} = 0 \\ \psi(0,x) = \psi_0 \\ \partial_t\psi(0,x) = -\psi_1 \end{cases} \tag{1.1}$$

where $\psi = \psi(t,x) \in \mathbb{R}$ with $x \in \mathbb{T}$, the mass $m > 0$ is a parameter, $(\psi_0, \psi_1) \in H^{s+1}(\mathbb{T};\mathbb{R}) \times H^{s-1}(\mathbb{T};\mathbb{R})$ having small size ε and $p \geq 3$. Since we are working in low regularity, it would make sense to consider $s = 1$.

Now I will state the main result we tend to reach at the end.

Theorem 1.1. *For almost all $m > 0$ and all $r > p \geq 3$, there exists $\beta_r > 0$ and $C_{m,r} > 0$ such that, for all $\psi_0 \in H^2(\mathbb{T};\mathbb{R})$ and all $\psi_1 \in L^2(\mathbb{T};\mathbb{R})$ with*

$$\varepsilon := ||\psi_0||_{H^2} + ||\psi_1||_{L^2} \leq \varepsilon_m$$

where ε_m is defined later, the global solution of 1.1 satisfies

$$|t| \leq \varepsilon^{-r} \implies \forall n \geq 1, |\mathcal{E}_n(\psi(t), \partial_t\psi(t)) - \mathcal{E}_n(\psi(0), \partial_t\psi(0))| \leq C_{m,r}\langle n\rangle^{\beta_r}\varepsilon^p$$

where the low Harmonic energies $\mathcal{E}_n$ of the Beam equation are given by the formula

$$\mathcal{E}_n(\psi(t), \partial_t\psi(t)) = \sqrt{n^4+m}\left|\int_0^{2\pi}\psi(x)e^{inx}\,dx\right|^2 + \frac{1}{\sqrt{n^4+m}}\left|\int_0^{2\pi}\partial_t\psi(x)e^{inx}\,dx\right|^2.$$

Remark.

- We deduce the almost global preservation of $\mathcal{E}_n$ for $\langle n\rangle \leq N$ with $N \geq 1$.
- This is a non-trivial result since r is arbitrarily large. It is trivial if $r = p - 2$.
- It is established as a dynamical corollary of the Birkhoff Normal Form Thoerem in

low regularity.

- In the proof $\beta_r >> 1$, however we conjecture that it shouldn't be.

In order to prove Theorem 1.1, let us place ourselves in the needed framework and highlight the outline of this report. In chapter 2, we state basic definitions and notations we will be using throughout our work. Furthermore, we give a quick recall of several important theorems.

Actually, the existence of resonance (rational dependency) allows an exchange of energy between modes. For instance, see [Grébert and Villegas-Blas, 2011]. In order to obtain our needed result, we introduce in chapter 3 a new non-resonance condition characterized by controlling the small divisors by the smallest index which is obviously stronger than the classical condition that controls those by the third largest index. The good news is that this new condition is easily satisfied by the Beam equation frequencies. After that, we generalize the definition to give a more suitable version for the Birkhoff normal form theorem.

In chapter 4, we set the Hamiltonian formalism defining a class of Hamiltonian functions satisfying particular properties custom-made for the Beam equation. In addition to the properties given by [Bernier and Grébert, 2021], we introduce the zero momentum condition which helped simplify different results. It turns out that these Hamiltonians are stable by Poisson bracket.

Chapter 5 states and proves the Birkhoff normal form theorem in low regularity. The proof uses the normal form process to remove the inessential part of the Hamiltonian that influences the dynamics of the low modes. This is possible due to the strong non-resonance condition. More precisely, the proof is done by induction and uses several techniques such as Taylor expansion and stability of Hamiltonians by Poisson bracket. Moreover, we establish a corollary of the theorem allowing us to obtain the almost global preservation of the low super-actions over very long time $|t| \leq \varepsilon^{-r}$.

As an interesting application, we study the behavior of low harmonic energies of the Beam equation which is known to be widely used (mainly in dimension 1) by scientists and engineers due to its physical importance in modeling the oscillations of a uniform beam. For instance, engineering of large structures like the Eiffel tower used the beam equation, see [Win,]. For this, we chose to apply our key result to this particular equation. We start chapter 6 by writing the equation in an appropriate Hamiltonian form and relating the terms to the class defined in chapter 4. Next, we highlight the fact that the frequencies satisfy the strong non-resonance condition. Finally in order to reach our goal, we prove the global well-posedness of the equation using Banach fixed point theorem, boundedness of the energy norm and the Hamiltonian conservation.

Chapter 2

Background Theory

2.1 Notations

We always consider the following set of notations:

- $\mathcal{R}(z)$, $\mathcal{I}(z)$ denote the real, imaginary part of z respectively.
- $\partial_{\overline{z}} := \frac{1}{2}(\partial_{\mathcal{R}(z)} + i\partial_{\mathcal{I}(z)})$ and $\partial_z := \frac{1}{2}(\partial_{\mathcal{R}(z)} - i\partial_{\mathcal{I}(z)})$.
- $w \in \mathbb{R}^{\mathbb{Z}}$ denotes $w \equiv (w_n)_{n\in\mathbb{Z}}$.
- For $x \in \mathbb{Z}$, the Japanese bracket is denoted by $\langle x\rangle := \sqrt{1+|x|^2}$.
- $\kappa_w(\sigma, n)$:=min$\left\{\langle n_j\rangle \text{ such that } j \in [\![1,p]\!] \text{ and } \sum_{\substack{k \\ w_{n_k}=w_{n_j}}} \sigma_k \neq 0.\right\}$
- The 1-dimensional torus is denoted by $\mathbb{T} = {}^{\mathbb{R}}\big/_{2\pi\mathbb{Z}}$.
- For $x, y \in \mathbb{R}$, we denote $x \lesssim_p$ if there exists a constant $c(p)$ depending on p such that $x \leq c(p)y$.
- $\mathscr{S}_r$ denotes the symmetric group of degree r.

- For $k \in \mathbb{Z}$, we write $e_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}$.

- For $s \in \mathbb{R}$, the discrete Sobolev space is written as

$$h^s(\mathbb{Z}) = \{u \in \mathbb{C} \quad | \quad ||u||_{h^s}^2 := \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |u_k|^2 < \infty\}.$$

- For $p \geq 1$, the Lebesgue space is written as

$$l^p(\mathbb{Z}) = \{u \in \mathbb{C} \quad | \quad ||u||_{l^p}^2 := \sum_{k \in \mathbb{Z}} |u_k|^p < \infty\}.$$

- We denote by $u_k^{+1} := u_k$ and $u_k^{-1} := \overline{u_k}$.

- for $\ell \in \mathbb{Z}^r$, we denote $|\ell|_1 = |\ell_1| + \cdots + |\ell_r|$.

- We say that $n \in \mathbb{Z}^r$ is injective if $n = (n_1, \cdots, n_r)$ with $n_i \neq n_j$, $\forall i \neq j$.

2.2 Definitions and basic tools

Definition 2.2.1. (natural scalar product) We equip $l^2(\mathbb{Z})$ with its natural real scalar product

$$\langle u, v \rangle_{l^2} := \sum_{k \in \mathbb{Z}} \mathcal{R}(\overline{u_k} v_k) = \sum_{k \in \mathbb{Z}} (\mathcal{R}(u_k)\mathcal{R}(v_k) + \mathcal{I}(u_k)\mathcal{I}(v_k)) \in \mathbb{R}$$

which can be extended when $u \in h^s$ and $v \in h^{-s}$.

Definition 2.2.2. (Gradient) Given a smooth function

$$\begin{aligned} H : h^s(\mathbb{Z}) &\to \mathbb{R} \\ u &\mapsto H(u), \end{aligned}$$

its gradient $\nabla H(u)$ is the element of $h^{-s}(\mathbb{Z})$ satisfying

$$\forall v \in h^s(\mathbb{Z}), \ \ \langle \nabla H(u), v\rangle_{l^2} = dH(u)(v)$$

with $\nabla H(u) = (2\partial_{\overline{u_k}} H(u))_{k\in\mathbb{Z}}$.

Definition 2.2.3. (Hamiltonian system) We associate to H the Hamiltonian vector field

$$X_H(u) = i\nabla H(u).$$

Then the Hamiltonian system reads

$$\partial_t u = X_H(u).$$

Definition 2.2.4. (Poisson bracket) Let $H, K : h^s(\mathbb{Z}) \to \mathbb{R}$, be two functions such that $\nabla H(u) \in h^s(\mathbb{Z})$. Then the Poisson bracket of H and K is defined by:

$$\{H, K\}(u) := \langle i\nabla H(u), \nabla K(u)\rangle_{l^2}.$$

Lemma 2.2.1. *We have*

$$\{H, K\}(u) = 2i \sum_{k\in\mathbb{Z}} \partial_{\overline{u_k}} H(u)\partial_{u_k} K(u) - \partial_{u_k} H(u)\partial_{\overline{u_k}} K(u).$$

Proof. To see this, we write using the definition

$$\begin{aligned}
\{H, K\}(u) &= \langle i\nabla H(u), \nabla K(u)\rangle_{l^2} \\
&= \langle 2i\partial_{\overline{u_k}} H(u), 2\partial_{\overline{u_k}} K(u)\rangle_{l^2} \\
&= \sum_{k\in\mathbb{Z}} \mathcal{R}[2i\partial_{\overline{u_k}} H(u)\overline{2\partial_{\overline{u_k}} K(u)}] \\
&= 4\sum_{k\in\mathbb{Z}} \mathcal{R}[i\partial_{\overline{u_k}} H(u)\overline{\partial_{\overline{u_k}} K(u)}].
\end{aligned}$$

By simple calculations, one can prove that

$$4\mathcal{R}[i\partial_{\overline{u_k}}H(u)\overline{\partial_{\overline{u_k}}K(u)}] = 2i[\partial_{\overline{u_k}}H(u)\partial_{u_k}K(u) - \partial_{u_k}H(u)\partial_{\overline{u_k}}K(u)]. \qquad \square$$

Definition 2.2.5. (adjoint of a linear map) Consider the linear operator $T : E \to F$ where E and F are Banach spaces. Then we define the adjoint operator as

$$T^* : F^* \to E^*$$

with E^* and F^* being the dual spaces of E and F respectively.

Lemma 2.2.2. *(bootstrap principle) Let I be a time interval, and for each $t \in I$ suppose we have 2 statements, a hypothesis $H(t)$ and a conclusion $C(t)$. Suppose we can verify the following assertions:*

(i) (Hypothesis implies Conclusion) If $H(t)$ is true for some time $t \in I$, then $C(t)$ is true for that time t.

(ii) (Conclusion is stronger than Hypothesis) If $C(t)$ is true for some $t \in I$, then $H(t')$ is true for all $t' \in I$ in a neighborhood of t.

(iii) (conclusion is closed) If $t_1, t_2, \cdots$ is a sequence of times in I which converges to another time $t \in I$, and $C(t_n)$ is true for all t_n, then $C(t)$ is true.

(iv) (Base case) $H(t)$ is true for at least one time $t \in I$.

Then $C(t)$ is true for all $t \in I$.

Proof. A small proof is found in [Tao, 2006] Chapter 1.3. $\square$

Definition 2.2.6. (symplectic map) Let $s \geq 0$, $\mathcal{C}$ an open set of $h^s(\mathbb{Z})$ and a C^1 map $\tau : \mathcal{C} \to h^s(\mathbb{Z})$. We say that τ is a symplectic map if it preserves the canonical symplectic form:

$$\forall u \in \mathcal{C}, \forall v, w \in h^s(\mathbb{Z}),\ \ \langle iv, w\rangle_{l^2} = \langle id\tau(u)(v), d\tau(u)(w)\rangle_{l^2}.$$

Theorem 2.2.3. *(Schwarz theorem) for a function $f : \mathcal{C} \to \mathbb{R}$ defined on an open set $\mathcal{C} \subset \mathbb{R}^n$, if $p \in \mathbb{R}^n$ is a point such that some neighborhood of p is contained in $\mathcal{C}$ and f has continuous second partial derivatives at the point p, then for all $i, j \in \{1, 2, \dots, n\}$*

$$\frac{\partial^2}{\partial_i \partial_j} f(p) = \frac{\partial^2}{\partial_j \partial_i} f(p).$$

Theorem 2.2.4. *(Gronwall's inequality) Suppose that $\alpha(t)$ and $\beta(t)$ are 2 continuous function on an interval I with $\alpha(t) \geq 0$. If $\forall t \in I$ we have*

$$\beta(t) \leq C + \int_0^t \alpha(s)\beta(s)\, ds$$

where C is a constant, then

$$\beta(t) \leq C e^{\int_0^t \alpha(s)\, ds}.$$

Definition 2.2.7. (operator norm) For E real normed vector space, the vector space $\mathscr{L}(E; E)$ of bounded linear maps from E to E is endowed with the operator norm:

$$||T|| = \sup\left\{\frac{||Tv||}{||v||} : v \in E \text{ with } v \neq 0\right\}.$$

Theorem 2.2.5. *(Mean value inequality) Let E, F be two Banach spaces, $U \subset E$ open and $f : U \to F$ continuous. Assume that f is differentiable at each point of the segment $[a, b] \subset U$, then*

$$\frac{|f(b) - f(a)|}{|b - a|} \leq \sup_{x \in [a,b]} |df(x)|.$$

Lemma 2.2.6. *For a quadratic function q, consider the associated quadratic form $q(u) = b(u, u)$. If b is continuous, then we have*

$$|q(u) - q(v)| \leq ||u - v||_{l^2}(||u||_{l^2} + ||v||_{l^2}).$$

Proof. We write the associated quadratic form Then we get,

$$\begin{aligned} |q(u) - q(v)| &= |b(u,u) - b(v,v)| \\ &= |b(u-v,u) - b(v-u,v)| \\ &\leq |b(u-v,u)| + |b(v-u,v)| \\ &\leq C\|u-v\|_{l^2}\|u\|_{l^2} + C\|v-u\|_{l^2}\|v\|_{l^2} \qquad \text{since } |b(u,v)| \leq C\|u\|_{l^2}\|v\|_{l^2} \\ &= C\|u-v\|_{l^2}(\|u\|_{l^2} + \|v\|_{l^2}). \qquad \square \end{aligned}$$

Theorem 2.2.7. *(Banach fixed point theorem) Let E be a Banach space and suppose that $f : E \to E$ satisfies that for all $x, y \in E$, there exists $0 \leq C < 1$ such that*

$$\|f(x) - f(y)\|_E \leq C\|x - y\|_E.$$

Then f has a unique fixed point in E.

Chapter 3

Strong Non-Resonance Condition

In this chapter, we introduce a new non-resonance condition satisfied by the frequencies obtained from the quadratic Hamiltonian later denoted by Z_2.

3.1 Particular Case

Definition 3.1.1. (strong non-resonance) The frequencies $w \in \mathbb{R}^{\mathbb{Z}}$ are strongly non-resonant if for all $r > 0$ there exists $\gamma_r > 0$, $\alpha_r > 0$ such that for all $r_* \leq r$, all $\ell_1, \cdots, \ell_{r_*} \in \mathbb{Z}^*$, and all $n \in \mathbb{Z}^{r_*}$ injective with $|\ell_1| + \cdots + |\ell_{r_*}| \leq r$ and $\langle n_1 \rangle \leq \cdots \leq \langle n_{r_*} \rangle$, we have

$$\left| \ell_1 w_{n_1} + \cdots + \ell_{r_*} w_{n_{r_*}} \right| \geq \gamma_r \langle n_1 \rangle^{-\alpha_r}.$$

Proposition 3.1.1. *Let $r \geq 1$ and $w \in \mathbb{R}^{\mathbb{Z}}$. Suppose that:*

(i) there exists $\alpha, \gamma > 0$ such that for all $r_* \leq r$, all $\ell \in (\mathbb{Z}^*)^{r_*}$, and all $n \in \mathbb{Z}^{r_*}$ injective

with $|\ell|_1 \leq r$ *and* $\langle n_1\rangle \leq \cdots \leq \langle n_{r_*}\rangle$, *we have*

$$\forall k \in \mathbb{Z}, \left|k + \ell_1 w_{n_1} + \cdots + \ell_{r_*} w_{n_{r_*}}\right| \geq \gamma \langle n_{r_*}\rangle^{-\alpha}, \tag{3.1}$$

(ii) the frequencies accumulate polynomially fast on $\mathbb{Z}$.

i.e. there exists $C > 0$ *and* $\nu > 0$ *such that*

$$\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, \quad |w_n - k| \leq C\langle n\rangle^{\nu}.$$

Then w *is strongly non-resonant.*

Proof. Fix r and r_* satisfying the given assumptions. We prove by induction on $r_\flat \leq r_*$ that there exists $\beta_{r_\flat} > 0$ (depending on α, ν, r) and $\eta_{r_\flat} > 0$ (depending on α, ν, C, γ, r) such that

$$\forall k \in \mathbb{Z}, \left|k + \sum_{1\leq j\leq r_\flat} \ell_j w_{n_j}\right| \geq \eta_{r_\flat}\langle n_1\rangle^{-\beta_{r_\flat}}.$$

Initial Step: For $r_\flat = 1$.

Using the first assumption with $r_* = 1$, we get

$$|k + \ell_1 w_{n_1}| \geq \gamma\langle n_1\rangle^{-\alpha}.$$

Hence, we obtain the result for $\eta_1 = \gamma$ and $\beta_1 = \alpha$.

Induction Step: Assume that it is true for $r_\flat < r_*$. Prove it for $r_\flat + 1$.

Using the second assumption with $n := n_{r_\flat+1}$,

$$\exists k_\flat \in \mathbb{Z}, \left|w_{n_{r_\flat+1}} - k_\flat\right| \leq C\langle n_{r_\flat+1}\rangle^{-\nu}. \tag{3.2}$$

Next, we have that

$$\left|k + \sum_{1\leq j\leq r_\flat+1} \ell_j w_{n_j}\right| \geq \left|k + \ell_{r_\flat+1}k_\flat + \sum_{1\leq j\leq r_\flat} \ell_j w_{n_j}\right| - |\ell_{r_\flat+1}|\left|w_{n_{r_\flat+1}} - k_\flat\right|. \tag{3.3}$$

Indeed, using the triangular inequality we can write

$$\begin{aligned}
&\left|k + \ell_{r_\flat+1}k_\flat + \sum_{1\le j\le r_\flat} \ell_j w_{n_j}\right| - |\ell_{r_\flat+1}|\left|w_{n_{r_\flat+1}} - k_\flat\right| \\
&= \left|k + \sum_{1\le j\le r_\flat} \ell_j w_{n_j} + \ell_{r_\flat+1}w_{n_{r_\flat+1}} - \ell_{r_\flat+1}w_{n_{r_\flat+1}} + \ell_{r_\flat+1}k_\flat\right| - |\ell_{r_\flat+1}|\left|w_{n_{r_\flat+1}} - k_\flat\right| \\
&= \left|k + \sum_{1\le j\le r_\flat+1} \ell_j w_{n_j} - \ell_{r_\flat+1}w_{n_{r_\flat+1}} + \ell_{r_\flat+1}k_\flat\right| - |\ell_{r_\flat+1}|\left|w_{n_{r_\flat+1}} - k_\flat\right| \\
&\le \left|k + \sum_{1\le j\le r_\flat+1} \ell_j w_{n_j}\right| + |\ell_{r_\flat+1}|\left|-w_{n_{r_\flat+1}} + k_\flat\right| - |\ell_{r_\flat+1}|\left|w_{n_{r_\flat+1}} - k_\flat\right| \\
&= \left|k + \sum_{1\le j\le r_\flat+1} \ell_j w_{n_j}\right|.
\end{aligned}$$

Now, notice that

- $|\ell_{r_\flat+1}| \le |\ell|_1 \le r$ (given) $\implies -|\ell_{r_\flat+1}| \ge -r$,
- using the induction hypothesis with $k := k + \ell_{r_\flat+1}k_\flat$, we get

$$\left|k + \ell_{r_\flat+1}k_\flat + \sum_{1\le j\le r_\flat} \ell_j w_{n_j}\right| \ge \eta_{r_\flat}\langle n_1\rangle^{-\beta_{r_\flat}}.$$

Back to 3.3, we get

$$\begin{aligned}
\left|k + \sum_{1\le j\le r_\flat+1} \ell_j w_{n_j}\right| &\ge \eta_{r_\flat}\langle n_1\rangle^{-\beta_{r_\flat}} - r\left|w_{n_{r_\flat+1}} - k_\flat\right| \\
&\ge \eta_{r_\flat}\langle n_1\rangle^{-\beta_{r_\flat}} - Cr\langle n_{r_\flat+1}\rangle^{-\nu} \qquad \text{by 3.2.}
\end{aligned}$$

Finally, we distinguish between 2 cases:

- For $2Cr\langle n_{r_\flat+1}\rangle^{-\nu} \le \eta_{r_\flat}\langle n_1\rangle^{-\beta_{r_\flat}}$, the result is direct.
- Otherwise, we have

$$\langle n_{r_\flat+1}\rangle^{-\nu} \ge \frac{1}{2Cr}\eta_{r_\flat}\langle n_1\rangle^{-\beta_{r_\flat}} \implies \langle n_{r_\flat+1}\rangle \le (2Cr\langle n_1\rangle^{\beta_{r_\flat}})^{1/\nu}\eta_{r_\flat}^{-1/\nu}$$

$$\implies \langle n_{r_\flat+1}\rangle \le (2Cr\eta_{r_\flat}^{-1})^{1/\nu}\langle n_1\rangle^{\beta_{r_\flat}/\nu}. \tag{3.4}$$

Also, applying first assumption with $r_* := r_\flat + 1$, we get

$$\begin{aligned}\left|k + \sum_{1\le j\le r_\flat+1} \ell_j w_{n_j}\right| &\ge \gamma\langle n_{r_\flat+1}\rangle^{-\alpha} \\ &\ge \gamma(2Cr\eta_{r_\flat}^{-1})^{-\alpha/\nu}\langle n_1\rangle^{-\alpha\beta_{r_\flat}/\nu} && \text{by 3.4} \\ &= \gamma\left(\frac{\eta_{r_\flat}}{2Cr}\right)^{\alpha/\nu}\langle n_1\rangle^{-\alpha\beta_{r_\flat}/\nu}. && \square\end{aligned}$$

3.2 Suitable Generalization

The sequence of frequencies w may not be injective and yet strongly non-resonant. Therefore, we extend Definition 3.1.1 and choose a suitable formalism for the Birkhoff Normal Form Theorem.

Definition 3.2.1. (Genaralized Strong Non-Resonance) A family of frequencies $w \in \mathbb{R}^{\mathbb{Z}}$ is strongly non-resonant up to any order, if for all $r \ge 1$ there exists $\gamma_r > 0$ and $\beta_r > 0$ such that for all $n \in \mathbb{Z}^r$, $\sigma \in \{-1,+1\}^r$, we have either

$$\left|\sum_{j=1}^{r} \sigma_j w_{n_j}\right| \ge \gamma_r \kappa_w(\sigma, n)^{-\beta_r}$$

or r is even and there exists $\rho \in \mathscr{S}_r$ such that for all $j \in [1, \frac{r}{2}]$, we have

$$\sigma_{\rho_{2j-1}} = -\sigma_{\rho_{2j}} \text{ and } w_{n_{\rho_{2j-1}}} = w_{n_{\rho_{2j}}}.$$

Now, we show that this is indeed an extension.

