Mémoire Master 2 MFA - 2018

Gelfand-Fuchs Cohomology of Surface Diffeomorphism Groups

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Introduction

Let Σ_g be a connected compact oriented Riemann surface without boundary of genus $g \geq 0$. We denote $G = Diff_0^+(\Sigma_g)$ the identity's connected component in the group of orientation preserving diffeomorphisms of Σ_g and $\mathfrak{g} = Vect(\Sigma_g)$ the Lie algebra of all vector fields on Σ_g. The purpose of this master thesis is to obtain some informations about the differential Gelfand-Fuchs cohomology, $H^{\cdot}_{diff}$ of G and $\mathfrak{g}$, with coefficient in $\mathbb{R}$. More precisely, we will establish:

Theorem 0.1.
(i)If $g = 0$:

$$\begin{cases} H^1_{diff}(G,\mathbb{R}) = H^1_{diff}(\mathfrak{g},\mathbb{R}) = \{0\} \\ H^2_{diff}(G,\mathbb{R}) = H^2_{diff}(\mathfrak{g},\mathbb{R}) = \{0\} \\ H^5_{diff}(G,\mathbb{R}) \cong H^5_{diff}(\mathfrak{g},\mathbb{R}) \end{cases}$$

(ii)If $g = 1$:

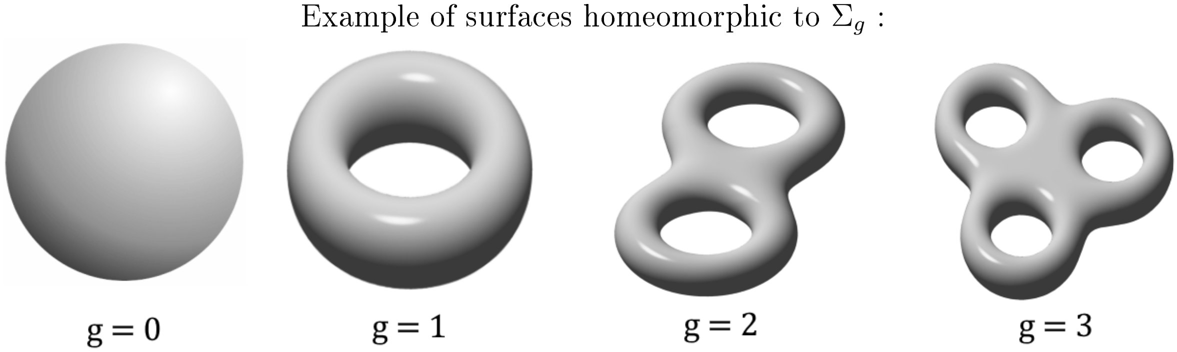
$$\begin{cases} H^1_{diff}(G,\mathbb{R}) = H^1_{diff}(\mathfrak{g},\mathbb{R}) = \{0\} \\ H^2_{diff}(G,\mathbb{R}) \cong \mathbb{R}^2 \\ H^2_{diff}(\mathfrak{g},\mathbb{R}) = \{0\} \end{cases}$$

(iii)If $g \geq 2$:

$$\begin{cases} H^1_{diff}(G,\mathbb{R}) = H^1_{diff}(\mathfrak{g},\mathbb{R}) = \{0\} \\ H^2_{diff}(G,\mathbb{R}) = H^2_{diff}(\mathfrak{g},\mathbb{R}) = \{0\} \\ H^q_{diff}(G,\mathbb{R}) \cong H^q_{diff}(\mathfrak{g},\mathbb{R}) \ \forall q \geq 0 \end{cases}$$

These results will be proved in the third section of this paper, mainly using spectral sequences and double complexes techniques, adapting the work of L.Guieu and C.Roger done for the group $Diff_0^+(\mathbb{S}^1)$. Before that, in the first section, we will state everything that we need to know about G (how to see it as a Lie Group), about the cohomologies that we will use and about double complexes and spectral sequences. In the second section we will discuss the notion of minimal models and we will see why we are unlikely to find a useful minimal model to study this cohomology.

Example of surfaces homeomorphic to Σ_g :



Remerciements

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Contents

1 Definitions and basic results

Let us start by setting up everything that we will need later.

1.1 About $Diff_0^+(\Sigma_g)$

The purpose of this subsection is to say a few words about how to define a Lie group structure on $G = Diff_0^+(\Sigma_g)$, which is in fact a pretty hard task.