Lemma 3.2.1. *If $w \in \mathbb{R}^{\mathbb{Z}}$ is injective and strongly non-resonant according to Definition 3.1.1, then it is strongly non-resonant according to Definition 3.2.1.*

Proof. Suppose that w is strongly non-resonant according to Definition 3.1.1, then we arrange the small divisors:

$$\sigma_1 w_{n_1} + \cdots + \sigma_r w_{n_r} = \ell_1 w_{m_1} + \cdots + \ell_{r_*} w_{n_{r_*}}$$

where $r_* \leq r$, $\kappa_w(\sigma, n) = \langle m_1 \rangle \leq \cdots \leq \langle m_{r_*} \rangle$ and $\ell_j = \sum\limits_{\substack{k \\ w_{n_k} = w_{m_j}}} \sigma_k \neq 0$. It is clear that

$$\begin{aligned}
|\ell_1| + \cdots + |\ell_{r_*}| &= \left| \sum_{\substack{k \\ w_{n_k} = w_{m_1}}} \sigma_k \right| + \cdots + \left| \sum_{\substack{k \\ w_{n_k} = w_{m_{r_*}}}} \sigma_k \right| \\
&\leq \sum_{\substack{k \\ w_{n_k} = w_{m_1}}} 1 + \cdots + \sum_{\substack{k \\ w_{n_k} = w_{m_{r_*}}}} 1 \\
&= 1 + \cdots + 1 && \text{since } w \text{ is injective} \\
&= r_* \\
&\leq r.
\end{aligned}$$

So, by assumption, there exists $\gamma_r > 0$ and $\beta_r > 0$ such that

$$|\sigma_1 w_{n_1} + \cdots + \sigma_r w_{n_r}| = \left| \ell_1 w_{m_1} + \cdots + \ell_{r_*} w_{n_{r_*}} \right| \geq \gamma_r \langle m_1 \rangle^{-\beta_r} = \gamma_r \kappa_w(\sigma, n)^{-\beta_r}.$$

Thus, w is strongly non-resonant according to Definition 3.2.1. $\square$

Chapter 4

Class of Hamiltonian Functions

In this chapter, our goal is to establish main properties of the following Hamiltonian class.

4.1 Properties of a Class of Hamiltonian Functions

Definition 4.1.1. We denote by $\mathscr{H}^r$ the set of inhomogeneous Hamiltonians of degree $r \geq 2$, written as

$$H(u) = \sum_{\substack{\sigma \in \{-1,+1\}^r \\ n \in \mathbb{Z}^r}} H_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_r}^{\sigma_r},$$

and satisfying:

(i) $H_n^\sigma \in \mathbb{C}$

(ii) the *zero momentum condition*: $\sigma_1 n_1 + \cdots + \sigma_r n_r = 0$

(iii) the *symmetry condition*: $\forall \phi \in \mathscr{S}_r$, $H_{n_1,\cdots,n_r}^{\sigma_1,\cdots,\sigma_r} = H_{n_{\phi_1},\cdots,n_{\phi_r}}^{\sigma_{\phi_1},\cdots,\sigma_{\phi_r}}$

(iv) the *reality condition*: $H_n^{-\sigma} = \overline{H_n^\sigma}$

(v) the *bound*: $|H_n^\sigma| \lesssim_r \prod_{j=1}^r \|H\| \langle n_j \rangle^{-1}$ or $\|H\| < \infty$ where the norm of H is given by:

$$\|H\| = \sup_{\substack{\sigma \in \{-1,+1\}^r \\ n \in \mathbb{Z}^r}} |H_n^\sigma| \prod_{j=1}^r \langle n_j \rangle.$$

Lemma 4.1.1. *The polynomials of $\mathscr{H}^r$ define naturally smooth real-valued functions on $h^s(\mathbb{Z})$ for $s \geq 0$. In other words, if $H \in \mathscr{H}^r$ and $u^{(1)}, \cdots, u^{(r)} \in h^s(\mathbb{Z})$, then*

$$\sum_{\substack{\sigma \in \{-1,+1\}^r \\ n \in \mathbb{Z}^r \\ \sigma_1 n_1 + \cdots + \sigma_r n_r = 0}} \left| H_n^\sigma u_{n_1}^{(1),\sigma_1} \cdots u_{n_r}^{(r),\sigma_r} \right| \lesssim_{r,s} \|H\| \prod_{j=1}^r \|u^{(j)}\|_{h^s}.$$

Proof. Let $H \in \mathscr{H}^r$ and $u^{(1)}, \cdots, u^{(r)} \in h^s(\mathbb{Z})$. Then

$$\begin{aligned}
&\sum_{\substack{\sigma \in \{-1,+1\}^r \\ n \in \mathbb{Z}^r \\ \sigma_1 n_1 + \cdots + \sigma_r n_r = 0}} \left| H_n^\sigma u_{n_1}^{(1),\sigma_1} \cdots u_{n_r}^{(r),\sigma_r} \right| \\
&\leq \sum_{\substack{\sigma \in \{-1,+1\}^r \\ n \in \mathbb{Z}^r \\ \sigma_1 n_1 + \cdots + \sigma_r n_r = 0}} \left| u_{n_1}^{(1),\sigma_1} \right| \cdots \left| u_{n_r}^{(r),\sigma_r} \right| \|H\| \prod_{j=1}^r \langle n_j \rangle^{-1} \qquad \text{by the bound (v)} \\
&= \sum_{\substack{\sigma \in \{-1,+1\}^r \\ n \in \mathbb{Z}^r \\ \sigma_1 n_1 + \cdots + \sigma_r n_r = 0}} \|H\| \prod_{j=1}^r \left| u_{n_j}^{(j),\sigma_j} \right| \langle n_j \rangle^{-1}.
\end{aligned}$$

Now, since $|u| = |\overline{u}|$ and for $\sigma \in \{-1,+1\}^r$ we have 2^r terms, then

$$\begin{aligned}
\sum_{\substack{\sigma \in \{-1,+1\}^r \\ n \in \mathbb{Z}^r \\ \sigma_1 n_1 + \cdots + \sigma_r n_r = 0}} \left| H_n^\sigma u_{n_1}^{(1),\sigma_1} \cdots u_{n_r}^{(r),\sigma_r} \right| &\leq 2^r \|H\| \sum_{n \in \mathbb{Z}^r} \prod_{j=1}^r \left| u_{n_j}^{(j)} \right| \langle n_j \rangle^{-1} \\
&= 2^r \|H\| \sum_{n \in \mathbb{Z}^r} \langle n_1 \rangle^{-1} \left| u_{n_1}^{(1)} \right| \cdots \langle n_r \rangle^{-1} \left| u_{n_r}^{(r)} \right| \\
&= 2^r \|H\| \left(\sum_{n_1 \in \mathbb{Z}} \langle n_1 \rangle^{-1} \left| u_{n_1}^{(1)} \right| \right) \cdots \left(\sum_{n_r \in \mathbb{Z}} \langle n_r \rangle^{-1} \left| u_{n_r}^{(r)} \right| \right)
\end{aligned}$$

Call each n_j, k to get

$$\sum_{\substack{\sigma \in \{-1,+1\}^r \\ n \in \mathbb{Z}^r \\ \sigma_1 n_1 + \cdots + \sigma_r n_r = 0}} \left| H_n^\sigma u_{n_1}^{(1),\sigma_1} \cdots u_{n_r}^{(r),\sigma_r} \right|$$

$$
\begin{aligned}
&\le 2^r \|H\| \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{-1} \left|u_k^{(1)}\right| \right) \cdots \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{-1} \left|u_k^{(r)}\right| \right) \\
&= 2^r \|H\| \prod_{j=1}^{r} \sum_{k\in\mathbb{Z}} \langle k\rangle^{-1} \left|u_k^{(j)}\right| \frac{\langle k\rangle^s}{\langle k\rangle^s} \\
&= 2^r \|H\| \prod_{j=1}^{r} \sum_{k\in\mathbb{Z}} \langle k\rangle^{s} \left|u_k^{(j)}\right| \langle k\rangle^{-s-1} \\
&\le 2^r \|H\| \prod_{j=1}^{r} \underbrace{\left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left|u_k^{(j)}\right|^2 \right)^{1/2}}_{\|u^{(j)}\|_{h^s}} \underbrace{\left(\langle k\rangle^{-2(s+1)} \right)^{1/2}}_{<\infty \text{ since } s\ge 0} \qquad \text{by Cauchy Schwartz} \\
&\lesssim_{r,s} \|H\| \prod_{j=1}^{r} \|u^{(j)}\|_{h^s}.
\end{aligned}
$$

We still need to check that H is real-valued. For this we write

$$
\overline{H(u)} = \sum_{\substack{\sigma\in\{-1,+1\}^r \\ n\in\mathbb{Z}^r}} \overline{H_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_r}^{\sigma_r}}
$$

with

- $\overline{H_n^\sigma} = H_n^{-\sigma}$ from the reality condition,
- If $\sigma_j = 1$, then $\overline{u_{n_j}^{\sigma_j}} = \overline{u_{n_j}} = u_{n_j}^{-1} = u_{n_j}^{-\sigma}$ (Similarly if $\sigma_j = -1$).

This implies that

$$
\begin{aligned}
\overline{H(u)} &= \sum_{\substack{\sigma\in\{-1,+1\}^r \\ n\in\mathbb{Z}^r}} H_n^{-\sigma} u_{n_1}^{-\sigma_1} \cdots u_{n_r}^{-\sigma_r} \\
&= \sum_{\substack{\sigma\in\{-1,+1\}^r \\ n\in\mathbb{Z}^r}} H_n^{\sigma} u_{n_1}^{\sigma_1} \cdots u_{n_r}^{\sigma_r} \qquad \text{since } \sigma \in \{-1,+1\}^r \\
&= H(u). \qquad \square
\end{aligned}
$$

Corollary 4.1.1. *We can permute derivatives with the sum defining H.*

Proof. From Lemma 4.1.1, we deduce that the multilinear map defining H is well-defined and smooth, so it's derivatives are easily computed. Now, notice that H is a composition of this multilinear map and a smooth map $u \mapsto (u, \cdots, u)$. Hence, H is regular and its derivatives are obtained by composition. □

Lemma 4.1.2. *Let $r \geq 3$, $s \geq 0$ and $k \in \mathbb{Z}$. Then for all $u \in h^s(\mathbb{Z})$, we have*

$$\sum_{k \in \mathbb{Z}} \left(\sum_{\substack{n \in \mathbb{Z}^{r-1} \\ \sigma \in \{-1,+1\}^{r-1} \\ \sigma_1 n_1 + \cdots + \sigma_{r-1} n_{r-1} = k}} \langle k \rangle^{s-1} \prod_{j=1}^{r-1} \langle n_j \rangle^{-1} \left| u_{n_j}^{\sigma_j} \right| \right)^2 \lesssim_{r,s} \|u\|_{h^s}^{2(r-1)}. \tag{4.1}$$

Proof. Without loss of generality, assume that $\langle n_1 \rangle \leq \cdots \leq \langle n_{r-1} \rangle$.

- First, we get $\langle k \rangle^s \leq (r-1)^s \langle n_{r-1} \rangle^s$.

  Indeed, the zero momentum condition implies that $|k| = |\sigma_1 n_1 + \cdots + \sigma_{r-1} n_{r-1}|$. Thus, $\langle k \rangle = \langle \sigma_1 n_1 + \cdots + \sigma_{r-1} n_{r-1} \rangle \leq \langle n_1 \rangle + \cdots + \langle n_{r-1} \rangle \leq (r-1) \langle n_{r-1} \rangle$. So, since $s \geq 0$, we get the inequality.

- Similarly, we get that $\langle k \rangle^{-1} \leq (r-1) \langle n_1 \rangle^{-1}$.

Now, using the above estimates, we write

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}} \left(\sum_{\substack{n \in \mathbb{Z}^{r-1} \\ \sigma \in \{-1,+1\}^{r-1} \\ \sigma_1 n_1 + \cdots + \sigma_{r-1} n_{r-1} = k}} \langle k \rangle^{s-1} \prod_{j=1}^{r-1} \langle n_j \rangle^{-1} \left| u_{n_j}^{\sigma_j} \right| \right)^2 \\
&\lesssim_{r,s} \sum_{k \in \mathbb{Z}} \left(\sum_{\substack{n \in \mathbb{Z}^{r-1} \\ \sigma \in \{-1,+1\}^{r-1} \\ \sigma_1 n_1 + \cdots + \sigma_{r-1} n_{r-1} = k}} \langle n_1 \rangle^{-2} \left| u_{n_1}^{\sigma_1} \right| \langle n_2 \rangle^{-1} \left| u_{n_2}^{\sigma_2} \right| \cdots \langle n_{r-1} \rangle^{s-1} \left| u_{n_{r-1}}^{\sigma_{r-1}} \right| \right)^2 \\
&= \sum_{k \in \mathbb{Z}} \left(\sum_{\substack{n \in \mathbb{Z}^{r-1} \\ \sigma \in \{-1,+1\}^{r-1} \\ \sigma_1 n_1 + \cdots + \sigma_{r-1} n_{r-1} = k}} \langle n_{r-1} \rangle^{s-1} \left| u_{n_{r-1}}^{\sigma_{r-1}} \right| \langle n_1 \rangle^{-2} \left| u_{n_1}^{\sigma_1} \right| \cdots \langle n_{r-2} \rangle^{-1} \left| u_{n_{r-2}}^{\sigma_{r-2}} \right| \right)^2 \\
&= 2^{2(r-1)} \sum_{k \in \mathbb{Z}} \Big(\langle . \rangle^{s-1} |u| * \langle . \rangle^{-2} |u| * \langle . \rangle^{-1} |u| \cdots \langle . \rangle^{-1} |u| (k) \Big)^2.
\end{aligned}$$

Next, we use Young's Convolution inequality: $l^2 * l^1 * l^1 * \cdots * l^1 \hookrightarrow l^\infty$, and we obtain

$$\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{n\in\mathbb{Z}^{r-1}\\ \sigma\in\{-1,+1\}^{r-1}\\ \sigma_1 n_1+\cdots+\sigma_{r-1}n_{r-1}=k}} \langle k\rangle^{s-1}\prod_{j=1}^{r-1}\langle n_j\rangle^{-1}\left|u_{n_j}^{\sigma_j}\right|\right)^2 \lesssim_{r,s} \|\langle . \rangle^{s-1}u\|_{l^2}^2\|\langle . \rangle^{-2}u\|_{l^1}^2\|\langle . \rangle^{-1}u\|_{l^1}^2\cdots\|\langle . \rangle^{-1}u\|_{l^1}^2. \tag{4.2}$$

Notice that we have the following:

- $\|\langle . \rangle^{s-1}u\|_{l^2} = \|u\|_{h^{s-1}} \le \|u\|_{h^s}$
- By Cauchy Schwartz, we have

$$\begin{aligned}\|\langle . \rangle^{-1}u\|_{l^1} &= \sum_{l\in\mathbb{Z}}\left|\langle . \rangle^{-1}u_l\right| \\ &= \sum_{l\in\mathbb{Z}}\left|\langle . \rangle^{-s-1}\langle . \rangle^{s}u_l\right| \\ &\le \underbrace{\left(\sum_{l\in\mathbb{Z}}\langle . \rangle^{-2(s+1)}\right)^{1/2}}_{<\infty \text{ since } s\ge 0}\underbrace{\left(\sum_{k\in\mathbb{Z}}\langle . \rangle^{2s}|u_l|^2\right)^{1/2}}_{\|u\|_{h^s}}.\end{aligned}$$

- A similar argument shows that $\|\langle . \rangle^{-2}u\|_{l^1} \lesssim \|u\|_{h^s}$.

Finally, putting these results back in equation 4.2, we get

$$\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{n\in\mathbb{Z}^{r-1}\\ \sigma\in\{-1,+1\}^{r-1}\\ \sigma_1 n_1+\cdots+\sigma_{r-1}n_{r-1}=k}} \langle k\rangle^{s-1}\prod_{j=1}^{r-1}\langle n_j\rangle^{-1}\left|u_{n_j}^{\sigma_j}\right|\right)^2 \lesssim_{r,s} \|u\|_{h^s}^{2(r-1)}. \qquad \square$$

Proposition 4.1.1. *Let $r \ge 3$, $s \ge 0$ and consider $H \in \mathscr{H}^r$. Then the gradient of H is a smooth function from $h^s(\mathbb{Z})$ into $h^s(\mathbb{Z})$, and we have*

$$\forall u \in h^s(\mathbb{Z}), \quad \|\nabla H(u)\|_{h^s} \lesssim_{r,s} \|u\|_{h^s}^{r-1}\|H\|.$$

Proof. From Lemma 4.1.1, we get that H is a smooth function on h^s. We are going to prove that its gradient is also smooth and belongs to h^s. For this, consider $u \in h^s(\mathbb{Z})$. Then by definition we have

$$\|\nabla H(u)\|_{h^s}^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |(\nabla H(u))_k|^2$$

with

$$\begin{aligned}
(\nabla H(u))_k &= 2\partial_{\overline{u_k}} H(u) && \text{by Definition 2.2.2}\\
&= 2\partial_{\overline{u_k}}\left(\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} H_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r}\right)\\
&= 2\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} H_n^\sigma \partial_{\overline{u_k}}(u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r}) && \text{by Corollary 4.1.1}
\end{aligned}$$

Now, call one of the u_{n_j}, u_k and set the associated $\sigma_j = -1$. Then,

$$\begin{aligned}
(\nabla H(u))_k = 2&\sum_{\substack{\sigma\in\{-1,+1\}^{r-1}\\ n\in\mathbb{Z}^{r-1}}} H_{n_1,\cdots,n_{r-1},k}^{\sigma_1,\cdots,\sigma_{r-1},-1} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}} + 2\sum_{\substack{\sigma\in\{-1,+1\}^{r-1}\\ n\in\mathbb{Z}^{r-1}}} H_{n_1,\cdots,k,n_{r-1}}^{\sigma_1,\cdots,-1,\sigma_{r-1}} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}}\\
&+\cdots+2\sum_{\substack{\sigma\in\{-1,+1\}^{r-1}\\ n\in\mathbb{Z}^{r-1}}} H_{k,n_1,\cdots,n_{r-1}}^{-1,\sigma_1,\cdots,\sigma_{r-1}} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}}.
\end{aligned}$$

Since H_n^σ satisfies the symmetry condition, we get $H_{n_1,\cdots,n_{r-1},k}^{\sigma_1,\cdots,\sigma_{r-1},-1} = \cdots = H_{k,n_1,\cdots,n_{r-1}}^{-1,\sigma_1,\cdots,\sigma_{r-1}}$. So,

$$(\nabla H(u))_k = 2r \sum_{\substack{\sigma\in\{-1,+1\}^{r-1}\\ n\in\mathbb{Z}^{r-1}}} H_{n,k}^{\sigma,-1} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}}. \tag{4.3}$$

Plugging this expression in the norm, we obtain

$$\begin{aligned}
\|\nabla H(u)\|_{h^s}^2 &= \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left|2r \sum_{\substack{\sigma\in\{-1,+1\}^{r-1}\\ n\in\mathbb{Z}^{r-1}}} H_{n,k}^{\sigma,-1} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}}\right|^2\\
&\le 4r^2 \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left(\sum_{\substack{\sigma\in\{-1,+1\}^{r-1}\\ n\in\mathbb{Z}^{r-1}}} \left|H_{n,k}^{\sigma,-1}\right| \left|u_{n_1}^{\sigma_1}\right|\cdots\left|u_{n_{r-1}}^{\sigma_{r-1}}\right|\right)^2
\end{aligned}$$

$$\lesssim_r \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \Bigg(\sum_{\substack{\sigma\in\{-1,+1\}^{r-1} \\ n\in\mathbb{Z}^{r-1}}} \|H\| \langle k\rangle^{-1} \langle n_1\rangle^{-1} \cdots \langle n_{r-1}\rangle^{-1} \big|u_{n_1}^{\sigma_1}\big| \cdots \big|u_{n_{r-1}}^{\sigma_{r-1}}\big| \Bigg)^2 \quad \text{by the bound (v)}$$

$$\lesssim_r \|H\|^2 \sum_{k\in\mathbb{Z}} \Bigg(\sum_{\substack{\sigma\in\{-1,+1\}^{r-1} \\ n\in\mathbb{Z}^{r-1}}} \langle k\rangle^{s-1} \langle n_1\rangle^{-1} \cdots \langle n_{r-1}\rangle^{-1} \big|u_{n_1}^{\sigma_1}\big| \cdots \big|u_{n_{r-1}}^{\sigma_{r-1}}\big| \Bigg)^2$$

$$\lesssim_{r,s} \|H\|^2 (\|u\|_{h^s}^{r-1})^2 \quad \text{by 4.1.}$$

Thus, this continuity estimate proves that $\nabla H(u) \in h^s$ is smooth. □

Proposition 4.1.2. *Let $r \geq 3$, $s \geq 0$ and consider $H \in \mathscr{H}^r$. Then we have*

$$\forall u \in h^s(\mathbb{Z}), \quad \|d\nabla H(u)\|_{\mathscr{L}(h^s)} \lesssim_{r,s} \|u\|_{h^s}^{r-2} \|H\|.$$

Proof. The proof is a direct consequence of Proposition 4.1.1 where we showed that the multilinear map associated with $\nabla H(u)$ is continuous and thus regular. □

As a corollary, we can extend the differential of ∇H to negative spaces which will be needed in the proof of some time differentiability later in Chapter 5.

Corollary 4.1.2. *Let $r \geq 3$, $s \geq 0$ and consider $H \in \mathscr{H}^r$. Then for all $u \in h^s(\mathbb{Z})$, $d\nabla H(u)$ admits a unique continuous extension from h^{-s} into h^{-s}. Furthermore, the map $u \mapsto d\nabla H(u) \in \mathscr{L}(h^{-s}(\mathbb{Z}))$ is smooth and bounded.*

Proof. Details of this proof are found in [Bernier and Grébert, 2021] pages 25-26. □

4.2 Relation With the Poisson Brackets

Now we prove that the class of Hamiltonians is stable by Poisson bracket.

Proposition 4.2.1. *Let $H \in \mathscr{H}^r$ and $K \in \mathscr{H}^{r'}$ with $r, r' \geq 2$. Then, there exists a Hamiltonian $N \in \mathscr{H}^{r+r'-2}$ such that $\{H, K\}(u) = N(u)$ for all $u \in h^s(\mathbb{Z})$ with $s \geq 0$.*

Proof. Let $u \in h^s$. We write

$$H(u) = \sum_{\substack{\sigma \in \{-1,+1\}^r \\ n \in \mathbb{Z}^r}} H_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_r}^{\sigma_r} \quad \text{and} \quad K(u) = \sum_{\substack{\sigma' \in \{-1,+1\}^{r'} \\ n' \in \mathbb{Z}^{r'}}} K_{n'}^{\sigma'} u_{n'_1}^{\sigma'_1} \cdots u_{n'_{r'}}^{\sigma'_{r'}}.$$

Then using Lemma 2.2.1, we have

$$\{H, K\}(u) = 2i \sum_{k \in \mathbb{Z}} \partial_{\overline{u_k}} H(u) \partial_{u_k} K(u) - \partial_{u_k} H(u) \partial_{\overline{u_k}} K(u),$$

with

$$\partial_{\overline{u_k}} H(u) \partial_{u_k} K(u) = rr' \sum_{\substack{(\sigma,\sigma') \in \{-1,+1\}^{r-1} \times \{-1,+1\}^{r'-1} \\ (n,n') \in \mathbb{Z}^{r-1} \times \mathbb{Z}^{r'-1}}} H_{n,k}^{\sigma,-1} u_{n_1}^{\sigma_1} \cdots u_{n_{r-1}}^{\sigma_{r-1}} K_{n',k}^{\sigma',1} u_{n'_1}^{\sigma'_1} \cdots u_{n'_{r'-1}}^{\sigma'_{r'-1}}$$

which is obtained by using 4.3. In what follows, we set $n'' := (n, n')$, $\sigma'' := (\sigma, \sigma')$ and $r'' := r + r' - 2$. After re-indexing, we can see that

$$\begin{aligned}
&\{H, K\}(u) \\
&= 2i \sum_{k \in \mathbb{Z}} \Big[rr' \Big(\sum_{\substack{\sigma'' \in \{-1,+1\}^{r''} \\ n'' \in \mathbb{Z}^{r''}}} H_{n,k}^{\sigma,-1} u_{n_1}^{\sigma_1} \cdots u_{n_{r-1}}^{\sigma_{r-1}} K_{n',k}^{\sigma',1} u_{n'_1}^{\sigma'_1} \cdots u_{n'_{r'-1}}^{\sigma'_{r'-1}} - H_{n,k}^{\sigma,1} u_{n_1}^{\sigma_1} \cdots u_{n_{r-1}}^{\sigma_{r-1}} K_{n',k}^{\sigma',-1} u_{n'_1}^{\sigma'_1} \cdots u_{n'_{r'-1}}^{\sigma'_{r'-1}} \Big) \Big] \\
&= \sum_{\substack{\sigma'' \in \{-1,+1\}^{r''} \\ n'' \in \mathbb{Z}^{r''}}} \Big[2irr' \sum_{k \in \mathbb{Z}} \Big(H_{n,k}^{\sigma,-1} K_{n',k}^{\sigma',1} - H_{n,k}^{\sigma,1} K_{n',k}^{\sigma',-1} \Big) u_{n''_1}^{\sigma''_1} \cdots u_{n''_{r''}}^{\sigma''_{r''}} \Big] \qquad (4.4) \\
&= \sum_{\substack{\sigma'' \in \{-1,+1\}^{r''} \\ n'' \in \mathbb{Z}^{r''}}} N_{n''}^{\sigma''} u_{n''_1}^{\sigma''_1} \cdots u_{n''_{r''}}^{\sigma''_{r''}} \\
&= N(u)
\end{aligned}$$

Now we prove that $N \in \mathscr{H}^{r''}$. Notice that N satisfies the following:

(i) $N_{n''}^{\sigma''} \in \mathbb{C}$,

(ii) the zero momentum condition: Since H and K satisfy the zero momentum condition, then we have that $n_1\sigma_1 + \cdots + n_{r-1}\sigma_{r-1} = k$ and $n'_1\sigma'_1 + \cdots + n'_{r'-1}\sigma'_{r'-1} = -k$. Then $n_1\sigma_1 + \cdots + n_{r-1}\sigma_{r-1} = -n'_1\sigma'_1 - \cdots - n'_{r'-1}\sigma'_{r'-1}$, and so $n''_1\sigma''_1 + \cdots + n''_{r''}\sigma''_{r''} = 0$,

(iii) the symmetry condition which follows directly since it is satisfied by H and K,

(iv) the reality condition: Also, since it is satisfied by H and K, we get

$$\begin{aligned}
\overline{\frac{N_{n''}^{-\sigma''}}{2rr'}} &= \overline{\frac{i}{2rr'}\sum_{k\in\mathbb{Z}}\left(H_{n,k}^{-\sigma,-1}K_{n',k}^{-\sigma',1} - H_{n,k}^{-\sigma,1}K_{n',k}^{-\sigma',-1}\right)} \\
&= \frac{-i}{2rr'}\sum_{k\in\mathbb{Z}}\left(H_{n,k}^{\sigma,1}K_{n',k}^{\sigma',-1} - H_{n,k}^{\sigma,-1}K_{n',k}^{\sigma',1}\right) \\
&= \frac{i}{2rr'}\sum_{k\in\mathbb{Z}}\left(H_{n,k}^{\sigma,-1}K_{n',k}^{\sigma',1} - H_{n,k}^{\sigma,1}K_{n',k}^{\sigma',-1}\right) \\
&= \frac{N_{n''}^{-\sigma''}}{2rr'},
\end{aligned}$$

(v) the bound: Using the bounds of H_n^σ and $K_{n'}^{\sigma'}$ we get

$$\begin{aligned}
\left|N_{n''}^{-\sigma''}\right| &= \left|2rr'\sum_{k\in\mathbb{Z}}\left(H_{n,k}^{\sigma,-1}K_{n',k}^{\sigma',1} - H_{n,k}^{\sigma,1}K_{n',k}^{\sigma',-1}\right)\right| \\
&\le 2rr'\sum_{k\in\mathbb{Z}}\left(\left|H_{n,k}^{\sigma,-1}K_{n',k}^{\sigma',1}\right| + \left|H_{n,k}^{\sigma,1}K_{n',k}^{\sigma',-1}\right|\right) \\
&\lesssim_{r''} \|H\|\|K\|\sum_{k\in\mathbb{Z}}\langle n_1\rangle^{-1}\cdots\langle n_{r-1}\rangle^{-1}\langle k\rangle^{-1}\langle n'_1\rangle^{-1}\cdots\langle n'_{r'-1}\rangle^{-1}\langle k\rangle^{-1} \\
&\lesssim_{r''} \|H\|\|K\| \underbrace{\sum_{k\in\mathbb{Z}}\langle k\rangle^{-2}}_{<\infty \text{ since } 2>1} \prod_{j=1}^{r''}\langle n''_j\rangle^{-1}.
\end{aligned} \tag{4.5}$$

Therefore $N \in \mathscr{H}^{r''}$. It remains to justify the interchange of summation in equation 4.4. Using the latter bound, we can see that

$$\sum_{\substack{\sigma''\in\{-1,+1\}^{r''}\\ n''\in\mathbb{Z}^{r''}}}\sum_{k\in\mathbb{Z}}\left|H_{n,k}^{\sigma,-1}K_{n',k}^{\sigma',1} - H_{n,k}^{\sigma,1}K_{n',k}^{\sigma',-1}\right|\left|u_{n''_1}^{\sigma''_1}\cdots u_{n''_{r''}}^{\sigma''_{r''}}\right| \lesssim_{r''} \|H\|\|K\| \sum_{\substack{\sigma''\in\{-1,+1\}^{r''}\\ n''\in\mathbb{Z}^{r''}}}\prod_{j=1}^{r''}\langle n''_j\rangle^{-1}\left|u_{n''_1}^{\sigma''_1}\cdots u_{n''_{r''}}^{\sigma''_{r''}}\right|.$$

Next, using similar arguments as in Lemma 4.1.1, we get

$$\sum_{\substack{\sigma''\in\{-1,+1\}^{r''}\\ n''\in\mathbb{Z}^{r''}}} \prod_{j=1}^{r''} \langle n''_j\rangle^{-1} \left| u_{n''_j}^{\sigma''_j} \right| \lesssim_{r,s} \|u\|_{h^s}^{r''} < \infty \text{ since } u \in h^s.$$

Finally, applying Fubini's theorem, we obtain

$$\sum_{\substack{\sigma''\in\{-1,+1\}^{r''}\\ n''\in\mathbb{Z}^{r''}}} \left[\sum_{k\in\mathbb{Z}} \left(H_{n,k}^{\sigma,-1} K_{n',k}^{\sigma',1} - H_{n,k}^{\sigma,1} K_{n',k}^{\sigma',-1} \right) u_{n''_1}^{\sigma''_1} \cdots u_{n''_{r''}}^{\sigma''_{r''}} \right]$$

$$= \sum_{k\in\mathbb{Z}} \left[\sum_{\substack{\sigma''\in\{-1,+1\}^{r''}\\ n''\in\mathbb{Z}^{r''}}} \left(H_{n,k}^{\sigma,-1} K_{n',k}^{\sigma',1} - H_{n,k}^{\sigma,1} K_{n',k}^{\sigma',-1} \right) u_{n''_1}^{\sigma''_1} \cdots u_{n''_{r''}}^{\sigma''_{r''}} \right] \qquad \square$$

Lemma 4.2.1. *Let $r \geq 3, s \geq 0$ and $H \in \mathscr{H}^r$. Consider the quadratic Hamiltonian $Z_2 : h^s(\mathbb{Z}) \to \mathbb{R}$ written in the form*

$$Z_2(u) = \sum_{n\in\mathbb{Z}} w_n |u_n|^2,$$

where $w_n \in \mathbb{R}$ and $(\langle n\rangle^{-2s} w_n)_{n\in\mathbb{Z}}$ is bounded. Then for all $u \in h^s(\mathbb{Z})$, we have

$$\{H, Z_2\}(u) = 2i \sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} (\sigma_1 w_{n_1} + \cdots + \sigma_r w_{n_r}) H_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_r}^{\sigma_r}.$$

Proof. To start, note that from Proposition 4.1.1 and Definition 2.2.2, we have that $\nabla H(u) \in h^s(\mathbb{Z})$ and $\nabla Z_2(u) \in h^{-s}(\mathbb{Z})$ respectively. Thus, their poisson bracket is well-defined. Then, notice that

$$\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} (\sigma_1 w_{n_1} + \cdots + \sigma_r w_{n_r}) H_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_r}^{\sigma_r}$$

$$= \sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} \sigma_1 w_{n_1} H_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_r}^{\sigma_r} + \cdots + \sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} \sigma_r w_{n_r} H_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_r}^{\sigma_r}.$$

Using the symmetry condition of H, we apply a permutation on the coefficients:

$$\begin{aligned}
&\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} \sigma_1 w_{n_1} H_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r} + \cdots + \sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} \sigma_r w_{n_r} H_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r}\\
&= \sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} \sigma_1 w_{n_1} H_{n_2,\cdots,n_r,n_1}^{\sigma_2,\cdots,\sigma_r,\sigma_1} u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r} + \cdots + \sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} \sigma_r w_{n_r} H_{n_1,\cdots,n_r}^{\sigma_1,\cdots,\sigma_r} u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r}.
\end{aligned}$$

After re-indexing, the r sums coincide and we obtain

$$\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} (\sigma_1 w_{n_1} + \cdots + \sigma_r w_{n_r}) H_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r} = r \sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^r}} \sigma_r w_{n_r} H_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r}. \tag{4.6}$$

Next, recall that

$$\{H, Z_2\}(u) = 2i \sum_{k\in\mathbb{Z}} \partial_{\overline{u_k}} H(u)\partial_{u_k} Z_2(u) - \partial_{u_k} H(u)\partial_{\overline{u_k}} Z_2(u)$$

with $\partial_{u_k} Z_2(u) = w_k \overline{u_k}$ and $\partial_{u_k} H(u) = r \sum_{\substack{\sigma\in\{-1,+1\}^{r-1}\\ n\in\mathbb{Z}^{r-1}}} H_{n,k}^{\sigma,-1} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}}$. Putting the results together, we get

$$\begin{aligned}
\{H, Z_2\}(u) &= 2ir \sum_{k\in\mathbb{Z}} \Big[\sum_{\substack{\sigma\in\{-1,+1\}^{r-1}\\ n\in\mathbb{Z}^{r-1}}} \Big(\underbrace{H_{n,k}^{\sigma,-1} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}} w_k \overline{u_k}}_{\sigma_r=-1} - \underbrace{H_{n,k}^{\sigma,1} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}} w_k u_k}_{\sigma_r=1} \Big)\Big]\\
&= -2ir \sum_{k\in\mathbb{Z}} \Big[\sum_{\substack{\sigma\in\{-1,+1\}^{r}\\ n\in\mathbb{Z}^{r-1}}} H_{n,k}^{\sigma} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}} \sigma_r w_k u_k^{\sigma_r} \Big]\\
&= -2ir \sum_{\substack{\sigma\in\{-1,+1\}^{r}\\ n\in\mathbb{Z}^{r-1}}} \Big[\sum_{k\in\mathbb{Z}} H_{n,k}^{\sigma} u_{n_1}^{\sigma_1}\cdots u_{n_{r-1}}^{\sigma_{r-1}} \sigma_r w_k u_k^{\sigma_r} \Big] \qquad (4.7)\\
&= -2ir \sum_{\substack{\sigma\in\{-1,+1\}^{r}\\ n\in\mathbb{Z}^{r}}} H_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r} \sigma_r w_{n_r} \qquad n_r := k\\
&= -2i \sum_{\substack{\sigma\in\{-1,+1\}^{r}\\ n\in\mathbb{Z}^{r}}} (\sigma_1 w_{n_1} + \cdots + \sigma_r w_{n_r}) H_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_r}^{\sigma_r} \qquad \text{by 4.6.}
\end{aligned}$$

We are left with justifying the interchange of sums in equation 4.7. Using the fact that

$\langle k\rangle^{-2s}w_k$ is bounded as well as the bound (v), we write

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^{r-1}}}\left|H^\sigma_{n,k}u^{\sigma_1}_{n_1}\cdots u^{\sigma_{r-1}}_{n_{r-1}}w_k u^{\sigma_r}_k\right|\\
&=\sum_{k\in\mathbb{Z}}\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^{r-1}}}\left|H^\sigma_{n,k}\right|\left|u^{\sigma_1}_{n_1}\right|\cdots\left|u^{\sigma_{r-1}}_{n_{r-1}}\right|\left|w_k\right|\left|u^{\sigma_r}_k\right|\\
&\lesssim_r\sum_{k\in\mathbb{Z}}\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^{r-1}}}\|H\|\langle n_1\rangle^{-1}\cdots\langle n_{r-1}\rangle^{-1}\langle k\rangle^{2s-1}\left|u^{\sigma_1}_{n_1}\right|\cdots\left|u^{\sigma_{r-1}}_{n_{r-1}}\right|\left|u^{\sigma_r}_k\right|\times\frac{\langle k\rangle^s}{\langle k\rangle^s}\\
&=\|H\|\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^{r-1}}}\langle n_1\rangle^{-1}\left|u^{\sigma_1}_{n_1}\right|\cdots\langle n_{r-1}\rangle^{-1}\left|u^{\sigma_{r-1}}_{n_{r-1}}\right|\langle k\rangle^s\right)\langle k\rangle^{s-1}\left|u^{\sigma_r}_k\right|\\
&\le\|H\|\left(\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{\sigma\in\{-1,+1\}^r\\ n\in\mathbb{Z}^{r-1}}}\langle n_1\rangle^{-1}\left|u^{\sigma_1}_{n_1}\right|\cdots\langle n_{r-1}\rangle^{-1}\left|u^{\sigma_{r-1}}_{n_{r-1}}\right|\langle k\rangle^s\right)^2\right)^{1/2}\left(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s-2}\left|u^2_k\right|\right)^{1/2}\\
&\lesssim_{r,s}\|u\|^{r-1}_{h^s}\left(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s-2}\left|u^2_k\right|\right)^{1/2}\|H\| \qquad \text{by 4.1}\\
&\lesssim_{r,s}\|u\|^{r-1}_{h^s}\underbrace{\left(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\left|u_k\right|^2\right)^{1/2}}_{\|u\|_{h^s}}\|H\|\\
&\lesssim_{r,s}\|u\|^r_{h^s}\|H\|<\infty.
\end{aligned}$$

Finally, we conclude using Fubini. □

Chapter 5

Birkhoff Normal Form Theorem

This chapter aims to prove the important theorems. We will state Birkhoff normal form theorem in low regularity and provide a rigorous proof, followed by a corollary, the key result of the work.

5.1 Birkhoff Normal Form Theorem

Proposition 5.1.1. *Let* $s \geq 0$. $r \geq 3$ *and* $\chi \in \mathscr{H}^r$. *Then there exists* $\varepsilon_1 = (K\|\chi\|)^{-1/(r-2)}$ *with* K *depending on* (s, r), *and there exists a smooth map*

$$\begin{aligned} \phi_\chi : [-1, 1] \times B_{h^s(\mathbb{Z})}(0, \varepsilon_1) &\to h^s(\mathbb{Z}) \\ (t, u) &\mapsto \phi_\chi^t(u), \end{aligned}$$

such that it satisfies the following:

1. *solves the equation* $-i\partial_t \phi_\chi = (\nabla \chi) \circ \phi_\chi$,

2. $\forall\ t \in [-1,1]$, ϕ_χ^t *is close to the identity:*

$$\forall u \in B_{h^s(\mathbb{Z})}(0,\varepsilon_1), \quad \|\phi_\chi^t(u) - u\|_{h^s} \lesssim_{r,s} \|u\|_{h^s}^{r-1}\|\chi\|,$$

3. $\forall\ t \in [-1,1]$, ϕ_χ^t *is intertible:* $\|\phi_\chi^t(u)\|_{h^s} < \varepsilon_1 \implies \phi_\chi^{-t} \circ \phi_\chi^t(u) = u$,

4. $\forall\ t \in [-1,1], \quad \phi_\chi^t$ *is symplectic,*

5. its differential admits a unique continuous extension from $h^{-s}(\mathbb{Z})$ *into* $h^{-s}(\mathbb{Z})$.

Moreover, the map $u \in B_{h^s(\mathbb{Z})}(0,\varepsilon_1) \mapsto d\phi_\chi^t(u) \in \mathscr{L}(h^{-s}(\mathbb{Z}))$ *is continuous and we have*

$$\forall u \in B_{h^s(\mathbb{Z})}(0,\varepsilon_1), \forall \sigma \in \{-1,+1\}, \ \|d\phi_\chi^t(u)\|_{\mathscr{L}(h^{\sigma s})} \le 2.$$

Proof. ▷ Since $\chi \in \mathscr{H}^r$, by Proposition 4.1.1, $\nabla\chi$ is a smooth function on h^s. So, Cauchy Lipschitz theorem proves that $-i\partial_t\phi_\chi = (\nabla\chi)\circ\phi_\chi$ admits a unique smooth local solution $\phi_\chi^t(u)$. Let I_u be the maximal interval on which $\phi_\chi^t(u)$ is well-defined. For $t \in I_u$,

$$\begin{aligned}
-i\partial_t\phi_\chi^t(u) = (\nabla\chi)\circ\phi_\chi^t(u) &\implies \int_0^t \partial_\tau\phi_\chi^\tau(u)\,d\tau = i\int_0^t (\nabla\chi)\circ\phi_\chi^\tau(u)\,d\tau \\
&\implies \phi_\chi^t(u) - \phi_\chi^0(u) = i\int_0^t (\nabla\chi)\circ\phi_\chi^\tau(u)\,d\tau \\
&\implies \phi_\chi^t(u) - u = i\int_0^t (\nabla\chi)\circ\phi_\chi^\tau(u)\,d\tau.
\end{aligned}$$

Consequently, if $t \in [-1,1]$, we get

$$\begin{aligned}
\|\phi_\chi^t(u) - u\|_{h^s} &= \left\| \int_0^t (\nabla\chi)\circ\phi_\chi^\tau(u)\,d\tau \right\|_{h^s} \\
&\le \int_0^t \|(\nabla\chi)\circ\phi_\chi^\tau(u)\|_{h^s}\,d\tau \\
&\le \sup_{\tau\in(0,t)} \|(\nabla\chi)\circ\phi_\chi^\tau(u)\|_{h^s} \underbrace{\int_0^t d\tau}_{\le 1} \\
&\lesssim_{r,s} \|\chi\| \sup_{\tau\in(0,t)} \|\phi_\chi^\tau(u)\|_{h^s}^{r-1} \qquad \text{by Proposition 4.1.1.}
\end{aligned}$$

Now we aim at using a bootstrap argument. Let $J_u \subset I_u$ such that for all $t \in J_u \cap [-1,1]$, $\phi_\chi^t(u)$ is well-defined ($\phi_\chi^t(u) \in h^s$) with $\|\phi_\chi^t(u)\|_{h^s} \leq 3\|u\|_{h^s}$. We need to prove that $\phi_\chi^t(u)$ is well-defined on $[-1,1]$. For this, let $t \in J_u \cap [-1,1]$, then

$$\|\phi_\chi^t(u) - u\|_{h^s} \lesssim_{r,s} \|\chi\|(3\|u\|_{h^s})^{r-1} \leq C_{s,r}\|\chi\|3^{r-1}\|u\|_{h^s}^{r-1}$$

where $C_{s,r}$ is the maximum of the 2 constants obtained from Proposition 4.1.1 and 4.1.2. Thus, it would be sufficient to choose $\varepsilon_1 = (3^{r-1}C_{s,r}\|\chi\|)^{-1/r-2}$. Since $\|u\|_{h^s} \leq \varepsilon_1$ it follows that

$$\begin{aligned}
&\|\phi_\chi^t(u) - u\|_{h^s} \leq \varepsilon_1^{-(r-2)}\|u\|_{h^s}^{r-1} \leq \left(\frac{\|u\|_{h^s}}{\varepsilon_1}\right)^{r-2}\|u\|_{h^s} \leq \|u\|_{h^s} \\
&\implies \|\phi_\chi^t(u)\|_{h^s} - \|u\|_{h^s} \leq \|\phi_\chi^t(u) - u\|_{h^s} \leq \|u\|_{h^s} \\
&\implies \|\phi_\chi^t(u)\|_{h^s} \leq 2\|u\|_{h^s}.
\end{aligned}$$

Hence, using Lemma 2.2.2, we get that $\phi_\chi^t(u)$ is well-defined for $t \in [-1,1]$ with

$$\|\phi_\chi^t(u)\|_{h^s} \leq 2\|u\|_{h^s} \leq 3\|u\|_{h^s} \tag{5.1}$$

and close to the identity (properties 1. and 2. are satisfied).

▷ Moreover, ϕ_χ^t is invertible. Indeed, suppose that $\|\phi_\chi^t(u)\|_{h^s} < \varepsilon_1$. Then since $-t \in [-1,1]$, we get that $\phi_\chi^{-t} \circ \phi_\chi^t(u)$ is a solution of $-i\partial_t\phi_\chi = (\nabla\chi) \circ \phi_\chi$ with initial condition $\phi_\chi^0 \circ \phi_\chi^0(u) = u$. Also, $\phi_\chi^{-t+t}(u) = \phi_\chi^0(u)$ is another solution with initial condition u. Thus, by uniqueness of solutions we must have $\phi_\chi^{-t} \circ \phi_\chi^t(u) = u$.