First, the group structure is easy to obtain. The product of two orientation preserving diffeomorphisms of Σ_g, f_1 and f_2 is simply defined by the composition: $(f_1.f_2)(x) = (f_1 \circ f_2)(x)$ for all $x \in \Sigma_g$. Then, $G = Diff_0^+(\Sigma_g)$ is stable under this product and it has every property required to be a group.

It is much harder to define a manifold structure on G. The main reason is that it will be an infinite dimensional manifold. Here, manifolds are modelled on Fréchet manifolds. In $[GR]$ there is a specific study of the Lie group structure of $Diff_0^+(\mathbb{S}^1)$ but for Σ_g we will need something more general. Let $\widetilde{G} := Diff(\Sigma_g)$. First, we see that if we prove that if $\widetilde{G}$ is a manifold, then $G = Diff_0^+(\Sigma_g)$ will be a submanifold of it because it is simply one of its connected components.

The proof that $Diff(M)$ is a Lie group for all smooth manifold M can be found in $[KM]$, p.454. More precisely, they prove that it is a regular Lie group, which is a more natural definition, that allows the existence of an exponential map. This implies the notion of evolutions to be defined. See $[KM]$, p.410 for more details.

More simply, we can state some arguments to prove the following proposition:

Proposition 1.1.
$Diff(M)$ is a submanifold of $C^\infty(M, M)$ for every smooth manifold M.

Proof. To prove that, we introduce a smooth curve $c : \mathbb{R} \longrightarrow C^\infty(M, M)$ such that $c(0) \in Diff(M)$.

The proof is based on the fact that $c(t)$ is a diffeomorphism for t in a neighbourhood of 0. Thus, we can get that $M = c(t)(M)$ for all t close enough to 0 and we finally obtain that $Diff(M)$ is a open submanifold of $C^\infty_{prop}(M, M)$, which is the submanifold of $C^\infty(M, M)$ of all proper smooth diffeomorphisms of M. □

Remark. A chart structure on $C^\infty(M, M)$ is given in $[KM]$, p.439. It is possible to use it to construct a chart structure on $Diff(M)$, and by restriction, on G. Also, it is quite interesting to see how a chart structure can be explicitly defined on $Diff_0^+(\mathbb{S}^1)$ in $[GR]$, p.199.

Now, in order to prove that $Diff(M)$ is a Lie group, there is still to verify that both the composition and the inversion of diffeomorphisms are smooth. In $[KM]$, the smoothness of the composition is given by seeing it as an evaluation mapping of the form $ev : C^\infty(M, M) \times M \longrightarrow M$, and by proving that such an evaluation mapping is always smooth.

Concerning the inversion inv of diffeomorphisms, we need again to introduce a smooth curve $c : \mathbb{R} \longrightarrow Diff(M)$. Let us associate to c the map $c^\wedge : \mathbb{R} \times M \longrightarrow M$. It is possible to show that for all $t \in \mathbb{R}$ and for all $m \in M$, the map $(inv \circ c)^\wedge$ is solution of the implicit equation $c^\wedge(t, (inv \circ c)^\wedge(t, m)) = m$. Then, the conclusion comes from the finite-dimensional implicit function theorem, giving the smoothness of $(inv \circ c)^\wedge$, and we can deduce that inv is smooth.

We finally obtained that G is a Lie group. It is pretty easy to see that the corresponding Lie algebra is $\mathfrak{g} = Vect(\Sigma_g)$, the Lie algebra of all vector fields on Σ_g. Our next step is now to define some cohomologies for groups and for Lie algebras, in order to study G and $\mathfrak{g}$.

1.2 Group cohomology

Here we will quickly present a cohomology theory for Lie groups. There are two possible definitions. We will mostly use the Eilenberg-MacLane cohomology point of view but it is also equivalent to the Alexander-Spanier cohomology that we will also define. See for example $[N]$, p.60 for a more complete description.

1.2.1 Eilenberg-MacLane cohomology

Let us start by the definition of the complex:

Definition 1.2. Let G be a Lie group and A a G-module (ie: an abelian group with an action of G on it). We define the Eilenberg-MacLane cohomological complex as the group of all smooth maps between G^q and A: $C^q(G, A) := C^\infty(G^q, A)$ for all positive integer q.