▷ Now, we check that ϕ_χ^t is symplectic. Since ϕ_χ is a smooth solution of $\partial_t\phi_\chi = i(\nabla\chi) \circ \phi_\chi$, then taking the differential of the 2 sides of the equation, we get

$$\begin{aligned}
\partial_t(d\phi_\chi^t(u)(v)) &= d(\partial_t\phi_\chi^t(u))(v) && \text{by Theorem 2.2.3} \\
&= id((\nabla\chi) \circ \phi_\chi^t(u))(v)
\end{aligned}$$

$$= i(d(\nabla\chi)\circ\phi^t_\chi(u))d\phi^t_\chi(u)(v) \tag{5.2}$$

with $d\phi^0_\chi(u)(v) = v$. Now for ϕ^t_χ to be symplectic, we need to prove that for $u \in B_{h^s}(0,\varepsilon_1)$ and $v, w \in h^s$ we have $\langle iv, w\rangle_{l^2} = \langle id\phi^t_\chi(u)(v), d\phi^t_\chi(u)(w)\rangle_{l^2}$. For this define

$$W(t) = \langle id\phi^t_\chi(u)(v), d\phi^t_\chi(u)(w)\rangle_{l^2}$$

and notice that

$$W(0) = \langle id\phi^0_\chi(u)(v), d\phi^0_\chi(u)(w)\rangle_{l^2} = \langle iv, w\rangle_{l^2}.$$

Thus it would be sufficient to prove that $\frac{d}{dt}W(t) = 0$. Indeed,

$$\begin{aligned}\frac{d}{dt}W(t) &= \langle i\partial_t d\phi^t_\chi(u)(v), d\phi^t_\chi(u)(w)\rangle_{l^2} + \langle id\phi^t_\chi(u)(v), \partial_t d\phi^t_\chi(u)(w)\rangle_{l^2}\\ &= \langle -(d(\nabla\chi)\circ\phi^t_\chi(u))d\phi^t_\chi(u)(v), d\phi^t_\chi(u)(w)\rangle_{l^2}\\ &+ \langle d\phi^t_\chi(u)(v), (d(\nabla\chi)\circ\phi^t_\chi(u))d\phi^t_\chi(u)(w)\rangle_{l^2} && \text{by 5.2.}\end{aligned}$$

Using the definition of the differential, it is easy to see that these 2 terms will vanish since

$$\begin{aligned}&\langle d\phi^t_\chi(u)(v), (d(\nabla\chi)\circ\phi^t_\chi(u))d\phi^t_\chi(u)(w)\rangle_{l^2}\\ &= d\Big[\langle d\phi^t_\chi(u)(v), \nabla\chi\circ\phi^t_\chi(u)\rangle_{l^2}\Big]d\phi^t_\chi(u)(w)\\ &= d\Big[d\chi\circ\phi^t_\chi(u)(d\phi^t_\chi(u)(v))\Big]d\phi^t_\chi(u)(w) && \text{by definition}\\ &= d^2\chi\circ\phi^t_\chi(u)(d\phi^t_\chi(u)(v))(d\phi^t_\chi(u)(w))\\ &= d^2\chi\circ\phi^t_\chi(u)(d\phi^t_\chi(u)(w))(d\phi^t_\chi(u)(v)) && \text{by Theorem 2.2.3}\\ &= d\Big[\langle d\phi^t_\chi(u)(w), \nabla\chi\circ\phi^t_\chi(u)\rangle_{l^2}\Big]d\phi^t_\chi(u)(v)\\ &= \langle d\phi^t_\chi(u)(w), (d(\nabla\chi)\circ\phi^t_\chi(u))d\phi^t_\chi(u)(v)\rangle_{l^2}\\ &= \langle (d(\nabla\chi)\circ\phi^t_\chi(u))d\phi^t_\chi(u)(v), d\phi^t_\chi(u)(w)\rangle_{l^2}.\end{aligned}$$

▷ Finally, we prove the estimates. From 5.2 we can write

$$d\phi_\chi^t(u)(v) = v + i\int_0^t (d(\nabla\chi)\circ\phi_\chi^\tau(u))d\phi_\chi^\tau(u)(v)\,d\tau.$$

As a result, we get

$$\begin{aligned}
\|d\phi_\chi^t(u)\|_{\mathscr{L}(h^s)} &\le 1 + \int_0^t \|d\nabla\chi(\phi_\chi^\tau(u))\|_{\mathscr{L}(h^s)}\|d\phi_\chi^\tau(u)\|_{\mathscr{L}(h^s)}\,d\tau \\
&\le 1 + \int_0^t C_{r,s}\|\phi_\chi^\tau(u))\|_{h^s}^{r-2}\|\chi\|\|d\phi_\chi^\tau(u)\|_{\mathscr{L}(h^s)}\,d\tau \qquad \text{by Proposition 4.1.2} \\
&\le 1 + \int_0^t C_{r,s}\|\chi\|3^{r-2}\|u\|_{h^s}^{r-2}\|d\phi_\chi^\tau(u)\|_{\mathscr{L}(h^s)}\,d\tau \qquad \text{by 5.1.}
\end{aligned}$$

By definition of ε_1, we have

$$3^{r-2}\|\chi\|C_{s,r}\|u\|_{h^s}^{r-2} \le 3^{r-2}\|\chi\|C_{s,r}\varepsilon_1^{r-2} = 3^{r-2}\|\chi\|C_{s,r}(3^{r-1}\|\chi\|C_{s,r})^{-r+2/r-2} = \frac{1}{3}.$$

So,

$$\|d\phi_\chi^t(u)\|_{\mathscr{L}(h^s)} \le 1 + \frac{1}{3}\int_0^t \|d\phi_\chi^\tau(u)\|_{\mathscr{L}(h^s)}\,d\tau.$$

By Gronwall's Lemma, we conclude that

$$\|d\phi_\chi^t(u)\|_{\mathscr{L}(h^s)} \le e^{1/3} \le 2.$$

With Corollary 4.1.2 and similar arguments as in the proof of ϕ_χ^t invertible, we can prove that its differential admits a unique extension from h^{-s} to h^{-s}. □

Remark. Keep in mind that $\|\phi_\chi^t(u)\|_{h^s} \le 2\|u\|_{h^s}$ for $u \in B_{h^s}(0,\varepsilon_1)$, because it has been used several times in this chapter.

Lemma 5.1.1. *If G is a smooth function, then*

(i) $\frac{d}{dt}(G\circ\phi_\chi^t) = \{\chi, G\}\circ\phi_\chi^t$

(ii) $G\circ\phi_\chi^t = \sum_{j=0}^{k}\frac{G_j}{j!} + \frac{1}{k!}\int_0^1 (1-t)^k G_{k+1}\circ\phi_\chi^t\,dt$ *with* $G_j = \mathrm{ad}_\chi^j G$.

Proof. (i) We have that

$$\frac{d}{dt}(G\circ\phi_\chi^t) = \langle i\nabla G(\phi_\chi^t), i\partial_t\phi_\chi^t\rangle_{l^2} = \langle i\nabla G(\phi_\chi^t), -i(\nabla\chi)\circ\phi_\chi^t\rangle_{l^2}$$
$$= -\{G,\chi\}\circ\phi_\chi^t = \{\chi, G\}\circ\phi_\chi^t$$

(ii) We write the Taylor expansion between 0 and 1. Then,

$$G\circ\phi_\chi^t = \sum_{j=0}^{k}\frac{(G\circ\phi_\chi^t)^{(j)}(0)}{j!} + \int_0^1 \frac{(G\circ\phi_\chi^t)^{(k+1)}}{k!}(1-t)^k\,dt.$$

Using (i), we deduce that $(G\circ\phi_\chi^t)^{(j)} = G_j\circ\phi_\chi^t$. So back to our expansion,

$$G\circ\phi_\chi^t = G + \{\chi, G\} + \sum_{j=2}^{k}\frac{1}{j!}\{\chi, G_{j-1}\} + \frac{1}{k!}\int_0^1 \frac{G_{k+1}\circ\phi_\chi^t}{k!}(1-t)^k\,dt. \qquad \square$$

Theorem 5.1.2. *Let $s\geq 0$ and $r>p\geq 3$. Let $Z_2 : h^s(\mathbb{Z})\to\mathbb{R}$ be a quadratic Hamiltonian of the form $Z_2(u) = \frac{1}{2}\sum_{n\in\mathbb{Z}} w_n|u_n|^2$ where $(\langle n\rangle^{-2s}w_n)_{n\in\mathbb{Z}}$ is bounded and the sequence of frequencies w is strongly non-resonant up to any order. Let $P : h^s(\mathbb{Z})\mapsto\mathbb{R}$ be a Hamiltonian polynomial of the form $P(u) = \sum_{p\leq j\leq r-1} P^{(j)}(u)$ with $P^{(j)}\in\mathscr{H}^j$ satisfying $\|P^{(j)}\|\leq c_j$ and $(c_j)_{p\leq j\leq r-1}$ is a sequence of positive constants. Then, there exists positive constants C depending on (r,s,γ,c) and b depending on (β,r) (β and γ are the constants obtained from the strong non-resonance condition) such that $\forall N\geq 1$, there exists $\varepsilon_0\geq\frac{1}{CN^b}$ and there exists two smooth symplectic close to the identity maps $\tau^{(0)}$ and $\tau^{(1)}$*

$$\forall\sigma\in\{0,1\},\ \|u\|_{h^s}<2^\sigma\varepsilon_0 \implies \|\tau^{(\sigma)}(u)-u\|_{h^s}\leq\left(\frac{\|u\|_{h^s}}{2^\sigma\varepsilon_0}\right)^{p-2}\|u\|_{h^s} \tag{5.3}$$

making the diagram commute

$$\begin{tikzcd} B_{h^s(\mathbb{Z})}(0,\varepsilon_0) \arrow[r, "\tau^{(0)}"] \arrow[rr, bend right, "id_{h^s}"'] & B_{h^s(\mathbb{Z})}(0,2\varepsilon_0) \arrow[r, "\tau^{(1)}"] & h^s(\mathbb{Z}) \end{tikzcd}$$

such that $(Z_2+P)\circ\tau^{(1)}$ *admits on* $B_{h^s(\mathbb{Z})}(0,2\varepsilon_0)$ *the decomposition*

$$(Z_2+P)\circ\tau^{(1)} = Z_2+Q+R$$

where Q *is a Hamoltonian polynomial constructed in the proof and commuting with the low super-actions given by* $J_n(u) = \sum_{w_k=w_n} |u_k|^2$

$$\forall n\in\mathbb{Z}, \langle n\rangle \le N \implies \{J_n, Q\} = 0,$$

and the remainder term R *is a smooth function on* $B_{h^s(\mathbb{Z})}(0,2\varepsilon_0)$ *satisfying*

$$\|\nabla R(u)\|_{h^s} \le CN^b\|u\|_{h^s}^{r-1}.$$

Moreover, for $\sigma\in\{0,1\}$ *and* $u\in B_{h^s(\mathbb{Z})}(0,2^\sigma\varepsilon_0)$, $d\tau^{(\sigma)}(u)$ *admits a unique continuous extension from* $h^{-s}(\mathbb{Z})$ *to* $h^{-s}(\mathbb{Z})$ *depending continuously on* u *and satisfying*

$$\|d\tau^{(\sigma)}(u)\|_{\mathscr{L}(h^s)} \le 2^{r-p} \quad \textit{and} \quad \|d\tau^{(\sigma)}(u)\|_{\mathscr{L}(h^{-s})} \le 2^{r-p}. \tag{5.4}$$

Proof. We will do the proof using induction on $r_*\in[\![p,r]\!]$.

Initial Step: For $r_*=p$.

We set $C=b=0$ and $\tau^{(0)}=\tau^{(1)}=id_{h^s}$. Consequently, we get $\varepsilon_0=+\infty$, $\tau^{(0)}$ and $\tau^{(1)}$ two symplectic maps and the decomposition $(Z_2+P)\circ\tau^{(1)}=Z_2+Q$ with $Q=P$ Hamiltonian (by assumption) and $R=0$.

Induction Step: Assume that it is true for r_*, and prove it for r_*+1.

In other words, assume that there exists non-negative constants $b_1,(b_{3,j})_{p\le j\le r}$ depending on (β,r_*) and b_2 depending on (β,r_*,r), as well as $C_1,(C_{3,j})_{p\le j\le r}$ depending on (r_*,s,γ) and C_2 depending on (r_*,s,γ,r) such that for all $N\ge 1$, there exists $\varepsilon_0\ge\frac{1}{C_1N^{b_1}}$ and there exists two smooth symplectic close to the identity maps $\tau^{(0)}$ and $\tau^{(1)}$ making the above diagram commute, such that $(Z_2 + P) \circ \tau^{(1)}$ admits on $B_{h^s(\mathbb{Z})}(0, 2\varepsilon_0)$ the decomposition

$$(Z_2 + P) \circ \tau^{(1)} = Z_2 + Q^{(p)} + \cdots + Q^{(r-1)} + R$$

where $Q^{(j)} \in \mathscr{H}^j$ satisfies $\|Q^{(j)}\| \leq C_{3,j} N^{b_{3,j}}$ and having the first polynomials commute with the low super-actions

$$|j| < r_* \quad \text{and} \quad \langle n \rangle \leq N \implies \{J_n, Q^{(j)}\} = 0,$$

and the remainder term R is a smooth function on $B_{h^s(\mathbb{Z})}(0, 2\varepsilon_0)$ satisfying

$$\|\nabla R(u)\|_{h^s} \leq C_2 N^{b_2} \|u\|_{h^s}^{r-1}.$$

Moreover, for $\sigma \in \{0, 1\}$ and $u \in B_{h^s(\mathbb{Z})}(0, 2^\sigma \varepsilon_0)$, $d\tau^{(\sigma)}(u)$ admits a unique continuous extension from $h^{-s}(\mathbb{Z})$ to $h^{-s}(\mathbb{Z})$ depending continuously on u and satisfying

$$\|d\tau^{(\sigma)}(u)\|_{\mathscr{L}(h^s)} \leq 2^{r_* - p} \quad \text{and} \quad \|d\tau^{(\sigma)}(u)\|_{\mathscr{L}(h^{-s})} \leq 2^{r_* - p}.$$

Now, we prove this result for $r_* + 1$ and we we distinguish between the terms associated to r_* and the ones associated to $r_* + 1$ by a symbol #. First, we will state and prove two lemmas.

Lemma 5.1.3. *We can decompose*

$$Q^{(r_*)} = L + U$$

where $L, U \in \mathscr{H}^{r_*}$ *and* U *commutes with the low super-actions.*

Proof. We write

$$
\begin{aligned}
Q^{(r_*)} &= \sum_{\substack{\sigma\in\{-1,+1\}^{r_*}\\ n\in\mathbb{Z}^{r_*}}} Q_n^{(r_*),\sigma} u_{n_1}^{\sigma_1}\cdots u_{n_{r_*}}^{\sigma_{r_*}} \\
&= \sum_{\substack{\sigma\in\{-1,+1\}^{r_*}\\ n\in\mathbb{Z}^{r_*}}} (L_n^\sigma + U_n^\sigma) u_{n_1}^{\sigma_1}\cdots u_{n_{r_*}}^{\sigma_{r_*}} \\
&= L + U
\end{aligned}
$$

with $L_n^\sigma = \begin{cases} Q_n^{(r_*),\sigma} & \text{if } \kappa_w(n,\sigma) \le N, \\ 0 & \text{otherwise} \end{cases}$ and $U_n^\sigma = \begin{cases} 0 & \text{if } \kappa_w(n,\sigma) \le N, \\ Q_n^{(r_*),\sigma} & \text{otherwise.} \end{cases}$

Obviously, L and $U \in \mathscr{H}^{r_*}$. Now, we check that U commutes with J_m. For instance, for $\langle m\rangle \le N$ and $u \in h^s$, we apply Lemma 4.2.1 and we get

$$
\begin{aligned}
\{J_m, U\}(u) &= \{\sum_{w_n = w_m} |u_n|^2, \sum_{\substack{\sigma\in\{-1,+1\}^{r_*}\\ n\in\mathbb{Z}^{r_*}}} U_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_{r_*}}^{\sigma_{r_*}}\} \\
&= \{\sum_{n\in\mathbb{Z}} \mathbb{1}_{w_n = w_m} |u_n|^2, \sum_{\substack{\sigma\in\{-1,+1\}^{r_*}\\ n\in\mathbb{Z}^{r_*}}} U_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_{r_*}}^{\sigma_{r_*}}\} \\
&= 2i \sum_{\substack{\sigma\in\{-1,+1\}^{r_*}\\ n\in\mathbb{Z}^{r_*}}} (\sigma_1 \mathbb{1}_{w_{n_1} = w_m} + \cdots + \sigma_{r_*} \mathbb{1}_{w_{n_{r_*}} = w_m}) U_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_{r_*}}^{\sigma_{r_*}} \\
&= 2i \sum_{\substack{\sigma\in\{-1,+1\}^{r_*}\\ n\in\mathbb{Z}^{r_*}}} \left(\sum_{\substack{k=1,\cdots,r_*\\ w_{n_k} = w_m}} \sigma_k \right) U_n^\sigma u_{n_1}^{\sigma_1}\cdots u_{n_{r_*}}^{\sigma_{r_*}}
\end{aligned}
$$

Notice that

- If $U_n^\sigma \neq 0$, then by definition of U, we know that $\kappa_w(\sigma,n) > N$. But $\langle m\rangle \le N < \kappa_w(\sigma,n)$, so $\langle m\rangle < \kappa_w(\sigma,n)$ which is a defined minimum. Hence, we cannot have $\sum_{w_{n_k} = w_m} \sigma_k \neq 0$. Thus $\sum_{w_{n_k} = w_m} \sigma_k = 0$.
- If $\sum_{w_{n_k} = w_m} \sigma_k \neq 0$, then $\kappa_w(\sigma,n) \le \langle m\rangle \le N$. Hence, $U_n^\sigma = 0$ again by definition of U.

We deduce that either $\sum_{w_{n_k}=w_m} \sigma_k = 0$ or $U_n^\sigma = 0$. Therefore, $\{J_m, U\}(u) = 0$. ∎

Lemma 5.1.4. *Recall L from the Lemma 5.1.3 and let $\chi \in \mathscr{H}^{r_*}$ be the Hamiltonian*

$$\chi_n^\sigma = \begin{cases} \frac{L_n^\sigma}{i(\sigma_1 w_{n_1} + \cdots + \sigma_{r_*} w_{n_{r_*}})} & \text{if } \kappa_w(\sigma, n) \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

Then, χ has the bound

$$\|\chi\| \leq \gamma_{r_*}^{-1} C_{3,r_*} N^{\beta_{r_*} + b_{3,r_*}}$$

and satisfies the Homological Equation given by

$$\{\chi, Z_2\} + L = 0. \tag{5.5}$$

Proof. First, note that for $\kappa_w(n, \sigma) \leq N$ we have due to the strong non-resonance condition

$$\left|\sigma_1 w_{n_1} + \cdots + \sigma_{r_*} w_{n_{r_*}}\right| \geq \gamma_{r_*} \kappa_w(\sigma, n)^{-\beta_{r_*}} \geq \gamma_{r_*} N^{-\beta_{r_*}} \neq 0.$$

So, χ is well-defined and satisfies

$$\begin{aligned}
\|\chi\| &= \sup_{\substack{\sigma \in \{-1,+1\}^{r_*} \\ n \in \mathbb{Z}^{r_*}}} |\chi_n^\sigma| \prod_{j=1}^{r_*} \langle n_j \rangle \\
&= \sup_{\substack{\sigma \in \{-1,+1\}^{r_*} \\ n \in \mathbb{Z}^{r_*}}} \left| \frac{L_n^\sigma}{i(\sigma_1 w_{n_1} + \cdots + \sigma_{r_*} w_{n_{r_*}})} \right| \prod_{j=1}^{r_*} \langle n_j \rangle \\
&\leq \sup_{\substack{\sigma \in \{-1,+1\}^{r_*} \\ n \in \mathbb{Z}^{r_*}}} \frac{|L_n^\sigma|}{\gamma_{r_*} N^{-\beta_{r_*}}} \prod_{j=1}^{r_*} \langle n_j \rangle \\
&= \gamma_{r_*}^{-1} N^{\beta_{r_*}} \|L\| \\
&\leq \gamma_{r_*}^{-1} N^{\beta_{r_*}} \|Q^{(r_*)}\| \\
&\leq \gamma_{r_*}^{-1} N^{\beta_{r_*}} C_{3,r_*} N^{b_{3,r_*}} && \text{by induction hypothesis} \\
&= \gamma_{r_*}^{-1} C_{3,r_*} N^{\beta_{r_*} + b_{3,r_*}}.
\end{aligned}$$

Next, to prove that it satisfies 5.5, we consider two cases.

- Case 1: $\kappa_w(\sigma, n) > N$. It is obvious to see that

$$\{\chi, Z_2\} + L = \{0, Z_2\} + L = 0.$$

- Case 2: $\kappa_w(\sigma, n) \le N$. Using Lemma 4.3 we obtain

$$\begin{aligned}
&\{\chi, Z_2\} + L \\
&= \Big\{ \sum_{\substack{\sigma \in \{-1,+1\}^{r_*} \\ n \in \mathbb{Z}^{r_*}}} \frac{L_n^\sigma}{i(\sigma_1 w_{n_1} + \cdots + \sigma_{r_*} w_{n_{r_*}})} u_{n_1}^{\sigma_1} \cdots u_{n_{r_*}}^{\sigma_{r_*}}, Z_2 \Big\} + \sum_{\substack{\sigma \in \{-1,+1\}^{r_*} \\ n \in \mathbb{Z}^{r_*}}} L_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_{r_*}}^{\sigma_{r_*}} \\
&= -\frac{2i}{2} \sum_{\substack{\sigma \in \{-1,+1\}^{r_*} \\ n \in \mathbb{Z}^{r_*}}} \frac{(\sigma_1 w_{n_1} + \cdots + \sigma_{r_*} w_{n_{r_*}}) L_n^\sigma}{i(\sigma_1 w_{n_1} + \cdots + \sigma_{r_*} w_{n_{r_*}})} u_{n_1}^{\sigma_1} \cdots u_{n_{r_*}}^{\sigma_{r_*}} + \sum_{\substack{\sigma \in \{-1,+1\}^{r_*} \\ n \in \mathbb{Z}^{r_*}}} L_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_{r_*}}^{\sigma_{r_*}} \\
&= -\sum_{\substack{\sigma \in \{-1,+1\}^{r_*} \\ n \in \mathbb{Z}^{r_*}}} L_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_{r_*}}^{\sigma_{r_*}} + \sum_{\substack{\sigma \in \{-1,+1\}^{r_*} \\ n \in \mathbb{Z}^{r_*}}} L_n^\sigma u_{n_1}^{\sigma_1} \cdots u_{n_{r_*}}^{\sigma_{r_*}} \\
&= 0. \qquad \blacksquare
\end{aligned}$$

After that, will deal with the existence of the new variables. By Proposition 5.1.1, we get $\varepsilon_1 = (K\|\chi\|)^{-1/(r_*-2)}$ and a smooth map

$$\begin{aligned}
\phi_\chi \ : [-1, 1] \times B_{h^s(\mathbb{Z})}(0, \varepsilon_1) &\to h^s(\mathbb{Z}) \\
(t, u) &\mapsto \phi_\chi^t(u),
\end{aligned}$$

such that it satisfies the following:

1. solves the equation $-i\partial_t \phi_\chi = (\nabla \chi) \circ \phi_\chi$,

2. $\forall\ t \in [-1, 1]$, ϕ_χ^t is close to the identity: $\forall u \in B_{h^s(\mathbb{Z})}(0, \varepsilon_1)$,

$$\|\phi_\chi^t(u) - u\|_{h^s} \lesssim_{r,s} \|u\|_{h^s}^{r_*-1} \|\chi\| \le K \|u\|_{h^s}^{r_*-1} \|\chi\| = \frac{1}{\varepsilon_1^{r_*-2}} \|u\|_{h^s}^{r_*-2} \|u\|_{h^s}$$

$$= \left(\frac{\|u\|_{h^s}}{\varepsilon_1}\right)^{r_*-2} \|u\|_{h^s} \tag{5.6}$$

3. $\forall\ t \in [-1,1]$, ϕ_χ^t is intertible: $\|\phi_\chi^t(u)\|_{h^s} < \varepsilon_1 \implies \phi_\chi^{-t} \circ \phi_\chi^t(u) = u$,

4. $\forall\ t \in [-1,1]$, ϕ_χ^t is symplectic,

5. its differential admits a unique continuous extension from $h^{-s}(\mathbb{Z})$ into $h^{-s}(\mathbb{Z})$.