Let us denote $g \cdot a$ the action of an element $g \in G$ on an element $a \in A$, induced by the G-module structure on A. We have now to define the coboundary operator:

Definition 1.3. For all positive integer q we define the map $\delta^q : C^q(G, A) \longrightarrow C^{q+1}(G, A)$ by the formula:

$$\begin{aligned}\delta^q c(g_0, g_1, \ldots, g_q) := {} & g_0 \cdot c(g_1, \ldots, g_q) \\ & + \sum_{i=0}^{q-1} (-1)^{i+1} c(g_0, \ldots, g_i g_{i+1}, \ldots g_q) \\ & + (-1)^{q+1} c(g_0, \ldots, g_{q-1})(g)\end{aligned}$$

Sometimes, in order to have lighter notations we will simply denote δ instead of δ^q.

Lemma 1.4. *We have $\delta^q \circ \delta^{q-1} = 0$ for all positive integer q.*

This lemma allows us to define the Eilenberg-MacLane cohomology:

Definition 1.5. For all positive integer q, we define $H^q_{diff}(G, A) := Ker(\delta^q)/Im(\delta^{q-1})$.

Remark. If G is just a group, but not a Lie group, it is still possible to define a similar cohomology by setting $C^q(G, A) := Map(G^q, A)$ and taking the same coboundary operator. We will not need that in this paper.

1.2.2 Alexander-Spanier cohomology

The Alexander-Spanier cohomology will give us an alternative point of view for the group cohomology which can be often more convenient.

Definition 1.6. The Alexander-Spanier complex is defined by the formula $\widetilde{C^q}(G, A) := \{c : G^{q+1} \longrightarrow C^\infty(G) |\ c \ is \ C^\infty \ and \ G - equivariant\}$
for all positive integer q.

In this definition, we say that a cochain $c : G^{q+1} \longrightarrow C^\infty(G)$ is G-equivariant if for any $(g_0, ..., g_{q-1}) \in G^q$ and for any $g \in G$, we have the relation: $g \cdot c(g_0, \ldots, g_{q-1})(h) = c(g^{-1}g_0, \ldots, g^{-1}g_{q-1})(h)$ for all $h \in G$.

Definition 1.7. The Alexander-Spanier differential is the map $\widetilde{\delta^q} : \widetilde{C^q}(G, A) \longrightarrow \widetilde{C^{q+1}}(G, A)$ given by the expression $\widetilde{\delta^q}(c)(g_0, \ldots, g_q)(g) := \sum\limits_{i=0}^{q} (-1)^{i+1} c(g_0, \ldots, \overset{\wedge}{g_i}, \ldots, g_q)(g)$, where $\overset{\wedge}{g_i}$ denotes the omission of the term g_i.

Lemma 1.8. *For all q, we have $\widetilde{\delta^q} \circ \widetilde{\delta^{q-1}} = 0$*

Then $(\widetilde{C^q}(G, A), \widetilde{\delta})$ is a cochain complex which is isomorphic to the Eilenberg-MacLane complex C^q.

This isomorphism, $\Phi_q : C^q(G, A) \longrightarrow \widetilde{C^q}(G, A)$ is given by:
$\Phi_q(c)(g_0, \ldots, g_q) := g_0 \cdot c(g_0^{-1}g_1, g_1^{-1}g_2, \ldots, g_{q-1}^{-1}g_q)$ for all $c \in C^q$.

Its inverse $\Phi_q^{-1} : \widetilde{C^q} \longrightarrow C^q$ is given by:
$\Phi_q^{-1}(c)(g_1, \ldots, g_q) := c(1, g_1, g_1g_2, \ldots, g_1 \ldots g_n)$

Remark. There is an alternative definition of the Alexander-Spanier complex that consists in defining $\widetilde{C^q}(G, A)$ as the set of all functions $c : G^{q+1} \longrightarrow C^\infty(G)$ which are G-equivariant and smooth in a neighbourhood of the diagonal. See $[N]$, p.61. Here we will not be using this second condition.

1.3 Gelfand-Fuchs cohomology

The Gelfand-Fuchs cohomology sets a cohomology theory for Lie-Fréchet algebras. There are two variations of this it. The cochains are either supposed to be continuous or smooth in the sense of the topology of the Lie-Fréchet algebra. We will denote $H^{\cdot}_{GF}$ the cohomology obtained for continuous cochains and $H^{\cdot}_{diff}$ (and often call it "differential cohomology") the cohomology obtained for smooth cochains. We will use both in section 3. These cohomologies are detailed in $[F]$. Here we will summarize the few things that we need to know about it.

First, we need to define a specific topology on the tensor product of Fréchet spaces.

Definition 1.9. Let E and F be two Fréchet spaces. On the space $E \bigotimes F$, we define the π-topology as the finest topology such that the bilinear form $\phi : E \times F \longrightarrow E \bigotimes F$ defined by $\phi(e, f) = e \otimes f$, is continuous.

Now for all positive integer p and for all Fréchet space E, let us denote $\hat{\bigotimes}^p_\pi E$ the completion of the space $\bigotimes^p E$ equipped with the π-topology and let $\hat{\Lambda}^p_\pi E$ be the subspace of alternate forms on $\hat{\bigotimes}^p_\pi E$. We can now define the Gelfand-Fuchs cohomology:

Definition 1.10. Let $\mathfrak{h}$ be a Lie algebra and V a $\mathfrak{h}$-Fréchet module. Then, for all positive integer p, we define the continuous Gelfand-Fuchs cochain complex as the space of continuous homomorphisms between $\hat{\Lambda}^p_\pi \mathfrak{h}$ and V: $C^p_{GF}(\mathfrak{h}, V) := Hom_{cont}(\hat{\Lambda}^p_\pi \mathfrak{h}, V)$

Similarly we define the smooth Gelfand-Fuchs cochain complex as the space of smooth homomorphisms between $\hat{\Lambda}^p_\pi \mathfrak{h}$ and V: $C^p_{diff}(\mathfrak{h}, V) := Hom_{diff}(\hat{\Lambda}^p_\pi \mathfrak{h}, V)$

Remark. In this definition we call $\mathfrak{h}$-Fréchet module a $\mathbb{C}$-vector space V equipped with a map $\phi : \mathfrak{h} \underset{x \longmapsto \phi_x}{\longrightarrow} End(V)$ such that for all x,$y \in \mathfrak{h}$ we have: $\phi_x \circ \phi_y - \phi_y \circ \phi_x = \phi_{[x,y]}$ where $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{h}$.

Definition 1.11. For both continuous and smooth complexes, the coboundary operator is the Chevalley-Eilenberg operator, which is, for all positive integer p, the map d^p : $C^p_{GF}(\mathfrak{h}, V) \longrightarrow C^{p+1}_{GF}(\mathfrak{h}, V)$ (resp.$d^p : C^p_{diff}(\mathfrak{h}, V) \longrightarrow C^{p+1}_{diff}(\mathfrak{h}, V)$) defined by:

$$\begin{aligned}(d^p c)(x_1, \dots, x_{p+1}) :=& \sum_{1\leq i,j\leq p+1} (-1)^{i+j} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{p+1}) \\ &- \sum_{i=1}^{p+1} (-1)^i \phi_{x_i} c(x_1, \dots, \hat{x}_i, \dots, x_{p+1})\end{aligned}$$

where $\hat{x}_i$ denotes the omission of the term x_i.

Using Cartan operations to make the computations easier, we can prove:

Lemma 1.12. *For all $p \geq 1$ and for all $c \in C^p_{GF}(\mathfrak{h}, V)$ (resp. $C^p_{diff}(\mathfrak{h}, V)$), we have :* $(d^p \circ d^{p-1})(c) = 0$.

This lemma shows that we can naturally define the corresponding cohomologies:

Definition 1.13. For all positive integer p we define $H^p_{GF}(\mathfrak{h}, V)$ (resp. $H^p_{diff}(\mathfrak{h}, V)$ as the space $Ker(d^p)/Im(d^{p-1})$ for $d^p : C^p_{GF}(\mathfrak{h}, V) \longrightarrow C^{p+1}_{GF}(\mathfrak{h}, V)$ (resp. $d^p : C^p_{diff}(\mathfrak{h}, V) \longrightarrow C^{p+1}_{diff}(\mathfrak{h}, V)$).

Let us finish this section with two important result about the Gelfand-Fuchs cohomology.

Firstly, Gelfand and Fuchs found a way to compute the cohomology of the Lie algebra of formal vector fields by linking it to the singular cohomology of a certain manifold.

Theorem 1.14. *(Gelfand-Fuchs, 1968)*

For all positive integer n, let W_n be the Lie algebra of formal vector fields with n variables. Let $(G(n, \infty))$ be the Grassmann manifold of n-planes in the space $\mathbb{C}^\infty = \bigcup\limits_{i \geq 0} \mathbb{C}^i$ and let X_n be the manifold defined as an open neighbourhood of the inverse image $\pi^{-1}(sk_{2n}(G(n, \infty))$ in the $U(n)$-universal bundle $\pi : EU(n) \longrightarrow BU(n) = G(n, \infty)$ of the $2n$-skeleton (given by Schubert cells of dimension inferior to $2n$) of $G(n, \infty)$.

Then, for all positive integer q, $H^q_{GF}(W_n) \cong H^q_{sing}(X_n)$.