Moreover, the map $u \in B_{h^s(\mathbb{Z})}(0,\varepsilon_1) \mapsto d\phi_\chi^t(u) \in \mathscr{L}(h^{-s}(\mathbb{Z}))$ is continuous and we have

$$\forall u \in B_{h^s(\mathbb{Z})}(0,\varepsilon_1), \forall \sigma \in \{-1,+1\}, \quad \|d\phi_\chi^t(u)\|_{\mathscr{L}(h^{\sigma s})} \leq 2. \tag{5.7}$$

Notice that

$$\varepsilon_1 = (K\|\chi\|)^{-1/(r_*-2)} \geq (K\gamma_{r_*}^{-1} C_{3,r_*} N^{\beta_{r_*}+b_{3,r_*}})^{-1/(r_*-2)} \geq \frac{6}{C_1^\# N^{b_1^\#}} := 6\varepsilon_0^\# \tag{5.8}$$

where $C_1^\# = 6\max(C_1, (K\gamma_{r_*}^{-1} C_{3,r_*})^{1/r_*-2}, 1)$ and $b_1^\# = \max(b_1, \frac{\beta_{r_*}+b_{3,r_*}}{r_*-2})$. After that, define

$$\tau_\#^{(1)} := \tau^{(1)} \circ \phi_\chi^1 \text{ on } B_{h^s}(0, 2\varepsilon_0^\#) \quad \text{and} \quad \tau_\#^{(0)} := \phi_\chi^{-1} \circ \tau^{(0)} \text{ on } B_{h^s}(0,\varepsilon_0^\#).$$

It is easy to see that the 2 maps are smooth being the composition of 2 smooth maps and that $\tau_\#^{(1)} \circ \tau_\#^{(0)} = \tau^{(1)} \circ \phi_\chi^1 \circ \phi_\chi^{-1} \circ \tau^{(0)} = \tau^{(1)} \circ \tau^{(0)} = \mathrm{id}_{h^s}$ by induction hypothesis, which indicates that the diagram commutes. We are going to prove next that $\tau_\#^{(1)}$ is indeed close to the identity, symplectic and has a continuous extension (Similar properties and computations apply to $\tau_\#^{(0)}$).

- Close to the identity: Let $u \in B_{h^s}(0, 2\varepsilon_0^\#)$. We need to prove that

$$\|\tau_\#^{(1)} u - u\|_{h^s} \leq \left(\frac{\|u\|_{h^s}}{2\varepsilon_0^\#}\right)^{p-2} \|u\|_{h^s}.$$

We have the following:

$$\begin{aligned}\|\tau_{\#}^{(1)}u-u\|_{h^s} &= \|\tau^{(1)}\circ\phi_\chi^1(u)-u\|_{h^s}\\ &= \|\tau^{(1)}\circ\phi_\chi^1(u)-\phi_\chi^1(u)+\phi_\chi^1(u)-u\|_{h^s}\\ &\le \|\tau^{(1)}\circ\phi_\chi^1(u)-\phi_\chi^1(u)\|_{h^s}+\|\phi_\chi^1(u)-u\|_{h^s}\end{aligned}$$

For the first term, $\|\phi_\chi^1(u)\|_{h^s}\le 2\|u\|_{h^s}\le 2(2\varepsilon_0^{\#})<2(6\varepsilon_0^{\#})\le 2\varepsilon_0$ by definition of $\varepsilon_0^{\#}$. Thus, applying 5.3, we get

$$\begin{aligned}\|\tau^{(1)}\circ\phi_\chi^1(u)-\phi_\chi^1(u)\|_{h^s} &\le \left(\frac{\|\phi_\chi^1(u)\|_{h^s}}{2\varepsilon_0}\right)^{p-2}\|\phi_\chi^1(u)\|_{h^s}\le\left(\frac{2\|u\|_{h^s}}{2\varepsilon_0}\right)^{p-2}2\|u\|_{h^s}\\ &\le 2\left(\frac{\|u\|_{h^s}}{6\varepsilon_0^{\#}}\right)^{p-2}\|u\|_{h^s}\le\frac{2}{3}\left(\frac{\|u\|_{h^s}}{2\varepsilon_0^{\#}}\right)^{p-2}\|u\|_{h^s}. \qquad (5.9)\end{aligned}$$

For the second term, $\|u\|_{h^s}\le 2\varepsilon_0^{\#}<6\varepsilon_0^{\#}\le\varepsilon_1$ by 5.8. Thus, applying 5.6, we get

$$\begin{aligned}\|\phi_\chi^1(u)-u\|_{h^s} &\le \left(\frac{\|u\|_{h^s}}{\varepsilon_1}\right)^{r_*-2}\|u\|_{h^s}\le\left(\frac{\|u\|_{h^s}}{6\varepsilon_0^{\#}}\right)^{r_*-2}\|u\|_{h^s}\\ &=\frac{1}{3^{r_*-2}}\left(\frac{\|u\|_{h^s}}{2\varepsilon_0^{\#}}\right)^{r_*-2}\|u\|_{h^s}\le\frac{1}{3}\left(\frac{\|u\|_{h^s}}{2\varepsilon_0^{\#}}\right)^{r_*-2}\|u\|_{h^s}. \qquad (5.10)\end{aligned}$$

Therefore, using 5.9 and 5.10, we get

$$\begin{aligned}\|\tau_{\#}^{(1)}u-u\|_{h^s} &\le \frac{2}{3}\left(\frac{\|u\|_{h^s}}{2\varepsilon_0^{\#}}\right)^{p-2}\|u\|_{h^s}+\frac{1}{3}\Bigg(\underbrace{\frac{\|u\|_{h^s}}{2\varepsilon_0^{\#}}}_{\le 1}\Bigg)^{r_*-2}\|u\|_{h^s}\\ &\le \frac{2}{3}\left(\frac{\|u\|_{h^s}}{2\varepsilon_0^{\#}}\right)^{p-2}\|u\|_{h^s}+\frac{1}{3}\left(\frac{\|u\|_{h^s}}{2\varepsilon_0^{\#}}\right)^{p-2}\|u\|_{h^s} \qquad r_*-2>p-2\\ &=\left(\frac{\|u\|_{h^s}}{2\varepsilon_0^{\#}}\right)^{p-2}\|u\|_{h^s}.\end{aligned}$$

- Symplectic: Let $u\in B_{h^s}(0,2\varepsilon_0^{\#})$ and $v,w\in h^s(\mathbb{Z})$. Then,

$$\langle id\tau_{\#}^{(1)}(u)(v),d\tau_{\#}^{(1)}(u)(w)\rangle_{l^2}$$

$$
\begin{aligned}
&= \langle id(\tau^{(1)} \circ \phi_\chi^1)(u)(v), d(\tau^{(1)} \circ \phi_\chi^1)(u)(w)\rangle_{l^2} \\
&= \langle i((d\tau^{(1)}) \circ \phi_\chi^1) d\phi_\chi^1(u)(v), ((d\tau^{(1)}) \circ \phi_\chi^1) d\phi_\chi^1(u)(w)\rangle_{l^2} \\
&= \langle id\phi_\chi^1(u)(v), d\phi_\chi^1(u)(w)\rangle_{l^2} && \text{since } \tau^{(1)} \text{ is symplectic} \\
&= \langle iv, w\rangle_{l^2} && \text{since } \phi_\chi^1 \text{ is symplectic.}
\end{aligned}
$$

- Continuous extension: The existence of the continuous extension of $d\phi_\chi^t$ and $d\tau^{(1)}$ ensures the existence of such extension for $d\tau_\#^{(1)}$. We are left with proving that

$$
\|d\tau_\#^{(1)}(u)\|_{\mathscr{L}(h^{\sigma s})} \leq 2^{r_*+1-p}.
$$

For $u \in B_{h^s}(0, 2\varepsilon_0^\#)$, we have $\|u\|_{h^s} \leq 2\varepsilon_0^\# \leq \varepsilon_1$ and

$$
\begin{aligned}
\|d\tau_\#^{(1)}(u)\|_{\mathscr{L}(h^{\sigma s})} &= \|d(\tau^{(1)} \circ \phi_\chi^1)(u)\|_{\mathscr{L}(h^{\sigma s})} \\
&= \|((d\tau^{(1)}) \circ \phi_\chi^1)(u)) d\phi_\chi^1(u)\|_{\mathscr{L}(h^{\sigma s})} \\
&\leq \|(d\tau^{(1)}) \circ \phi_\chi^1(u)\|_{\mathscr{L}(h^{\sigma s})} \|d\phi_\chi^1(u)\|_{\mathscr{L}(h^{\sigma s})} \\
&\leq 2\|(d\tau^{(1)}) \circ \phi_\chi^1(u)\|_{\mathscr{L}(h^{\sigma s})} && \text{by 5.7} \\
&\leq 2^{1+r_*-p}. && \text{by induction hypothesis}
\end{aligned}
$$

Our goal now is to decompose $(Z_2 + P) \circ \tau_\#^{(1)}$. Let $u \in B_s(0, 2\varepsilon_0^\#)$. By definition, we have

$$
\begin{aligned}
(Z_2 + P) \circ \tau_\#^{(1)} &= (Z_2 + P) \circ \tau^{(1)} \circ \phi_\chi^1 \\
&= (Z_2 + Q^{(p)} + \cdots + Q^{(r-1)} + R) \circ \phi_\chi^1 && \text{by induction hypothesis} \\
&= Z_2 \circ \phi_\chi^1 + \sum_{j=p}^{r-1} Q^{(j)} \circ \phi_\chi^1 + R \circ \phi_\chi^1.
\end{aligned}
$$

Applying Lemma 5.1.1 with $t = 1$, we get

$$
(Z_2 + P) \circ \tau_\#^{(1)}
$$

$$
\begin{aligned}
&= Z_2 + \{\chi, Z_2\} + \sum_{k=2}^{m_{r_*}+1} \frac{1}{k!}\{\chi, Z_{2_{k-1}}\} + \int_0^1 \frac{1}{(m_{r_*}+1)!} Z_{2_{m_{r_*}+2}} \circ \phi_\chi^t (1-t)^{m_{r_*}+1}\, dt \\
&+ \sum_{j=p}^{r-1} \Big[Q^{(j)} + \sum_{k=1}^{m_j} \frac{1}{k!}\{\chi, Q^{(j)}_{k-1}\} + \int_0^1 \frac{1}{m_j!} Q^{(j)}_{m_j+1} \circ \phi_\chi^t (1-t)^{m_j}\, dt \Big] + R \circ \phi_\chi^1 \\
&= Z_2 + \{\chi, Z_2\} + \sum_{k=1}^{m_{r_*}} \frac{1}{(k+1)!} \mathrm{ad}_\chi^{k+1} Z_2 + \int_0^1 \frac{(1-t)^{m_{r_*}+1}}{(m_{r_*}+1)!} \mathrm{ad}_\chi^{m_{r_*}+2} Z_2 \circ \phi_\chi^t\, dt \\
&+ \sum_{j=p}^{r-1} \Big[Q^{(j)} + \sum_{k=1}^{m_j} \frac{1}{k!} \mathrm{ad}_\chi^k Q^{(j)} + \int_0^1 \frac{(1-t)^{m_j}}{m_j!} \mathrm{ad}_\chi^{m_j+1} Q^{(j)} \circ \phi_\chi^t\, dt \Big] + R \circ \phi_\chi^1
\end{aligned}
$$

where m_j is the smallest integer such that $j + m_j(r_* - 2) < r$. From equation 5.5 we have $\{\chi, Z_2\} + L = 0$, then

$$
\mathrm{ad}_\chi^{k+1} Z_2 = \underbrace{\{\chi, \{\chi, \cdots, \{\chi, Z_2\}\cdots\}\}}_{k+1 \text{ times}} = \{\chi, \{\chi, \cdots, -L \cdots\}\} = -\underbrace{\{\chi, \{\chi, \cdots, \{\chi, L\}\cdots\}\}}_{k \text{ times}} = -\mathrm{ad}_\chi^k L.
$$

Similarly, we get

$$
\mathrm{ad}_\chi^{m_{r_*}+2} Z_2 = -\mathrm{ad}_\chi^{m_{r_*}+1} L.
$$

So, we write

$$
\begin{aligned}
&(Z_2 + P) \circ \tau_\#^{(1)} \\
&= Z_2 + \{\chi, Z_2\} - \sum_{k=1}^{m_{r_*}} \frac{1}{(k+1)!} \mathrm{ad}_\chi^k L - \int_0^1 \frac{(1-t)^{m_{r_*}+1}}{(m_{r_*}+1)!} \mathrm{ad}_\chi^{m_{r_*}+1} L \circ \phi_\chi^t\, dt \\
&+ \sum_{j=p}^{r-1} \Big[Q^{(j)} + \sum_{k=1}^{m_j} \frac{1}{k!} \mathrm{ad}_\chi^k Q^{(j)} + \int_0^1 \frac{(1-t)^{m_j}}{m_j!} \mathrm{ad}_\chi^{m_j+1} Q^{(j)} \circ \phi_\chi^t\, dt \Big] + R \circ \phi_\chi^1 \\
&= Z_2 + \sum_{j=p}^{r_*-1} Q^{(j)} + Q^{(r_*)} + \{\chi, Z_2\} + \sum_{j=r_*+1}^{r-1} Q^{(j)} + \sum_{j=p}^{r-1} \sum_{k=1}^{m_j} \frac{1}{k!} \mathrm{ad}_\chi^k Q^{(j)} - \sum_{k=1}^{m_{r_*}} \frac{1}{(k+1)!} \mathrm{ad}_\chi^k L \\
&+ R \circ \phi_\chi^1 - \int_0^1 \frac{(1-t)^{m_{r_*}+1}}{(m_{r_*}+1)!} \mathrm{ad}_\chi^{m_{r_*}+1} L \circ \phi_\chi^t\, dt + \sum_{j=p}^{r-1} \int_0^1 \frac{(1-t)^{m_j}}{m_j!} \mathrm{ad}_\chi^{m_j+1} Q^{(j)} \circ \phi_\chi^t\, dt.
\end{aligned}
$$

Using the induction hypothesis and Proposition 4.2.1, it is easy to see that

- $Q^{(j)}$ is of order j,
- $\{\chi, Z_2\}$ is of order r_*,

- $\mathrm{ad}_\chi^k Q^{(j)}$ is of order $j + k(r_* - 2) > r_*$,
- $\mathrm{ad}_\chi^k L$ is of order $r_* + k(r_* - 2) > r_*$.

As a result, re-ordering the sums, it would make sense to set:

$$Q_\#^{(j)} = \begin{cases} Q^{(j)} & \text{for } j < r_*, \\ Q^{(r_*)} + \{\chi, Z_2\} & \text{for } j = r_*, \\ Q^{(j)} + \sum\limits_{\substack{j_*,k \\ j_*+k(r_*-2)=j}} \frac{1}{k!}\mathrm{ad}_\chi^k Q^{(j_*)} - \sum\limits_{\substack{k \\ r_*+k(r_*-2)=j}} \frac{1}{(k+1)!}\mathrm{ad}_\chi^k L & \text{for } r_* < j < r \end{cases}$$

and

$$R_\# = R \circ \phi_\chi^1 - \int_0^1 \left(\frac{(1-t)^{m_{r_*}+1}}{(m_{r_*}+1)!}\mathrm{ad}_\chi^{m_{r_*}+1} L \circ \phi_\chi^t - \sum_{j=p}^{r-1} \frac{(1-t)^{m_j}}{m_j!}\mathrm{ad}_\chi^{m_j+1} Q^{(j)} \circ \phi_\chi^t \right) dt.$$

Note that for $j < r_*$, $Q_\#^{(j)}$ commutes with the low super-actions by induction hypothesis, and $Q_\#^{(r_*)} = Q^{(r_*)} + \{\chi, Z_2\} = Q^{(r_*)} - L = U$ also commutes with J_n by Lemma 5.1.3. Hence,

$$|j| < r_* + 1 \quad \text{and} \quad \langle n \rangle \le N \implies \{J_n, Q_\#^{(j)}\} = 0.$$

Moreover, by construction we have $Q_\#^{(j)} \in \mathscr{H}^j$ for $p \le j < r$ satisfying the following needed bounds: For $j \le r_*$, we easily have by induction hypothesis

$$\|Q_\#^{(j)}\| \le \|Q^{(j)} \le C_{3,j} N^{b_{3,j}}.$$

For $j > r_*$, we notice that

$$\begin{aligned} \|\mathrm{ad}_\chi^k Q^{(j_*)}\| &\lesssim_{r_*,j_*,j} \|\chi\|^k \|Q^{(j_*)}\| && \text{by } 4.5 \\ &\le K_{r_*,j_*,j} (\gamma_{r_*}^{-1} C_{3,r_*} N^{\beta_{r_*}+b_{3,r_*}})^k (C_{3,j_*} N^{b_{3,j_*}}) && (5.11) \\ &\le K_{r_*,j_*,j} \gamma_{r_*}^{-1} (C_{3,r_*})^k C_{3,j_*} N^{k(\beta_{r_*}+b_{3,r_*})+b_{3,j_*}}. \end{aligned}$$

Since $\mathrm{ad}_\chi^k Q^{(r_*)}$ and $\mathrm{ad}_\chi^k L$ enjoy the same estimate, then we deduce that

$$\begin{aligned}\|Q_{\#}^{(j)}\| &\leq 2 \sum_{\substack{j_*,k \\ j_*+k(r_*-2)=j}} \frac{1}{k!}\|\mathrm{ad}_\chi^k Q^{(j_*)}\| \\ &\leq C_{3,j}^{\#} N^{b_{3,j}^{\#}}\end{aligned}$$

where $C_{3,j}^{\#} = 2 \sum_{\substack{j_*,k \\ j_*+k(r_*-2)=j}} \frac{1}{k!} K_{r_*,j_*,j}\gamma_{r_*}^{-1}(C_{3,r_*})^k C_{3,j_*}$ and $b_{3,j}^{\#} = \max_{j_*+k(r_*-2)=j}(k(\beta_{r_*}+b_{3,r_*})+b_{3,j_*})$.

Finally, we are left with proving the estimate of the remainder $R_{\#}$. For this fix $u \in B_{h^s}(0, 2\varepsilon_0^{\#})$ and we start by checking that $\nabla(R \circ \phi_\chi^1)(u) \in h^s$. By composition, we have

$$\nabla(R \circ \phi_\chi^1)(u) = (d\phi_\chi^1(u))^*(\nabla R) \circ \phi_\chi^1(u).$$

We know that $d\phi_\chi^1(u)$ admits a continuous extension from h^{-s} to h^{-s} with $\|d\phi_\chi^1(u)\|_{\mathscr{L}(h^{-s})} \leq 2$, then $(d\phi_\chi^1(u))^* : h^s \to h^s$ with $\|(d\phi_\chi^1(u))^*\|_{\mathscr{L}(h^s)} = \|d\phi_\chi^1(u)\|_{\mathscr{L}(h^{-s})} \leq 2$. Moreover, $(\nabla R) \circ \phi_\chi^1(u) \in h^s$ by induction hypothesis, so $(d\phi_\chi^1(u))^*(\nabla R) \circ \phi_\chi^1(u) \in h^s$ and we get

$$\begin{aligned}\|\nabla(R \circ \phi_\chi^1)(u)\|_{h^s} &= \|(d\phi_\chi^1(u))^*(\nabla R) \circ \phi_\chi^1(u)\|_{h^s} \\ &\leq \|(d\phi_\chi^1(u))^*\|_{\mathscr{L}(h^s)}\|(\nabla R) \circ \phi_\chi^1(u)\|_{h^s} \\ &\leq 2\|(\nabla R) \circ \phi_\chi^1(u)\|_{h^s} \\ &\leq 2C_2 N^{b_2}\|\phi_\chi^1(u)\|_{h^s}^{r-1} && \text{since } \|\phi_\chi^1(u)\|_{h^s} \leq 2\varepsilon_0 \\ &\leq 2 2^{r-1} C_2 N^{b_2}\|u\|_{h^s}^{r-1} && \text{since } \|\phi_\chi^1(u)\|_{h^s} \leq 2\|u\|_{h^s} \\ &= 2^r C_2 N^{b_2}\|u\|_{h^s}^{r-1}.\end{aligned}$$

Now for the terms of $R_{\#}$ inside the integral. Using same arguments as above, we have for $p \leq j \leq r-1$ and $t \in [0,1]$

$$\|\nabla(\mathrm{ad}_\chi^{m_j+1} Q^{(j)} \circ \phi_\chi^t)(u)\|_{h^s} \leq 2\|\nabla(\mathrm{ad}_\chi^{m_j+1} Q^{(j)}) \circ \phi_\chi^t(u)\|_{h^s}$$

where $\mathrm{ad}_\chi^{m_j+1}Q^{(j)}$ is a smooth function belonging to $\mathscr{H}^{r_j}$ where $r_j := j + (m_j+1)(r_* - 2)$. Thus, applying Proposition 4.1.1, we get

$$\begin{aligned}
&\|\nabla(\mathrm{ad}_\chi^{m_j+1}Q^{(j)})\circ\phi_\chi^t(u)\|_{h^s}\\
&\le M_{r_j}\|\phi_\chi^t(u)\|_{h^s}^{r_j-1}\|\mathrm{ad}_\chi^{m_j+1}Q^{(j)}\|\\
&\le M_{r_j}K_{r_*,j,r_j}(\gamma_{r_*}^{-1}C_{3,r_*}N^{\beta_{r_*}+b_{3,r_*}})^{m_j+1}(C_{3,j}N^{b_{3,j}})\|\phi_\chi^t(u)\|_{h^s}^{r_j-1} \qquad \text{by } 5.11\\
&= M_{r_j}K_{r_*,j,r_j}\gamma_{r_*}^{-m_j-1}(C_{3,r_*})^{m_j+1}C_{3,j}N^{(\beta_{r_*}+b_{3,r_*})(m_j+1)+b_{3,j}}\|\phi_\chi^t(u)\|_{h^s}^{r_j-1}
\end{aligned}$$

where M_{r_j} depends on (r_j, s). Consequently, we can write

$$\begin{aligned}
&\|\nabla(\mathrm{ad}_\chi^{m_j+1}Q^{(j)}\circ\phi_\chi^t)(u)\|_{h^s}\\
&\le 2M_{r_j}K_{r_*,j,r_j}\gamma_{r_*}^{-m_j-1}(C_{3,r_*})^{m_j+1}C_{3,j}N^{(\beta_{r_*}+b_{3,r_*})(m_j+1)+b_{3,j}}\|\phi_\chi^t(u)\|_{h^s}^{r_j-1}\\
&\le 2M_{r_j}K_{r_*,j,r_j}\gamma_{r_*}^{-m_j-1}(C_{3,r_*})^{m_j+1}C_{3,j}N^{(\beta_{r_*}+b_{3,r_*})(m_j+1)+b_{3,j}}2^{r_j-1}\|u\|_{h^s}^{r_j-1}\\
&= 2^{r_j}M_{r_j}K_{r_*,j,r_j}\gamma_{r_*}^{-m_j-1}(C_{3,r_*})^{m_j+1}C_{3,j}N^{(\beta_{r_*}+b_{3,r_*})(m_j+1)+b_{3,j}}\|u\|_{h^s}^{r_j-1}\frac{\|u\|_{h^s}^{r-1}}{\|u\|_{h^s}^{r-1}}\\
&= 2^{r_j}M_{r_j}K_{r_*,j,r_j}\gamma_{r_*}^{-m_j-1}(C_{3,r_*})^{m_j+1}C_{3,j}N^{(\beta_{r_*}+b_{3,r_*})(m_j+1)+b_{3,j}}\|u\|_{h^s}^{r_j-r}\|u\|_{h^s}^{r-1}.
\end{aligned}$$

Recall that $\|u\|_{h^s} \le 2\varepsilon_0^\# = \frac{2}{C_1^\# N_\#^{b_1}} \le \frac{2}{N_\#^{b_1}} \le 2 \implies \|u\|_{h^s}^{r_j-r} \le 2^{r_j-r}$. Therefore, we get

$$\begin{aligned}
&\|\nabla(\mathrm{ad}_\chi^{m_j+1}Q^{(j)}\circ\phi_\chi^t)(u)\|_{h^s}\\
&\le 2^{r_j}M_{r_j}K_{r_*,j,r_j}\gamma_{r_*}^{-m_j-1}(C_{3,r_*})^{m_j+1}C_{3,j}N^{(\beta_{r_*}+b_{3,r_*})(m_j+1)+b_{3,j}}2^{r_j-r}\|u\|_{h^s}^{r-1}\\
&= \underbrace{2^{2r_j-r}M_{r_j}K_{r_*,j,r_j}\gamma_{r_*}^{-m_j-1}(C_{3,r_*})^{m_j+1}C_{3,j}}_{:=A_j}N^{(\beta_{r_*}+b_{3,r_*})(m_j+1)+b_{3,j}}\|u\|_{h^s}^{r-1}.
\end{aligned} \tag{5.12}$$

Similarly, since $L \in \mathscr{H}^{r_*}$, we get

$$\begin{aligned}
&\|\nabla(\mathrm{ad}_\chi^{m_{r_*}+1}L\circ\phi_\chi^t)(u)\|_{h^s}\\
&\le \underbrace{2^{2r'-r}M_{r'}K_{r_*,r'}\gamma_{r_*}^{-m_{r_*}-1}(C_{3,r_*})^{m_{r_*}+1}C_{3,r_*}}_{:=A_{r'}}N^{(\beta_{r_*}+b_{3,r_*})(m_{r_*}+1)+b_{3,r_*}}\|u\|_{h^s}^{r-1}
\end{aligned} \tag{5.13}$$

with $r' := r_* + (m_{r_*} + 1)(r_* - 2)$. Next, using 5.12, 5.13 and the fact that the 2 functions are smooth, we are able to interchange the gradient and the integral by applying Leibniz integral rule. Hence, putting the results back together, we establish

$$\begin{aligned}
&\|\nabla R_{\#}(u)\|_{h^s} \\
&= \Big\| \nabla (R \circ \phi_\chi^1)(u) + \nabla \int_0^1 \Bigg(\frac{(1-t)^{m_{r_*}+1}}{(m_{r_*}+1)!} (\mathrm{ad}_\chi^{m_{r_*}+1} L \circ \phi_\chi^t)(u) \\
&- \sum_{j=p}^{r-1} \frac{(1-t)^{m_j}}{m_j!} (\mathrm{ad}_\chi^{m_j+1} Q^{(j)} \circ \phi_\chi^t)(u) \Bigg) dt \Big\| \\
&= \Big\| \nabla (R \circ \phi_\chi^1)(u) + \int_0^1 \Bigg(\frac{(1-t)^{m_{r_*}+1}}{(m_{r_*}+1)!} \nabla (\mathrm{ad}_\chi^{m_{r_*}+1} L \circ \phi_\chi^t)(u) \\
&- \sum_{j=p}^{r-1} \frac{(1-t)^{m_j}}{m_j!} \nabla (\mathrm{ad}_\chi^{m_j+1} Q^{(j)} \circ \phi_\chi^t)(u) \Bigg) dt \Big\| \\
&\le \|\nabla (R \circ \phi_\chi^1)(u)\|_{h^s} + \int_0^1 \Bigg(\Big\| \frac{1}{(m_{r_*}+1)!} \nabla (\mathrm{ad}_\chi^{m_{r_*}+1} L \circ \phi_\chi^t)(u) \Big\|_{h^s} \\
&+ \Big\| \sum_{j=p}^{r-1} \frac{1}{m_j!} \nabla (\mathrm{ad}_\chi^{m_j+1} Q^{(j)} \circ \phi_\chi^t)(u) \Big\|_{h^s} \Bigg) dt \\
&\le 2^r C_2 N^{b_2} \|u\|_{h^s}^{r-1} + \int_0^1 \Bigg(\frac{1}{(m_{r_*}+1)!} A_{r'} N^{(\beta_{r_*} + b_{3,r_*})(m_{r_*}+1) + b_{3,r_*}} \|u\|_{h^s}^{r-1} \\
&+ \sum_{j=p}^{r-1} \frac{1}{m_j!} A_j N^{(\beta_{r_*} + b_{3,r_*})(m_j+1) + b_{3,j}} \|u\|_{h^s}^{r-1} \Bigg) dt
\end{aligned}$$

Noticing that the bounds of $\|\nabla(\mathrm{ad}_\chi^{m_{r_*}+1} Q^{(r_*)} \circ \phi_\chi^t)(u)\|_{h^s}$ and $\|\nabla(\mathrm{ad}_\chi^{m_{r_*}+1} L \circ \phi_\chi^t)(u)\|_{h^s}$ coincide ($r_j = r'$ when $j = r_*$), we can see that

$$\frac{1}{(m_{r_*}+1)!} A_{r'} N^{(\beta_{r_*} + b_{3,r_*})(m_{r_*}+1) + b_{3,r_*}} \le \sum_{j=p}^{r-1} \frac{1}{m_j!} A_j N^{(\beta_{r_*} + b_{3,r_*})(m_j+1) + b_{3,j}}.$$

Finally, we get

$$\begin{aligned}
\|\nabla R_{\#}(u)\|_{h^s} &\le 2^r C_2 N^{b_2} \|u\|_{h^s}^{r-1} + 2 \sum_{j=p}^{r-1} \frac{1}{m_j!} A_j N^{(\beta_{r_*} + b_{3,r_*})(m_j+1) + b_{3,j}} \|u\|_{h^s}^{r-1} \\
&\le C_2^{\#} N^{b_2^{\#}} \|u\|_{h^s}^{r-1}
\end{aligned}$$

where we set $C_2^{\#} := 2^r C_2 + 2 \sum_{j=p}^{r-1} \frac{1}{m_j!} A_j$ and $b_2^{\#} := \max_{p \le j \le r-1} (b_2, (\beta_{r_*} + b_{3,r_*})(m_j + 1) + b_{3,j})$. $\square$

5.2 Dynamical Corollary

Lemma 5.2.1. *Consider* $u \in C_b^0(\mathbb{R}; h^s) \cap C^1(\mathbb{R}; h^{-s})$, $v(t) := \tau^{(0)}(u(t))$ *and* $L^{(0)}(u)$ *the continuous extension of* $d\tau^{(0)}(u)$ *to* $\mathscr{L}(h^{-s})$*. Then* v *is time differentiable in* h^{-s} *with*

$$\partial_t v(t) = L^{(0)}(u(t))(\partial_t u(t)).$$

Proof. Notice that $u \in C^1(\mathbb{R}; h^{-s})$ and $\tau^{(0)}$ is not defined a priori on h^{-s}. For this, we extend $d\tau^{(0)}(u) : h^s \to h^s$ to $L^{(0)}(u) : h^{-s} \to h^{-s}$ using Theorem 5.1.2. Now, we start by proving time differentiability of v. Fix $t \in \mathbb{R}$, and consider $h \in (-1,1) \setminus 0$. Then we write

$$\frac{v(t+h) - v(t)}{h} = \frac{\tau^{(0)}(u(t+h)) - \tau^{(0)}(u(t))}{h} = \int_0^1 L^{(0)}(u_{\nu,t,h})\, d\nu \left(\frac{u(t+h) - u(t)}{h}\right)$$

with $u_{\nu,t,h} = \nu u(t+h) + (1-\nu)u(t)$. So,

$$\begin{aligned}
&\left\|\frac{v(t+h) - v(t)}{h} - L^{(0)}(u(t))(\partial_t u(t))\right\|_{h^{-s}} \\
&= \left\|\int_0^1 L^{(0)}(u_{\nu,t,h})\, d\nu \left(\frac{u(t+h) - u(t)}{h}\right) - L^{(0)}(u(t))(\partial_t u(t))\right\|_{h^{-s}}.
\end{aligned}$$

Next, add and subtract $\int_0^1 L^{(0)}(u_{\nu,t,h})\, d\nu(\partial_t u(t))$, to get

$$\begin{aligned}
&\left\|\frac{v(t+h) - v(t)}{h} - L^{(0)}(u(t))(\partial_t u(t))\right\|_{h^{-s}} \\
&\le \left\|\int_0^1 L^{(0)}(u_{\nu,t,h})\, d\nu \left(\frac{u(t+h) - u(t)}{h} - \partial_t u(t)\right)\right\|_{h^{-s}} \\
&+ \left\|\left(\int_0^1 L^{(0)}(u_{\nu,t,h}) - L^{(0)}(u(t))\, d\nu\right)\partial_t u(t)\right\|_{h^{-s}} \\
&\le \left\|\frac{u(t+h) - u(t)}{h} - \partial_t u(t)\right\|_{h^{-s}} \int_0^1 \|L^{(0)}(u_{\nu,t,h})\|_{\mathscr{L}(h^{-s})}\, d\nu \\
&+ \|\partial_t u(t)\|_{h^{-s}} \int_0^1 \|L^{(0)}(u_{\nu,t,h}) - L^{(0)}(u(t))\|_{\mathscr{L}(h^{-s})}\, d\nu \\
&\le 2^{r-p}\left\|\frac{u(t+h) - u(t)}{h} - \partial_t u(t)\right\|_{h^{-s}} && \text{by 5.4} \\
&+ \|\partial_t u(t)\|_{h^{-s}} \int_0^1 \|L^{(0)}(u_{\nu,t,h}) - L^{(0)}(u(t))\|_{\mathscr{L}(h^{-s})}\, d\nu.
\end{aligned}$$

- For the first term, since $u \in C^1(\mathbb{R}; h^{-s})$, then $\lim\limits_{h\to 0} \frac{u(t+h)-u(t)}{h} = \partial_t u(t)$ in h^{-s}. Hence

$$\left\| \frac{u(t+h)-u(t)}{h} - \partial_t u(t) \right\|_{h^{-s}} \xrightarrow[h\to 0]{} 0.$$

- Now for the second term. We have that $u \in C_b^0(\mathbb{R}; h^s)$, so

$$u_{\nu,t,h} = \nu u(t+h) + (1-\nu)u(t) \xrightarrow[h\to 0]{} \nu u(t) + u(t) - \nu u(t) = u(t).$$

  Thus, since $L^{(0)}$ is continuous, we have

$$L^{(0)}(u_{\nu,t,h}) \xrightarrow[h\to 0]{} L^{(0)}(u(t)) \quad \text{in } \mathscr{L}(h^{-s}).$$

  Finally, by the dominated convergence theorem, we conclude that

$$\int_0^1 \|L^{(0)}(u_{\nu,t,h}) - L^{(0)}(u(t))\|_{\mathscr{L}(h^{-s})}\, d\nu \xrightarrow[h\to 0]{} 0.$$

Consequently, v is time derivable with $\partial_t v(t) = L^{(0)}(u(t))(\partial_t u(t))$ as needed. □

Lemma 5.2.2. *Given $u \in B_{h^s}(0, \varepsilon_0)$, we have that*

$$L^{(0)} i = i((d\tau^{(1)}) \circ \tau^{(0)})^*$$

where $((d\tau^{(1)}) \circ \tau^{(0)}(u))^* \in \mathscr{L}(h^{-s})$ denotes the adjoint of $(d\tau^{(1)}) \circ \tau^{(0)}(u)$.

Proof. First, I would like to mention that since $d\tau^{(1)} \circ \tau^{(0)}(u) : h^s \to h^s$ is a linear operator, then by Definition 2.2.5 its adjoint is defined as $(d\tau^{(1)} \circ \tau^{(0)}(u))^* : h^{-s} \to h^{-s}$ with h^{-s} being the dual space of h^s. Now, let $y, v, w \in h^s$. Because $\tau^{(1)}$ is symplectic (recall Definition 2.2.6), we have

$$\langle (d\tau^{(1)}(y))^* i (d\tau^{(1)})(y)(v), w\rangle = \langle i(d\tau^{(1)})(y)(v), d\tau^{(1)}(y)(w)\rangle = \langle iv, w\rangle$$

$$\implies (d\tau^{(1)}(y))^* i (d\tau^{(1)})(y) = i \tag{5.14}$$

which is true for all y, in particular for $y = \tau^{(0)}$. Now since the diagram in Theorem 5.1.2 commutes, we have $\tau^{(1)} \circ \tau^{(0)} = \mathrm{id}_{h^s}$ and consequently we obtain

$$d(\tau^{(1)} \circ \tau^{(0)}) = d(\mathrm{id}_{h^s}) = \mathrm{id}_{h^s} \implies ((d\tau^{(1)}) \circ \tau^{(0)}) d\tau^{(0)} = \mathrm{id}_{h^s}. \tag{5.15}$$

Finally, multiply by $d\tau^{(0)}$ both sides of equation 5.14 to get

$$\begin{aligned} i(d\tau^{(0)}) &= (d\tau^{(1)} \circ \tau^{(0)})^* i ((d\tau^{(1)}) \circ \tau^{(0)}) d\tau^{(0)} \\ &= (d\tau^{(1)} \circ \tau^{(0)})^* i (\mathrm{id}_{h^s}) && \text{by 5.15} \\ &= (d\tau^{(1)} \circ \tau^{(0)})^* i. \end{aligned}$$

Using the fact that $d\tau^{(0)} \in \mathscr{L}(h^s) \subset \mathscr{L}(h^s; h^{-s})$ and that h^s is dense in h^{-s}, we extend the last equation from $\mathscr{L}(h^s; h^{-s})$ to $\mathscr{L}(h^{-s}; h^{-s})$ and we write

$$L^{(0)} i = (d\tau^{(1)} \circ \tau^{(0)})^* i. \qquad \square$$

Lemma 5.2.3. *Let $u \in C^1(\mathbb{R}; h^{-s})$ and assume that the frequencies w_k are coercive (i.e. $|w_k| \to \infty$ as $|k| \to \infty$). Then J_n is a smooth function on h^{-s}.*

Proof. Fix n. Since the frequencies are coercive, then by definition

$$\forall M \geq 0,\ \exists N \text{ such that } \forall |k| \geq N,\ |w_k| > M.$$

In particular, for $M := |w_n| + 1$. Consequently, we can only have a finite number of $k's$ satisfying $w_k = w_n$. Moreover, $|u_k|^2$ is smooth on h^{-s}, so $J_n(u) = \sum_{\substack{k \\ w_k = w_n}} |u_k|^2$ is a finite sum of smooth functions on h^{-s}. Thus it is itself a smooth function on h^{-s}. $\square$

Theorem 5.2.4. *(Key Result) Let $s \geq 0$ and $r > p \geq 3$. With the same assumptions and* *notations as in Theorem 5.1.2, along with an arbitrary constant* $\varepsilon_1 > 0$, *if* $u \in C_b^0(\mathbb{R}; h^s) \cap C^1(\mathbb{R}; h^{-s})$ *is a global solution of*

$$i\partial_t u(t) = \nabla Z_2(u(t)) + \nabla P(u(t)) \tag{5.16}$$

where u *satisfies:* $\forall t \in \mathbb{R}, \|u(t)\|_{h^s} \leq \varepsilon_1$ *and the frequencies are coercive, then*

$$|t| < \varepsilon_1^{-(r-p)} \implies |J_n(u(t)) - J_n(u(0))| \leq M\langle n\rangle^{\flat}\varepsilon_1^p$$

with the constants M *and* $\flat$ *depending on* (r, s, γ) *and* (β, r) *respectively.*

Proof. First notice that if $\varepsilon_1 \geq \frac{1}{C\langle n\rangle^{\flat}}$ with C defined in Theorem 5.1.2, then

$$\begin{aligned}
|J_n(u(t)) - J_n(u(0))| &= \left| \sum_{w_k = w_n} |u_k(t)|^2 - \sum_{w_k = w_n} |u_k(0)|^2 \right| \\
&\leq \sum_{w_k = w_n} |u_k(t)|^2 + \sum_{w_k = w_n} |u_k(0)|^2 \\
&\leq \sum_{k \in \mathbb{Z}} |u_k(t)|^2 + \sum_{k \in \mathbb{Z}} |u_k(0)|^2 \\
&= \|u(t)\|_{l^2}^2 + \|u(0)\|_{l^2}^2 \\
&\leq \|u(t)\|_{h^s}^2 + \|u(0)\|_{h^s}^2 \\
&\leq 2\varepsilon_1^2.
\end{aligned}$$

Also, we have $2(C\langle n\rangle^b)^{p-2}\varepsilon_1^p = 2(C\langle n\rangle^b)^{p-2}\varepsilon_1^{p-2}\varepsilon_1^2 \geq 2\varepsilon_1^2(C\langle n\rangle^b)^{p-2}\left(\frac{1}{C\langle n\rangle^b}\right)^{p-2} = 2\varepsilon_1^2$.
Hence we get

$$|J_n(u(t)) - J_n(u(0))| \leq 2(C\langle n\rangle^b)^{p-2}\varepsilon_1^p,$$

and we obtain the estimate for $\flat := b(p-2)$ and $M := 2C^{p-2}$. For this, we focus on the case where $\varepsilon_1 < \frac{1}{C\langle n\rangle^b}$ and we set $N = \langle n\rangle$. By Theorem 5.1.2, we have

$$\forall t \in \mathbb{R}, \;\; \|u(t)\|_{h^s} \leq \varepsilon_1 < \frac{1}{C\langle n\rangle^b} = \frac{1}{CN^b} \leq \varepsilon_0.$$

So, looking at 5.3, it would make sense to consider $v(t) = \tau^{(0)}(u(t))$. Now, since u is a global solution of 5.16, then setting $H = Z_2 + P$ we get

$$\begin{aligned}
\partial_t v(t) &= L^{(0)}(u(t))(\partial_t u(t)) && \text{by Lemma 5.2.1}\\
&= L^{(0)}(u(t))(-i\nabla H(u(t)))\\
&= -L^{(0)}(u(t))(i\nabla H(u(t)))\\
&= -i((d\tau^{(1)}) \circ \tau^{(0)} u(t))^* (\nabla H(u(t))) && \text{by Lemma 5.2.2}\\
&= -i(d\tau^{(1)}(v(t)))^* (\nabla H(u(t))).
\end{aligned}$$

Remark that $\tau^{(1)} \circ \tau^{(0)}(u(t)) = u(t) \implies \tau^{(1)}(v(t)) = u(t)$. So replacing $u(t)$ and using composition we get

$$\begin{aligned}
\partial_t v(t) &= -i(d\tau^{(1)}(v(t)))^* ((\nabla H) \circ \tau^{(1)} v(t))\\
&= -i\nabla(H \circ \tau^{(1)})(v(t)).
\end{aligned}$$

Furthermore, using the decomposition of $H \circ \tau^{(1)}$, we have

$$\begin{aligned}
i\partial_t v(t) &= \nabla(Z_2 + Q + R)(v(t))\\
&= \nabla Z_2(v(t)) + \nabla Q(v(t)) + \nabla R(v(t)).
\end{aligned} \tag{5.17}$$

Next, we aim at estimating $\partial_t J_n(v(t))$ in order to apply the Mean Value Inequality. We already proved that J_n is a smooth function on h^{-s} (Lemma 5.2.3), this implies that its gradient is an element of $h^{-(-s)} = h^s$. Furthermore, since $\partial_t v \in h^{-s}$ their scalar product is defined.

$$\begin{aligned}
\partial_t J_n(v(t)) &= \frac{d}{dt}(J_n \circ v(t))\\
&= \langle \nabla J_n(v(t)), \partial_t v(t) \rangle_{l^2}\\
&= \langle i\nabla J_n(v(t)), i\partial_t v(t) \rangle_{l^2}
\end{aligned}$$

$$= \langle i\nabla J_n(v(t)), \nabla(Z_2 + Q + R)(v(t))\rangle_{l^2} \qquad \text{by (5.17)}$$
$$= \{J_n, Z_2 + Q + R\}(v(t))$$
$$= (\{J_n, Z_2\} + \{J_n, Q\} + \{J_n, R\})\,(v(t)).$$

From Theorem 5.1.2, we have that $\{J_n, Q\} = 0$. Moreover, we have

$$\begin{aligned}
\{J_n, Z_2\}(v(t)) &= 4 \sum_{w_k = w_n} \mathcal{R}[i\partial_{\overline{v_k}} J_n(v(t))\overline{\partial_{\overline{v_k}} Z_2(v(t))}] && \text{by definition} \\
&= 2 \sum_{w_k = w_n} \mathcal{R}[i v_k \overline{w_k v_k}] \\
&= 2 \sum_{w_k = w_n} w_k \mathcal{R}[i|v_k|^2] \\
&= 0.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
|\partial_t J_n(v(t))| &= |\{J_n, R\}(v(t))| \\
&= |\langle i\nabla J_n(v(t)), \nabla R(v(t))\rangle_{l^2}| \\
&\le \|\nabla J_n(v(t))\|_{l^2} \|\nabla R(v(t))\|_{l^2} && \text{by Cauchy Schwartz} \\
&\le \|\nabla J_n(v(t))\|_{h^s} \|\nabla R(v(t))\|_{h^s}. && (5.18)
\end{aligned}$$