This theorem can be proved by comparing the Hochschild-Serre spectral sequence for the sub-algebra $gl_n(\mathbb{R}) \subset W_n$ with the Lerray-Serre spectral sequence for the restriction of the $U(n)$ universal bundle to the $2n$-skeleton. See $[F]$, p.75.

Here is another important result:

Theorem 1.15. *(Haefliger-Bott-Segal, 1977)*

Let M be a compact manifold of dimension n. Then, there exists a fiber bundle E of M with typical fiber X_n such that for all positive integer q, $H^q_{GF}(Vect(M)) \cong H^q_{sing}(\Gamma(E))$, where $\Gamma(E)$ denotes the space of sections of E.

Remark. In this theorem, the space $\Gamma(E)$ is equipped with the compact-open topology.

1.4 Spectral sequences and double complexes

Here we will introduce the main tools of section 3. A double complex is the data of two different cohomological complexes such that they can be arranged to obtain a total cohomological complex. A very important tool to study double complexes is spectral sequences. Let us start this section with a few words about it. See $[W]$ for more informations about spectral sequences.

Definition 1.16. A cohomological spectral sequence is a family of vector spaces $(E^{p,q}_r)_{r \geq 0}$ with, for all integers p and q and for all $r \in \mathbb{N}$, a linear map $d^{p,q}_r : E^{p,q}_r \longrightarrow E^{p+r,q-r+1}_r$ such that for all $r \geq 0$, $d_r \circ d_r = 0$ and such that the cohomology of E_r relative to d_r is isomorphic to E_{r+1}.

The E_r are called the pages of the spectral sequence. This definition means that every page holds a coboundary operator and its cohomology is given by the following page. Also, we call first quadrant spectral sequence a spectral sequence $(E_r^{p,q})_r$ such that $E_r^{p,q} = \{0\}$ whenever $p < 0$ or $q < 0$. There is also the important concept of bounded spectral sequences:

Definition 1.17. A spectral sequence $(E_r^{p,q})_r$ is said to be bounded if for all $n \in \mathbb{N}$ there exists only a finite number of $(p,q) \in \mathbb{Z}^2$ such that $E_0^{p,q} \neq \{0\}$.

Proposition 1.18. *Let $(E_r^{p,q})_r$ be a bounded spectral sequence. Then, for all $(p,q) \in \mathbb{Z}^2$ there exists a rank $r_0 \in \mathbb{N}$ such that $E_r^{p,q} = E_{r_0}^{p,q}$ for all $r \geq r_0$. $E_{r_0}^{p,q}$ is very often denoted by $E_\infty^{p,q}$.*

A common way to obtain a spectral sequence is to have a cohomological complex with a filtration. More precisely, let (C,d) be a cohomological complex and let $\mathfrak{F}C := \{\mathfrak{F}^pC\}_{p\in\mathbb{Z}}$ such that for all p, $\mathfrak{F}^pC \subset \mathfrak{F}^{p+1}C$. Let us assume that there exists $n \in \mathbb{N}$ such that $\mathfrak{F}^nC = \{0\}$ and that $\mathfrak{F}^0C = C$. Then, if for all $p \in \mathbb{Z}$, $d(\mathfrak{F}^pC) \subset \mathfrak{F}^pC$, we can define a spectral sequence by recursion by setting $E_0^{p,q} := \mathfrak{F}^pC^{p+q}/\mathfrak{F}^{p+1}C^{p+q}$ and by defining each page E_{r+1} as the cohomology of the previous page for the coboundary operator d_r, induced by d on C. Such a spectral sequence always converges to the cohomology of the complex C. The convergence of spectral sequences can be defined as follows:

Definition 1.19. We say that a bounded spectral sequence $(E_r^{p,q})_r$ converges to $H^\cdot$ if for all $n \in \mathbb{N}$ there exists a finite filtration $\{0\} = F^tH^n \subset \ldots \subset F^{k+1}H^n \subset F^kH^n \subset \ldots \subset F^sH^n = H^n$ such that for all $(p,q) \in \mathbb{Z}^2$, $E_\infty^{p,q}$ is isomorphic to $F^pH^{p+q}/F^{p+1}H^{p+q}$.

As mentioned earlier, among the most common spectral sequences used in algebraic topology, we can name the Lerray-Serre spectral sequence, that concerns the singular cohomology, or the Hochschild-Serre spectral sequence for cohomologies of Lie algebras. Details about it can be found in $[F]$, for example.