Recalling that $\|u(t)\|_{h^s} \le \varepsilon_1$ and using the fact that

$$\begin{aligned}
\|v(t)\|_{h^s} &= \|\tau^{(0)}(u(t)) - u(t) + u(t)\|_{h^s} \\
&\le \|u(t)\|_{h^s} + \|\tau^{(0)}(u(t)) - u(t)\|_{h^s} \\
&\le \|u(t)\|_{h^s} + \left(\frac{\|u(t)\|_{h^s}}{\varepsilon_0}\right)^{p-2} \|u(t)\|_{h^s} && \text{by 5.3} \\
&\le 2\|u(t)\|_{h^s} && \text{since } \|u(t)\|_{h^s} < \varepsilon_0 \\
&\le 2\varepsilon_1,
\end{aligned}$$

we can see that

- $\|\nabla J_n(v(t))\|_{h^s}^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |(\nabla J_n(v(t)))_k|^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |2\partial_{\overline{v_k}} J_n(v(t))|^2 = 4 \sum_{w_k=w_n} \langle k\rangle^{2s} |v_k|^2$
$\le 4 \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |v_k|^2 = 4\|v(t)\|_{h^s}^2 \le 16\varepsilon_1^2.$
- $\|\nabla R(v(t))\|_{h^s} \le CN^b \|v(t)\|_{h^s}^{r-1} \le CN^b (2\varepsilon_1)^{r-1} = 2^{r-1} C\langle n\rangle^b \varepsilon_1^{r-1}.$

Putting the estimates back in equation 5.18, we obtain

$$\begin{aligned} |\partial_t J_n(v(t))| &\le 2^{r-1} 4C\langle n\rangle^b \varepsilon_1^r \\ &= 2^{r+1} C\langle n\rangle^b \varepsilon_1^r \\ &= M^{\#} \langle n\rangle^b \varepsilon_1^r \end{aligned}$$

where $M^{\#} := 2^{r+1}C$. Now, we apply Theorem 2.2.5 on $[0,t]$, and we deduce that

$$\begin{aligned} |t| < \varepsilon_1^{-(r-p)} \implies & \frac{|J_n(v(t)) - J_n(v(0))|}{|t-0|} \le |\partial_t J_n(v(t))| \\ \implies & |J_n(v(t)) - J_n(v(0))| \le |t||\partial_t J_n(v(t))| \le \varepsilon_1^{-(r-p)} M^{\#} \langle n\rangle^b \varepsilon_1^r = M^{\#} \langle n\rangle^b \varepsilon_1^p. \end{aligned}$$

In order to conclude, we need to get this result for $u(t)$. So, since for all t we have

$$\begin{aligned} \|u(t) - v(t)\|_{h^s} &\le \left(\frac{\|u(t)\|_{h^s}}{\varepsilon_0}\right)^{p-2} \|u(t)\|_{h^s} && \text{by 5.3} \\ &= \underbrace{\|u(t)\|_{h^s}^{p-1}}_{\le \varepsilon_1^{p-1}} \underbrace{\frac{1}{\varepsilon_0^{p-2}}}_{\le C^{p-2}\langle n\rangle^{b(p-2)}} \\ &\le C^{p-2} \langle n\rangle^{b(p-2)} \varepsilon_1^{p-1}, \end{aligned} \tag{5.19}$$

and since J_n is quadratic

$$\begin{aligned} |J_n(v(t)) - J_n(u(t))| &\le (\|v(t)\|_{l^2} + \|u(t)\|_{l^2}) \|u(t) - v(t)\|_{l^2} && \text{by Lemma 2.2.6} \\ &\le (\|v(t)\|_{h^s} + \|u(t)\|_{h^s}) \|u(t) - v(t)\|_{h^s} \end{aligned}$$

$$\begin{aligned}
&\leq (2\varepsilon_1 + \varepsilon_1) C^{p-2} \langle n \rangle^{b(p-2)} \varepsilon_1^{p-1} && \text{by } 5.19 \\
&= 3 C^{p-2} \langle n \rangle^{b(p-2)} \varepsilon_1^{p},
\end{aligned}$$

we finally deduce that

$$\begin{aligned}
|J_n(u(t)) - J_n(u(0))| &= |J_n(u(t)) - J_n(u(0)) - J_n(v(t)) + J_n(v(t)) - J_n(v(0)) + J_n(v(0))| \\
&\leq \underbrace{|J_n(v(t)) - J_n(v(0))|}_{\leq M^{\#} \langle n \rangle^{b} \varepsilon_1^{p}} + \underbrace{|J_n(v(t)) - J_n(u(t))|}_{\leq 3C^{p-2} \langle n \rangle^{b(p-2)} \varepsilon_1^{p}} + \underbrace{|J_n(v(0)) - J_n(u(0))|}_{\leq 3C^{p-2} \langle n \rangle^{b(p-2)} \varepsilon_1^{p}} \\
&\leq M \langle n \rangle^{\flat} \varepsilon_1^{p}
\end{aligned}$$

where we obtained the needed result for $M := M^{\#} + 6C^{p-2}$ and $\flat := b(p-2)$. $\square$

Chapter 6

Applications to the Beam Equation

In this chapter we are interested in applying the above results to the beam equation defined on the 1-dimensional torus in the introduction. Recall

$$\begin{cases} \partial_{tt}\psi + \partial_{xxxx}\psi + m\psi + p\psi^{p-1} = 0 \\ \psi(0,x) = \psi_0 \\ \partial_t \psi(0,x) = -\psi_1 \end{cases} \tag{6.1}$$

where $\psi = \psi(t,x) \in \mathbb{R}$ with $x \in \mathbb{T}$, the mass $m > 0$ is a parameter, $(\psi_0, \psi_1) \in H^{s+1}(\mathbb{T};\mathbb{R}) \times H^{s-1}(\mathbb{T};\mathbb{R})$ having small size ε and $p \geq 3$.

6.1 Hamiltonian Formalism

We start by identifying the Hamiltonian structure of the beam equation.
First, it is easy to check that for $s \in \mathbb{R}$ the following map is an isometry:

$$\mathcal{F} : H^s(\mathbb{T}) \to h^s(\mathbb{Z})$$

$$u \mapsto (\hat{u}(k))_{k\in\mathbb{Z}}.$$

Thus $H^s(\mathbb{T}) \equiv h^s(\mathbb{Z})$, and we write $u(x) = \sum\limits_{k\in\mathbb{Z}} u_k(t)e_k(x)$ where u_k denotes the Fourier coefficients of u. Now, we let

$$\Omega = (\partial_x^4 + m)^{1/2}$$

be a Fourier multiplier defined on $h^s(\mathbb{Z})$ by linearity as

$$\Omega u = \sum_{k\in\mathbb{Z}} u_k \Omega e_k,$$

$$\Omega e^{ikx} = w_k e^{ikx} \text{ with } w_k \coloneqq \sqrt{k^4 + m}.$$

Then the beam equation reads

$$\partial_{tt}\psi + \Omega^2\psi + p\psi^{p-1} = 0. \tag{6.2}$$

Moreover, we introduce a variable $-v = \partial_t\psi$ and we rewrite equation 6.2 as:

$$-v = \partial_t\psi \quad \text{and} \quad \partial_t v = \Omega^2\psi + p\psi^{p-1}.$$

As a result, it is easy to see that equation 6.2 can be written in the Hamiltonian form

$$\partial_t \begin{pmatrix} \psi \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H(\psi, v)$$

$$\text{with} \quad H(\psi, v) = \int_{\mathbb{T}} \frac{1}{2}v^2 + \frac{1}{2}(\Omega^2\psi)\psi + \psi^p \, dx. \tag{6.3}$$

Remark. As mentioned before, we will only apply the results to $s = 1$ which is possible since we are always in low regularity.

Proposition 6.1.1. *We pose a complex variable*

$$u(t,x) = \frac{1}{\sqrt{2}}\Big[\Omega^{1/2}\psi + i\Omega^{-1/2}v\Big]. \tag{6.4}$$

Then $(\psi, \partial_t\psi) \in C_b^0(\mathbb{R}; H^2 \times L^2)$ *is a solution of equation 6.1 if and only if* $u \in C_b^0(\mathbb{R}; h^1)$ *solves the equation*

$$\partial_t u = i\Omega u + \frac{ip}{\sqrt{2}}\Omega^{-1/2}\left(\Omega^{-1/2}\left(\frac{u+\overline{u}}{\sqrt{2}}\right)\right)^{p-1}. \tag{6.5}$$

Proof. Assume first that 6.1 is satisfied (similar calculations for the other direction). Notice that

$$\Omega^{-1/2}\left(\frac{u+\overline{u}}{\sqrt{2}}\right) = \Omega^{-1/2}\left(\frac{2}{2}\Omega^{1/2}\psi\right) = \psi.$$

So replacing u by its formula, we have

$$\begin{aligned}
i\Omega u + \frac{ip}{\sqrt{2}}\Omega^{-1/2}\left(\Omega^{-1/2}\left(\frac{u+\overline{u}}{\sqrt{2}}\right)\right)^{p-1} &= i\Omega\Big[\frac{1}{\sqrt{2}}\left(\Omega^{1/2}\psi + i\Omega^{-1/2}v\right)\Big] + \frac{i}{\sqrt{2}}\Omega^{-1/2}\underbrace{p\psi^{p-1}}_{-\partial_{tt}\psi-\Omega^2\psi} \\
&= \frac{i}{\sqrt{2}}\Omega^{3/2}\psi + \frac{\Omega^{1/2}}{\sqrt{2}}\partial_t\psi - \frac{i}{\sqrt{2}}\Omega^{-1/2}\partial_{tt}\psi - \frac{i}{\sqrt{2}}\Omega^{3/2}\psi \\
&= \frac{1}{\sqrt{2}}\Big[\Omega^{1/2}\partial_t\psi - i\Omega^{-1/2}\partial_{tt}\psi\Big] \\
&= \frac{1}{\sqrt{2}}\Big[\Omega^{1/2}\partial_t\psi + i\Omega^{-1/2}\partial_t v\Big] \\
&= \partial_t u.
\end{aligned}$$

□

Proposition 6.1.2. *For* $p \geq 3$ *the equation 6.5 can be written in the Hamiltonian form*

$$\partial_t u_k = i\frac{\partial H(u)}{\partial \overline{u_k}}$$

with

$$H(u) = Z_2 + H_p$$

where

$$Z_2(u) := \sum_{k\in\mathbb{Z}} w_k |u_k|^2 \qquad \textit{and} \qquad H_p(u) := \int_{\mathbb{T}} \left(\Omega^{-1/2}\left(\frac{u+\overline{u}}{\sqrt{2}}\right)\right)^p dx.$$

Proof. Passing to Fourier with $u(x) = \sum_{k\in\mathbb{Z}} u_k(t)e_k(x)$, we have that the beam equation is equivalent to

$$\partial_t u_k = i\frac{\partial H(u)}{\partial \overline{u_k}},$$

with

$$H(u) = \int_{\mathbb{T}} \overline{u}\Omega u \, dx + \int_{\mathbb{T}} \left(\Omega^{-1/2}\left(\frac{u+\overline{u}}{\sqrt{2}}\right)\right)^p dx.$$

To see this, we apply a change of variable to equation 6.3 to get

$$H(u) = \int_{\mathbb{T}} \frac{1}{2}\left(\Omega^{1/2}\left(\frac{u-\overline{u}}{i\sqrt{2}}\right)\right)^2 + \frac{1}{2}\left(\Omega^{3/2}\left(\frac{u+\overline{u}}{\sqrt{2}}\right)\right)\left(\Omega^{-1/2}\left(\frac{u+\overline{u}}{\sqrt{2}}\right)\right) + \left(\Omega^{-1/2}\left(\frac{u+\overline{u}}{\sqrt{2}}\right)\right)^p dx.$$

Then, expanding and using the fact that Ω is self-adjoint, we get the equation in terms on u and $\overline{u}$. Now, we write

$$\begin{aligned}
\int_{\mathbb{T}} \overline{u}\Omega u \, dx &= \int_{\mathbb{T}} (\sum_{k\in\mathbb{Z}} \overline{u_k} e_{-k})(\Omega \sum_{k\in\mathbb{Z}} u_k e_k) \, dx \\
&= \int_{\mathbb{T}} (\sum_{k\in\mathbb{Z}} \overline{u_k} e_{-k})(\sum_{k\in\mathbb{Z}} u_k \Omega e_k) \, dx && \text{by linearity of } \Omega \\
&= \sum_{k\in\mathbb{Z}} u_k w_k \int_{\mathbb{T}} \sum_{k\in\mathbb{Z}} \overline{u_k} e_{-k} e_k \, dx && \text{since } \Omega e_k = w_k e_k \\
&= \sum_{k\in\mathbb{Z}} u_k w_k \sum_{k\in\mathbb{Z}} \overline{u_k} \underbrace{\int_{\mathbb{T}} e_{-k} e_k \, dx}_{=1} \\
&= \sum_{k\in\mathbb{Z}} w_k |u_k|^2.
\end{aligned}$$

After this, define

$$H_p : u \to \int_{\mathbb{T}} \left(\Omega^{-1/2}\left(\frac{u+\overline{u}}{\sqrt{2}}\right)\right)^p dx,$$

Hence, we get the needed result. □

Our goal now is to identify H_p with a formal Hamiltonian:

Proposition 6.1.3. *Let H_p be defined as in Proposition 6.1.2. Then $H_p \in \mathscr{H}^p$.*

Proof. We have

$$\begin{aligned}
H_p(u) &= \int_{\mathbb{T}} \left(\Omega^{-1/2} \left(\frac{u + \overline{u}}{\sqrt{2}} \right) \right)^p dx \\
&= \int_{\mathbb{T}} \left(\Omega^{-1/2} \left(\frac{\sum_{k \in \mathbb{Z}} u_k e_k + \sum_{k \in \mathbb{Z}} \overline{u_k} e_{-k}}{\sqrt{2}} \right) \right)^p dx \\
&= \int_{\mathbb{T}} \left(\frac{\sum_{k \in \mathbb{Z}} u_k w_k^{-1/2} e_k + \sum_{k \in \mathbb{Z}} \overline{u_k} w_k^{-1/2} e_{-k}}{\sqrt{2}} \right)^p dx \\
&= \frac{1}{2^{p/2}} \int_{\mathbb{T}} \left(\sum_{k \in \mathbb{Z}} \frac{u_k e_k + \overline{u_k} e_{-k}}{w_k^{1/2}} \right)^p dx \\
&= \frac{1}{2^{p/2}} \int_{\mathbb{T}} \left(\sum_{\substack{k \in \mathbb{Z} \\ \sigma \in \{-1,+1\}}} \frac{u_k^\sigma e_k^\sigma}{w_k^{1/2}} \right)^p dx \\
&= \frac{1}{2^{p/2}} \int_{\mathbb{T}} \left(\sum_{\substack{k_1 \in \mathbb{Z} \\ \sigma_1 \in \{-1,+1\}}} \frac{u_{k_1}^{\sigma_1} e_{k_1}^{\sigma_1}}{w_{k_1}^{1/2}} \right) \cdots \left(\sum_{\substack{k_p \in \mathbb{Z} \\ \sigma_p \in \{-1,+1\}}} \frac{u_{k_p}^{\sigma_p} e_{k_p}^{\sigma_p}}{w_{k_p}^{1/2}} \right) dx \\
&= \frac{1}{2^{p/2}} \sum_{\substack{k_1, \cdots, k_p \in \mathbb{Z} \\ \sigma_1, \cdots, \sigma_p \in \{-1,+1\}}} \frac{1}{w_{k_1}^{1/2} \cdots w_{k_p}^{1/2}} u_{k_1}^{\sigma_1} \cdots u_{k_p}^{\sigma_p} \int_{\mathbb{T}} e_{k_1}^{\sigma_1} \cdots e_{k_p}^{\sigma_p} \, dx
\end{aligned}$$

where

$$\int_{\mathbb{T}} e_{k_1}^{\sigma_1} \cdots e_{k_p}^{\sigma_p} \, dx = \begin{cases} 1 & \text{if } \sigma_1 k_1 + \cdots + \sigma_p k_p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently we get that

$$\begin{aligned}
H_p(u) &= \frac{1}{2^{p/2}} \sum_{\substack{k_1, \cdots, k_p \in \mathbb{Z} \\ \sigma_1, \cdots, \sigma_p \in \{-1,+1\} \\ \sigma_1 k_1 + \cdots + \sigma_p k_p = 0}} \frac{1}{w_{k_1}^{1/2} \cdots w_{k_p}^{1/2}} u_{k_1}^{\sigma_1} \cdots u_{k_p}^{\sigma_p} \qquad (6.6) \\
&= \frac{1}{2^{p/2}} \sum_{\substack{k \in \mathbb{Z}^p \\ \sigma \in \{-1,+1\}^p \\ \sigma_1 k_1 + \cdots + \sigma_p k_p = 0}} (H_p)_k^\sigma u_{k_1}^{\sigma_1} \cdots u_{k_p}^{\sigma_p}
\end{aligned}$$

with $(H_p)_k^\sigma = \frac{1}{2^{p/2}} w_{k_1}^{-1/2} \cdots w_{k_p}^{-1/2}$, satisfying the bound:

$$\begin{aligned}|(H_p)_k^\sigma| &= \frac{1}{2^{p/2}} \left| w_{k_1}^{-1/2} \cdots w_{k_p}^{-1/2} \right| \\ &= \frac{1}{2^{p/2}} (k_1^4 + m)^{-1/4} \cdots (k_p^4 + m)^{-1/4}.\end{aligned}$$

Furthermore, using direct calculation, we prove that

$$\frac{(k_j^2 + m)^{1/2}}{(k_j^4 + m)^{1/4}} = O(1) \quad \text{and} \quad \frac{(k_j^2 + m)^{-1/2}}{(k_j^2 + 1)^{-1/2}} = O(1)$$

. Hence, we conclude that there exists a constant C_m such that

$$\begin{aligned}|(H_p)_k^\sigma| &\leq C_m \frac{2^{p/2}}{2^{p/2}} (k_1^2 + 1)^{-1/2} \cdots (k_p^2 + 1)^{-1/2} \\ &= C_m \prod_{j=1}^{p} \langle k_j \rangle^{-1}\end{aligned}$$

and $\|H_p\| \lesssim_m 1$. □

6.2 Strong Non-Resonance Condition

At this point, we will check that the frequencies of the defined beam equation satisfy the needed non-resonance condition.

Lemma 6.2.1. *For $k \in \mathbb{Z}$ and $m \geq 0$, the frequencies $w_k = \sqrt{m + k^4}$ are strongly non-resonant.*

Proof. Consider the frequencies $w'_n = \sqrt{m + n^2}$ of the Klein–Gordan equation. In his paper, [Bambusi, 2003] proves that these frequencies satisfy the assumption 3.1. Hence, the result remains true for subsequences $(n_k)_{k \in \mathbb{Z}}$. In particular, for $n_k := k^2$, we get that $w'_{n_k} = \sqrt{m + k^4} = w_k$ satisfy the assumption. It remains to prove that the frequencies accumulate polynomially fast on $\mathbb{Z}$. We have

$$\left|w_k - k^2\right| = \left|\underbrace{\sqrt{m+k^4} - k^2}_{\geq 0}\right| = (\sqrt{m+k^4} - k^2) \times \frac{\sqrt{m+k^4}+k^2}{\sqrt{m+k^4}+k^2} = \frac{m}{\sqrt{m+k^4}+k^2} \leq \frac{m}{\sqrt{k^4}+k^2} = \frac{m}{2k^2}.$$

Thus, applying Proposition 3.1.1, we obtain the desired result. □

6.3 Gobal Well-posedness

Our main goal in this section is to prove the global well-posedness of solutions to $\partial_t u_k = i\frac{\partial H(u)}{\partial \overline{u_k}}$ with initial data $u(0,x) = u_0$. For this favor, we introduce

$$\eta_k(t) = e^{-iw_k t} u_k(t).$$

Lemma 6.3.1. *We have that*

$$u \in C_b^0(\mathbb{R}; h^1) \text{ is a solution of } \begin{cases} \partial_t u_k = i\frac{\partial H(u)}{\partial \overline{u_k}} \\ u(0,x) = u_0 \end{cases} \iff \eta \in C_b^0(\mathbb{R}; h^1) \text{ is a solution of } \begin{cases} \dot{\eta} = X(t,\eta) \\ \eta(0) = \eta_0 \end{cases}$$

with $X(t,\eta)$ is to be determined.

Proof. Since $\eta_k(t) = e^{-iw_k t} u_k(t)$ and u is a solution of $\partial_t u_k = i\frac{\partial H(u)}{\partial \overline{u_k}}$, then

$$\begin{aligned}
\dot{\eta}_k(t) &= -iw_k e^{-iw_k t} u_k(t) + e^{-iw_k t} \partial_t u_k(t) \\
&= -iw_k e^{-iw_k t} u_k(t) + e^{-iw_k t} i \frac{\partial H(u)}{\partial \overline{u_k}} \\
&= -iw_k e^{-iw_k t} u_k(t) + ie^{-iw_k t} \partial_{\overline{u_k}} \Big[\sum_{n\in\mathbb{Z}} w_n |u_n|^2 + H_p \Big] && \text{by Proposition 6.2.2} \\
&= -iw_k e^{-iw_k t} u_k(t) + ie^{-iw_k t} \partial_{\overline{u_k}} \Big(\sum_{n\in\mathbb{Z}} w_n |u_n|^2 \Big)
\end{aligned}$$

$$+ ie^{-iw_k t}\partial_{\overline{u_k}}\left(\frac{1}{2^{p/2}}\sum_{\substack{n_1,\cdots,n_p\in\mathbb{Z}\\ \sigma_1,\cdots,\sigma_p\in\{-1,+1\}\\ \sigma_1 n_1+\cdots+\sigma_p n_p=0}}\frac{1}{w_{n_1}^{1/2}\cdots w_{n_p}^{1/2}}u_{n_1}^{\sigma_1}\cdots u_{n_p}^{\sigma_p}\right) \qquad \text{by 6.6}$$

$$= -iw_k e^{-iw_k t}u_k(t) + iw_k e^{-iw_k t}u_k(t)$$

$$+ \frac{i}{2^{p/2}}e^{-iw_k t}\sum_{\substack{n_1,\cdots,n_p\in\mathbb{Z}\\ \sigma_1,\cdots,\sigma_p\in\{-1,+1\}\\ \sigma_1 n_1+\cdots+\sigma_p n_p=0}}\frac{1}{w_{n_1}^{1/2}\cdots w_{n_p}^{1/2}}\partial_{\overline{u_k}}(u_{n_1}^{\sigma_1}\cdots u_{n_p}^{\sigma_p})$$

$$= \frac{ip}{2^{p/2}}e^{-iw_k t}\sum_{\substack{n\in\mathbb{Z}^{p-1}\\ \sigma\in\{-1,+1\}^{p-1}\\ \sigma_1 n_1+\cdots+\sigma_{p-1}n_{p-1}=k}}\frac{1}{w_{n_1}^{1/2}\cdots w_{n_{p-1}}^{1/2}w_k^{1/2}}u_{n_1}^{\sigma_1}\cdots u_{n_{p-1}}^{\sigma_{p-1}}.$$

Finally, replace $\eta_k(t)e^{iw_k t} = u_k(t)$ in order to obtain

$$\dot{\eta}(t) = \underbrace{\frac{ip}{2^{p/2}}\frac{e^{-iw_k t}}{w_k^{1/2}}\sum_{\substack{n\in\mathbb{Z}^{r-1}\\ \sigma\in\{-1,+1\}^{r-1}\\ \sigma_1 n_1+\cdots+\sigma_{p-1}n_{p-1}=k}}\frac{1}{w_{n_1}^{1/2}\cdots w_{n_{p-1}}^{1/2}}(\eta_{n_1}e^{iw_{n_1}t})^{\sigma_1}\cdots(\eta_{n_{p-1}}e^{iw_{n_{p-1}}t})^{\sigma_{p-1}}}_{X(t,\eta)}. \qquad \square$$

Proposition 6.3.1. *For $t\in\mathbb{R}$, $X(t,.) : h^1\to h^1$ is locally Lipschitz with respect to η.*

Proof. Let $R>0$ and assume that $\eta,\eta'\in B_{h^1}(0,R)$. We have to prove that for all t, there exists $C_R>0$ such that $\|X(t,\eta)-X(t,\eta')\|_{h^1}\le C_R\|\eta-\eta'\|_{h^1}$. For this, we write

$$\|X(t,\eta)-X(t,\eta')\|_{h^1}^2$$

$$= \sum_{k\in\mathbb{Z}}\langle k\rangle^2\left|\frac{e^{-iw_k t}}{w_k^{1/2}}\frac{p}{2^{p/2}}\sum_{\substack{n\in\mathbb{Z}^{p-1}\\ \sigma\in\{-1,+1\}^{p-1}\\ \sigma_1 n_1+\cdots+\sigma_{p-1}n_{p-1}=k}}\frac{1}{w_{n_1}^{1/2}\cdots w_{n_{p-1}}^{1/2}}(\eta_{n_1}e^{iw_{n_1}t})^{\sigma_1}\cdots(\eta_{n_{p-1}}e^{iw_{n_{p-1}}t})^{\sigma_{r-1}}\right.$$

$$\left.-\frac{e^{-iw_k t}}{w_k^{1/2}}\frac{p}{2^{p/2}}\sum_{\substack{n\in\mathbb{Z}^{p-1}\\ \sigma\in\{-1,+1\}^{p-1}\\ \sigma_1 n_1+\cdots+\sigma_{p-1}n_{p-1}=k}}\frac{1}{w_{n_1}^{1/2}\cdots w_{n_{p-1}}^{1/2}}(\eta'_{n_1}e^{iw_{n_1}t})^{\sigma_1}\cdots(\eta'_{n_{p-1}}e^{iw_{n_{p-1}}t})^{\sigma_{p-1}}\right|^2$$

$$\le \sum_{k\in\mathbb{Z}}\langle k\rangle^2\left|\frac{p}{2^{p/2}}\sum_{\substack{n\in\mathbb{Z}^{p-1}\\ \sigma\in\{-1,+1\}^{p-1}\\ \sigma_1 n_1+\cdots+\sigma_{p-1}n_{p-1}=k}}(\eta_{n_1}^{\sigma_1}\cdots\eta_{n_{p-1}}^{\sigma_{p-1}}-\eta'^{\sigma_1}_{n_1}\cdots\eta'^{\sigma_{p-1}}_{n_{p-1}})e^{-iw_k t}e^{i(w_{n_1}\sigma_1+\cdots w_{n_{p-1}}\sigma_{p-1})t}\right|^2$$

$$\leq \frac{p^2}{2^p}\sum_{k\in\mathbb{Z}}\langle k\rangle^2\left|\sum_{\substack{n\in\mathbb{Z}^{p-1}\\ \sigma\in\{-1,+1\}^{p-1}\\ \sigma_1 n_1+\cdots+\sigma_{p-1}n_{p-1}=k}}(\eta_{n_1}^{\sigma_1}\cdots\eta_{n_{p-1}}^{\sigma_{p-1}}-\eta'^{\sigma_1}_{n_1}\cdots\eta'^{\sigma_{p-1}}_{n_{p-1}})\right|^2$$

Using the fact that $\sum_{j_1+j_2=k}a(j_1)b(j_2)=\sum_{j_3\in\mathbb{Z}}a(j_3)b(k-j_3)=a*b(k)$, we write

$$\begin{aligned}
\|X(t,\eta)-X(t,\eta')\|_{h^1}^2 &\leq \frac{p^2}{2^p}\|\underbrace{\eta*\cdots*\eta}_{p-1\text{ terms}}-\underbrace{\eta'*\cdots*\eta'}_{p-1\text{ terms}}\|_{h^1}^2\\
&= \frac{p^2}{2^p}\|(\eta-\eta')*\eta*\cdots*\eta+\eta'*(\eta-\eta')*\eta*\cdots*\eta+\cdots+\eta'*\eta'*\cdots*(\eta-\eta')\|_{h^1}\\
&\leq \frac{p^2}{2^p}\|\eta-\eta'\|_{h^1}\underbrace{(\|\eta\|_{h^1}^{p-2}+\|\eta'\|_{h^1}\|\eta\|_{h^1}^{p-3}+\cdots+\|\eta'\|_{h^1}^{p-3}\|\eta\|_{h^1}+\|\eta'\|_{h^1}^{p-2})}_{\|\eta\|_{h^1},\|\eta'\|_{h^1}\leq R}\\
&\leq C_R\|\eta-\eta'\|_{h^1}. \qquad (6.7)
\end{aligned}$$

Hence, X is locally Lipschitz and continuous with respect to the second variable. □

Proposition 6.3.2. *Fix $T>0$ small enough depending only on $\|\eta_0\|_{h^1}$. Then there exists a unique local solution*

$$u\in C_b^0([0,T];h^1)\cap C^1([0,T];h^{-1})\quad \text{to the equation}\quad \begin{cases}\partial_t u_k = i\frac{\partial H(u)}{\partial \overline{u_k}}\\ u(0,x)=u_0.\end{cases}$$

Proof. Using Lemma 6.3.1, it is sufficient to prove that $\dot\eta = X(t,\eta)$ admits a local solution on $C_b^0([0,T];h^1)$. It is not difficult to see that η must satisfy

$$\eta(t)=\eta_0+\int_0^t X(s,\eta(s))\,ds \qquad \forall t\in[0,T]. \qquad (6.8)$$

In other words, we define

$$B_T := \{\Phi\in C_b^0([0,T];h^1) : \|\Phi\|_{C^0(h^1)}\leq K\}\subset C_b^0([0,T];h^1)$$

with K a constant to be determined, and we seek a Φ being the fixed point of the mapping

$$\begin{aligned}\Gamma : B_T &\to B_T \\ \Phi(t) &\mapsto \Gamma(\Phi)(t) = \eta_0 + \int_0^t X(s, \Phi(s))\, ds.\end{aligned}$$

In order to apply the Banach Fixed Point Theorem, we will prove that $\Gamma(\Phi)(t)$ is a contraction on B_T. First, let's check that Γ is well-defined. Given $\Phi(t) \in B_T$, we have

$$\begin{aligned}\|X(t,\Phi(t))\|_{h^1} &\leq \|X(t,\Phi(t)) - X(t,0)\|_{h^1} + \|X(t,0)\|_{h^1} \\ &\leq C_K \|\Phi(t)\|_{h^1} + \|X(t,0)\|_{h^1} && \text{by 6.7} \\ &\leq C_K \|\Phi(t)\|_{C^0(h^1)} + \|X(t,0)\|_{h^1} \\ &\leq C_K K + \|X(t,0)\|_{h^1} \\ &= C_K K && \text{by definition of } X(t,\eta).\end{aligned}$$

Thus, we can write

$$\begin{aligned}\|\Gamma(\Phi)(t)\|_{C^0(h^1)} &\leq \|\eta_0\|_{h^1} + \sup_{t\in[0,T]} \left\| \int_0^t X(s,\Phi(s))\, ds \right\|_{h^1} \\ &\leq \|\eta_0\|_{h^1} + \sup_{t\in[0,T]} \int_0^t \left\| X(s,\Phi(s)) \right\|_{h^1} ds \\ &\leq \|\eta_0\|_{h^1} + \sup_{t\in[0,T]} \int_0^t C_K K\, ds \\ &= \|\eta_0\|_{h^1} + \sup_{t\in[0,T]} C_K K t \\ &\leq \|\eta_0\|_{h^1} + C_K K T \\ &\leq K\end{aligned}$$

where we chose $K = 2\|\eta_0\|_{h^1}$ and $T \leq \frac{1}{2C_K}$. So, $\Gamma(\Phi) \in B_T$. Now we prove the contraction estimate. Let $\Phi, \Phi' \in B_T$, then

$$\|\Gamma(\Phi)(t) - \Gamma(\Phi')(t)\|_{C^0(h^1)} = \sup_{t\in[0,T]} \|\Gamma(\Phi)(t) - \Gamma(\Phi')(t)\|_{h^1}$$

$$
\begin{aligned}
&= \sup_{t\in[0,T]} \left\| \int_0^t X(s,\Phi(s))\,ds - \int_0^t X(s,\Phi'(s))\,ds \right\|_{h^1} \\
&\le \sup_{t\in[0,T]} \int_0^t \|X(s,\Phi(s)) - X(s,\Phi'(s))\|_{h^1}\,ds \\
&\le \sup_{t\in[0,T]} \int_0^t C_K \|\Phi(s) - \Phi'(s)\|_{h^1}\,ds \qquad \text{by } 6.7 \\
&\le TC_K \sup_{s\in[0,T]} \|\Phi(s) - \Phi'(s)\|_{h^1} \\
&= TC_K \|\Phi(s) - \Phi'(s)\|_{C^0(h^1)} \\
&\le \frac{1}{2} \|\Phi(s) - \Phi'(s)\|_{C^0(h^1)}.
\end{aligned}
$$

where $\frac{1}{2} < 1$ leading to a contraction. Next, since B_T is a closed subset of a Banach space then it is Banach. Applying Theorem 2.2.7, $\Gamma(\Phi)$ admits a unique fixed point $\Phi \in B_T$ and consequently in $C_b^0([0,T];h^1)$. Hence, due to uniqueness, we get $\eta \in C_b^0([0,T];h^1)$ satisfying $\begin{cases} \dot{\eta} = X(t,\eta) \\ \eta(0) = \eta_0 \end{cases}$. To conclude, it is easy to check that $\eta \in C^1([0,T];h^{-1})$ using 6.8. Indeed, using the fundamental theorem of calculus, we get that η is differentiable with a continuous derivative. □

In what follows, we give two results needed to conclude the global existence. We start with the ellipticity condition.

Lemma 6.3.2. *For $m > 0$, there exists $\varepsilon_m > 0$ and $\Lambda_m > 1$ such that for $\psi \in H^2(\mathbb{T})$ and $v \in L^2(\mathbb{T})$ satisfying $\|\psi\|_{H^2} \le 1$ and $\|\psi\|_{H^2} + \|v\|_{L^2} \le \varepsilon_m$, we have*

$$\Lambda_m^{-1}(\|\psi\|_{H^2} + \|v\|_{L^2})^2 \le H(\psi,v) \le \Lambda_m(\|\psi\|_{H^2} + \|v\|_{L^2})^2.$$

Proof. Using Sobolev inequality, there exists a universal constant C such that $\|\psi\|_{L^\infty} \le C\|\psi\|_{H^2}$. So, we get

$$\int_{\mathbb{T}} |\psi|^p\,dx \le 2\pi\|\psi\|_{L^\infty}^p \le 2\pi C^p\|\psi\|_{H^2}^p \le 2\pi C^p\|\psi\|_{H^2}^2 \le 2\pi C^p(\|\psi\|_{H^2}^2 + \|v\|_{L^2}^2)$$

$$\leq 2\pi C^p (\|\psi\|_{H^2} + \|v\|_{L^2})^2.$$

Consequently, by 6.3 we can see that

$$\begin{aligned}
H(\psi, v) &= \int_{\mathbb{T}} \frac{1}{2} v^2 + \frac{1}{2} (\Omega^2 \psi) \psi + \psi^p \, dx \\
&= \int_{\mathbb{T}} \frac{1}{2} v^2 + \frac{1}{2} (\partial_{xxxx} \psi) \psi + \frac{m}{2} \psi^2 + \psi^p \, dx \\
&= \int_{\mathbb{T}} \frac{1}{2} v^2 + \frac{1}{2} (\partial_{xx} \psi)^2 + \frac{m}{2} \psi^2 \, dx + \int_{\mathbb{T}} \psi^p \, dx \\
&\leq \int_{\mathbb{T}} \frac{1}{2} v^2 + \max(\frac{1}{2}, \frac{m}{2}) \Big((\partial_{xx} \psi)^2 + \psi^2 \Big) \, dx + \int_{\mathbb{T}} \psi^p \, dx \\
&\leq \max(\frac{1}{2}, \frac{m}{2}) (\|v\|_{L^2} + \|\psi\|_{H^2})^2 + 2\pi C^p (\|\psi\|_{H^2} + \|v\|_{L^2})^2 \\
&= \left(\max(\frac{1}{2}, \frac{m}{2}) + 2\pi C^p \right) (\|\psi\|_{H^2} + \|v\|_{L^2})^2.
\end{aligned}$$

On the other hand, we similiarly have

$$\begin{aligned}
H(\psi, v) &= \int_{\mathbb{T}} \frac{1}{2} v^2 + \frac{1}{2} (\partial_{xx} \psi)^2 + \frac{m}{2} \psi^2 \, dx + \int_{\mathbb{T}} \psi^p \, dx \\
&\geq \int_{\mathbb{T}} \frac{1}{2} v^2 + \min(\frac{1}{2}, \frac{m}{2}) \Big((\partial_{xx} \psi)^2 + \psi^2 \Big) \, dx + \int_{\mathbb{T}} \psi^p \, dx \\
&\geq \min(\frac{1}{2}, \frac{m}{2}) (\|v\|_{L^2} + \|\psi\|_{H^2})^2 - 2\pi C^p (\|\psi\|_{H^2} + \|v\|_{L^2})^2 \\
&= \left(\min(\frac{1}{2}, \frac{m}{2}) - 2\pi C^p \right) (\|\psi\|_{H^2} + \|v\|_{L^2})^2.
\end{aligned}$$

Thus, it would be sufficient to choose a $\Lambda_m > 1$ satisfying the needed result. □

Now, we prove the energy preservation.

Proposition 6.3.3. *We have that*

$$H(\psi, v) = H(\psi_0, \psi_1) \quad \forall t \in \mathbb{R}.$$

Proof. Using the formal definition of Poisson brackets, we can see that

$$\frac{d}{dt}H(\psi,v) = \left\langle \nabla H(\psi,v), \partial_t \begin{pmatrix} \psi \\ v \end{pmatrix} \right\rangle = \langle \nabla H(\psi,v), X_H(\psi,v)\rangle = \{H,H\} = 0,$$

where $\langle .,. \rangle$ denotes the canonical scalar product. Then, we conclude that

$$H(\psi(t,x), v(t,x)) = H(\psi(0,x), -\partial_t\psi(0,x)) = H(\psi_0,\psi_1). \qquad \square$$

Remark. We have done here formal calculations, however this is not trivial to justify. For more details, see [Cazenave and Haraux, 1998] chapter 6.

Theorem 6.3.3. *Let $m > 0$ and ε_m be given by Lemma 6.3.2 and assume that $\|\psi_0\|_{H^2} + \|\psi_1\|_{L^2} \leq \varepsilon_m$. Then there exists a unique global solution to the Beam Equation 6.1 given by*

$$(\psi, \partial_t\psi) \in C_b^0(\mathbb{R}; H^2 \times L^2) \cap C^1(\mathbb{R}; L^2 \times H^{-2}).$$

Proof. By Proposition 6.3.2, we were able to prove local existence on some time interval $[0,T]$ such that T depends on $\|\eta_0\|_{h^1} = \|u_0\|_{h^1}$. Repeating the same arguments using initial data $u(T,x)$, we obtain existence on the interval $[T, T+\alpha_1]$ where α_1 depends on $\|u(T,x)\|_{h^1}$. Thus, iterating this process, we can extend the existence interval to $[0, T + \sum_{i=1}^{\infty} \alpha_i]$. Notice that if $\|u\|_{h^1}$ approaches ∞, then α_i will approach 0. However, the good news is that the Hamiltonian conservation ensures the boundedness of $\|u\|_{h^1}$. For instance,

$$\begin{aligned}
(\|\psi\|_{H^2} + \|v\|_{L^2})^2 &\leq \Lambda_m H(\psi, v) && \text{by Lemma 6.3.2} \\
&= \Lambda_m H(\psi_0, \psi_1) && \text{by Proposition 6.3.3} \\
&\leq \Lambda_m^2 (\|\psi_0\|_{H^2} + \|\psi_1\|_{L^2})^2 && \text{by Lemma 6.3.2} \\
&\leq \Lambda_m^2 \varepsilon_m^2. && (6.9)
\end{aligned}$$

As a result, for all $t \in \mathbb{R}$ we have

$$
\begin{aligned}
||u||_{h^1}^2 &= \sum_{k\in\mathbb{Z}} \langle k\rangle^2 |u_k|^2 \\
&= \sum_{k\in\mathbb{Z}} \langle k\rangle^2 \left| \frac{1}{\sqrt{2}} w_k^{1/2}\psi_k + \frac{i}{\sqrt{2}w_k^{1/2}} v_k \right|^2 && \text{by 6.4} \\
&\leq \sum_{k\in\mathbb{Z}} \frac{\langle k\rangle^2}{2} \left(2\left| w_k^{1/2}\psi_k \right|^2 + 2\left| \frac{1}{w_k^{1/2}} v_k \right|^2 \right) \\
&= \sum_{k\in\mathbb{Z}} \langle k\rangle^2 \left(\sqrt{k^4+m}|\psi_k|^2 + \frac{1}{\sqrt{k^4+m}}|v_k|^2 \right).
\end{aligned}
$$

It is easy to see that $\frac{(1+k^2)\sqrt{k^4+m}}{(1+k^2)^2} = O(1)$ and $\frac{1+k^2}{\sqrt{k^4+m}} = O(1)$ as $|k|$ goes to infinity. So, there exists a constant $C_m > 0$ such that

$$
\begin{aligned}
||u||_{h^1}^2 &\leq \sum_{k\in\mathbb{Z}} \left((1+k^2)\sqrt{k^4+m}|\psi_k|^2 + \frac{1+k^2}{\sqrt{k^4+m}}|v_k|^2 \right) \\
&\leq C_m \sum_{k\in\mathbb{Z}} \left(\langle k\rangle^4 |\psi_k|^2 + |v_k|^2 \right) \\
&= C_m(||\psi||_{H^2}^2 + ||v||_{L^2}^2) \\
&\leq C_m(||\psi||_{H^2} + ||v||_{L^2})^2 \\
&\leq C_m \Lambda_m^2 \varepsilon_m^2 && \text{by 6.9.}
\end{aligned}
$$

Therefore, $||u||_{h^1}^2$ is bounded. Adding the term infinitely many times, we conclude that $\sum_{i=1}^{\infty} \alpha_i = \infty$ and consequently global existance. □

__To conclude:__ Having proved all the assumptions, apply Theorem 5.2.4 for

$$J_n(u) = |u_n|^2 + \left|u_{-n}^2\right|$$

to obtain the almost global preservation of the low Harmonic energies of the Beam equation given by

$$
\begin{aligned}
\mathcal{E}_n(\psi, v) &= \left|\frac{1}{\sqrt{2}} w_n^{1/2} \psi_n + \frac{i}{\sqrt{2} w_n^{1/2}} v_n\right|^2 + \left|\frac{1}{\sqrt{2}} w_n^{1/2} \psi_{-n} + \frac{i}{\sqrt{2} w_n^{1/2}} v_{-n}\right|^2 \\
&= \frac{1}{2}\left(w_n |\psi_n|^2 + \frac{1}{w_n}|v_n|^2 - i\psi_n v_{-n} + i\psi_{-n} v_n\right) + \frac{1}{2}\left(w_n |\psi_{-n}|^2 + \frac{1}{w_n}|v_{-n}|^2 - i\psi_{-n} v_n + i\psi_n v_{-n}\right) \\
&= \frac{1}{2}\sqrt{n^4 + m}|\psi_n|^2 + \frac{1}{2\sqrt{n^4 + m}}|v_n|^2 + \frac{1}{2}\sqrt{n^4 + m}|\psi_{-n}|^2 + \frac{1}{2\sqrt{n^4 + m}}|v_{-n}|^2 \\
&= \frac{1}{2}\sqrt{n^4 + m}\left(\left|\frac{1}{\sqrt{2}} \int_0^{2\pi} \psi(x) e^{-inx}\, dx\right|^2 + \left|\frac{1}{\sqrt{2}} \int_0^{2\pi} \psi(x) e^{inx}\, dx\right|^2\right) \\
&+ \frac{1}{2\sqrt{n^4 + m}}\left(\left|\frac{1}{\sqrt{2}} \int_0^{2\pi} v(x) e^{-inx}\, dx\right|^2 + \left|\frac{1}{\sqrt{2}} \int_0^{2\pi} v(x) e^{inx}\, dx\right|^2\right) \\
&= \frac{1}{2}\sqrt{n^4 + m}\left(\left|\int_0^{2\pi} \psi(x) e^{inx}\, dx\right|^2\right) + \frac{1}{2\sqrt{n^4 + m}}\left(\left|\int_0^{2\pi} v(x) e^{inx}\, dx\right|^2\right).
\end{aligned}
$$

Bibliography

[Win,] https://healthresearchfunding.org/euler-bernoulli-beam-theory-explained/.

[Bambusi, 2003] Bambusi, D. (2003). Birkhoff normal form for some nonlinear pdes. *Communications in mathematical physics*, 234(2):253–285.

[Bambusi and Grébert, 2006] Bambusi, D. and Grébert, B. (2006). Birkhoff normal form for partial differential equations with tame modulus. *Duke Mathematical Journal*, 135(3):507–567.

[Bernier and Grébert, 2021] Bernier, J. and Grébert, B. (2021). Birkhoff normal forms for hamiltonian pdes in their energy space. *arXiv preprint arXiv:2102.09852*.

[Cazenave and Haraux, 1998] Cazenave, T. and Haraux, A. (1998). *An introduction to semilinear evolution equations*, volume 13. Oxford University Press on Demand.

[Grébert and Villegas-Blas, 2011] Grébert, B. and Villegas-Blas, C. (2011). On the energy exchange between resonant modes in nonlinear schrödinger equations. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, 28(1):127–134.

[Tao, 2006] Tao, T. (2006). *Nonlinear dispersive equations: local and global analysis*. Number 106. American Mathematical Soc.