Now let us define double complexes and see how they are related to spectral sequences.

Definition 1.20. A double cohomological complex is, for all $(q,p) \in \mathbb{N}^2$, the data of a vector space $C^{q,p}$ and two coboundary operators :

$$\begin{cases} d : C^{q,p} \longrightarrow C^{q,p+1} \\ \delta : C^{q,p} \longrightarrow C^{q+1,p} \end{cases}$$

such that $d \circ d = 0$, $\delta \circ \delta = 0$ and $d \circ \delta + \delta \circ d = 0$. Then, we define the total complex $C^\cdot$ associated to this double complex by: $C^n := \bigoplus\limits_{q+p=n} C^{q,p}$ for all $n \in \mathbb{N}$, equipped with the coboundary operator $D := d + \delta$.

Remark. The conditions $d \circ d = 0$ and $\delta \circ \delta = 0$ insure that the partial complexes $C^{q,\cdot}$ and $C^{\cdot,p}$ are cohomological complexes. Thus, the differential d and δ are inducing partial cohomologies $H_\delta^q(C^{\cdot,\cdot})$ and $H_\delta^q(C^{\cdot,\cdot})$.

The third condition, $d \circ \delta + \delta \circ d = 0$ shows that D is a coboundary operator, because $D^2 = d^2 + d \circ \delta + \delta \circ d + \delta^2 = d^2 + \delta^2 = 0$. This gives us a total cohomology space $H_D^{\cdot}(C^{\cdot,\cdot})$, of the total complex, relative to D.

Let us finish this section with the following theorem which makes an important link between spectral sequences and double complexes:

Theorem 1.21. *Let $(C^{\cdot,\cdot}, \delta, d)$ be a double complex, with $D = d + \delta$. Then, there exists two spectral sequences $({}^{I}E_r^{p,q})_r$ and $({}^{II}E_r^{q,p})_r$, both converging to the total cohomology of $(C^{\cdot}, D)$, with second pages given by: ${}^{I}E_2^{p,q} = H_d^p(H_\delta^q(C^{\cdot,\cdot}))$ and ${}^{II}E_2^{q,p} = H_\delta^q(H_d^p(C^{\cdot,\cdot}))$, for all $(p, q) \in \mathbb{Z}^2$.*

Thanks to this theorem, to study some cohomology, we can try to find a double complex such that the second page of its spectral sequences are the cohomology that we want to study. Then, informations can be obtain by looking a the convergence of the spectral sequences. This will be the main idea of section 3.

2 Minimal models approach

One interesting idea to study a manifold M (like our Lie group G) and the Lie algebre $Vect(M)$ uses the fact that thanks to the Heafliger-Bott-Segal theorem, there exists a certain fiber bundle over M, E such that the Gelfand-Fuchs is isomorphic to the singular cohomology of $\Gamma(E)$. This singular cohomology can be related to the rational homotopy groups of $\Gamma(E)$.

When the homotopy groups of a topological space M are too hard to compute it is often smarter to study the rational homotopy groups of M, $\pi_q(M) \bigotimes \mathbb{Q}$, which are obtained from $\pi_q(M)$ by forgetting torsion. Minimal models are a good tool to obtain these rational homotopy groups. The most common models for this kind of problems are for example Quillen's models and Sullivan's models. A complete study of these models can be found in $[FOT]$.

2.1 Sullivan's models

Here we will introduce, as an example, Sullivan's concept of minimal models and we will state the basic results about it. Sullivan's models are simple to define an useful to study the rational homotopy groups of manifolds.

Let $A = \bigoplus\limits_{i \geq 0} A^i$ be a commutative differential graded algebra. This means that there is a map d on A which is an antiderivation of degree 1. That is to say that $d(a \cdot b) = (da) \cdot b + (-1)^{deg(a)} a \cdot (db)$ for all a and b in A, with a homogeneous (so that its degree is well defined).

Also, the commutativity of A means that for all homogeneous elements a and b in A, we have the relation $a \cdot b = (-1)^{deg(a) \cdot deg(b)} b \cdot a$.

A is said to be free if it does not satisfy any relation other than this commutativity and the associativity. We say that an element $a \in A$ is decomposable if it can be written as a sum of products of positive elements of A. Here is the definition models and minimal models on A that can be found in $[BT]$, p.259:

Definition 2.1. Let $\mathfrak{M}$ be a differential graded algebra. We say that it is a Sullivan model for A if:

(a) $\mathfrak{M}$ is free.

(b) There exist a chain map $f : \mathfrak{M} \longrightarrow A$ inducing an isomorphism in cohomology.

Then, the model $\mathfrak{M}$ is said to be minimal if it also verifies:

(c) For any generator m of $\mathfrak{M}$, dm is either zero or decomposable.

Also, we call Sullivan minimal model for a manifold M any Sullivan minimal model the algebra of forms $\Omega(M)$.

Here is a first important result about the existence of these minimal models:

Proposition 2.2. *If A is 1-connected and has a finite-dimensional cohomology, then it has a minimal model.*

See $[BT]$ p.260 for a proof of this proposition.

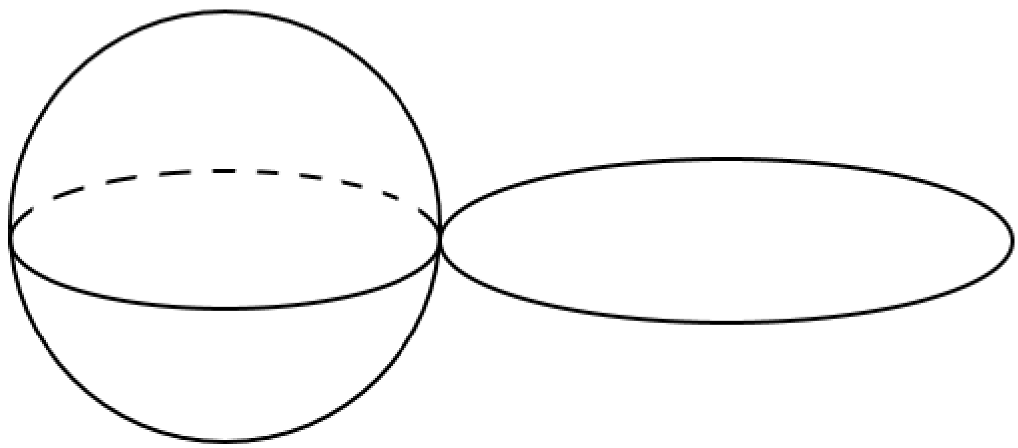
The following theorem is very important because it makes a link between minimal models and rational homotopy groups:

Theorem 2.3. *Let M be a simply connected manifold and $\mathfrak{M}$ a minimal model of M. Then for all $q \in \mathbb{N}$, the dimension of $\pi_q(M) \bigotimes \mathbb{Q}$ is equal to the number of q-dimensional generators of $\mathfrak{M}$ as an algebra.*

For example this theorem is useful to obtain homotopy groups of a wedge of spheres.

For two positive integers p and q, the wedge of the p-dimensional sphere $\mathbb{S}^p$ and the q-dimensional sphere $\mathbb{S}^q$, which is denoted by $\mathbb{S}^p \vee \mathbb{S}^q$ is the union of $\mathbb{S}^p$ and $\mathbb{S}^q$ with one single intersection point.

$\mathbb{S}^1 \vee \mathbb{S}^2$:



These wedges of sphere are important for our problem. For example, if W_2 denotes the Lie algebra of formal vector field in dimension 2, using theorem and spectral sequences methods, it is possible to link its Gelfand-Fuchs cohomology with $\mathbb{S}^5 \vee \mathbb{S}^5 \vee \mathbb{S}^7 \vee \mathbb{S}^8 \vee \mathbb{S}^8$.

Then, the cohomology of $\mathfrak{g} = Vect(\Sigma_g)$ can be studied, for example, by looking at the cohomology of $Map(\Sigma_g, \mathbb{S}^5 \vee \mathbb{S}^5 \vee \mathbb{S}^7 \vee \mathbb{S}^8 \vee \mathbb{S}^8)$ with coefficients in $\mathbb{Q}$.

For instance, the construction of a minimal model for $\mathbb{S}^2 \vee \mathbb{S}^2$ is done in $[BT]$, p.262 and some low-dimensional generator of this model and the rank of the few first homotopy groups of $\mathbb{S}^2 \vee \mathbb{S}^2$ are given. The problem is that it is easy to guess that this minimal model is not finitely generated. Thus, it is hard to compute the rank of any homotopy group of $\mathbb{S}^2 \vee \mathbb{S}^2$.

2.2 About Shibata's theorem

This problem of not having a finitely generated cohomology algebra for $Vect(\Sigma_g)$ is in fact more general.

While introducing their cohomology theory, D.B.Fuchs and I.Gelfand were stating that the cohomology of $\mathfrak{g}$ was finitely generated. Later this was proved to be wrong, for example in P.Trauber's doctoral thesis. See $[T]$.

More precisely, K.Shibata proved the following theorem in 1981. See $[Sh]$, p.380.

Theorem 2.4. *(Shibata)*

If M is a paracompact Hausdorff smooth manifold having a finite number of connected components, then the following assertions are equivalent:

(a) The Gelfand-Fuchs cohomology of $\mathfrak{g}$ is not finitely generated as a $\mathbb{R}$-graded algebra.

(b) $dim(M) \geq 2$ and $\bigoplus_{i\geq 1} H^i(M,\mathbb{R}) \neq \{0\}$.

Applying this theorem for $M = \Sigma_g$ shows that the cohomology of $\mathfrak{g}$ is not finitely generated. This means that we are unlikely to find an interesting model to describe it. Instead we will focus on the study of its low-dimensional cohomology groups using double complexes and spectral sequences methods.

3 Main study

In this section we are going to adapt the work of L.Guieu and C.Roger that consists in the study of a double complex and its spectral sequences to obtain informations about the differential cohomology of group of orientation preserving diffeomorphisms of the circle. See $[GR]$, p.218. We will do a similar study with the surface Σ_g instead of the circle.

3.1 Setting the double complex

For two integers p and q we denote $C^{q,p} := C^{\infty}(G^q, \Omega^p(G))$.

$C^{\cdot,\cdot}$ is a cohomology double complex for the two following differentials:

(a) The map $d : C^{q,p} \longrightarrow C^{q,p+1}$ defined by $d := (-1)^p d_{dR}$ where d_{dR} is the de Rham differential.

(b) The map $\delta : C^{q,p} \longrightarrow C^{q+1,p}$, the Eilenberg-MacLane differential.

Then, we can verify that $d \circ \delta + \delta \circ d = 0$, which means that the double complex is well defined.

Our goal is now to obtain some informations about the differential cohomology of the group G and the differential cohomology of its Lie algebra $\mathfrak{g}$ by using the two spectral sequences related to the double complex.
We will denote $({}^{I}E_r^{p,q})_r$ and $({}^{II}E_r^{q,p})_r$ those spectral sequences.
We know that their second pages are given by:
${}^{I}E_2^{p,q} = H_d^p(H_\delta^q(C^{\cdot,\cdot}))$ and ${}^{II}E_2^{q,p} = H_\delta^q(H_d^p(C^{\cdot,\cdot}))$.

Also, we know that both spectral sequences converge to the total cohomology of the complex $H_D(C^{\cdot,\cdot})$.

Remark. We can see that this double complex is multiplicative. There is a product $C^{q,p} \times C^{q',p'} \longrightarrow C^{q+q',p+p'}$ which comes form the exterior product of differential forms and the cup product of the Eilenberg-MacLane cochaines on G. In fact, the differentials d and δ are derivations for this multiplicative structure. From this, we get a multiplicative structure on the total cohomology related to D, but also on the partial cohomologies, related to d and δ

3.2 Preliminary results

The results of this subsection does not depend on the genus g of the surface.

3.2.1 Study of $({}^{I}E_r^{p,q})_r$

Let us start with an important lemma that is also stated in $[GR]$ p.219 and in $[F]$ p.294. Both have some mistakes in their proofs that we will correct.

Lemma 3.1. *Let $\mathfrak{g}^*$ be the topological dual of $\mathfrak{g}$ and $\Lambda^p_{diff}(\mathfrak{g}^*)$ be the set of alternate differential p-forms on $\mathfrak{g}$. Then, for all $q > 0$, $H_d^q(C^{\cdot,p}) = 0$ and $H_d^0(C^{\cdot,p}) = \Lambda^p_{diff}(\mathfrak{g}^*)$*

Proof. We first consider the case of $p = 0$.

Let us denote $C^q := (C^\infty(G^q, C^\infty(G)), d)$.

Here, the group G acts on G^{q+1} by $g \cdot (g_0, \dots, g_q) := (gg_0, \dots, gg_q)$ for all $g \in G$.

Moreover, for all $c : G^q \longrightarrow C^\infty(G)$, G acts on $c(g_0, \dots, g_{q-1})$ by the formula:
$(g \cdot c)(g_0, \dots, g_{q-1})(h) := c(g_0, \dots, g_{q-1})(g^{-1}h)$ for all $h \in G$.

The idea of this proof is to define a homotopy $h_q : C^\infty(G^q, C^\infty(G)) \longrightarrow C^\infty(G^{q-1}, C^\infty(G))$ which will show that the complex $C^{\cdot,p}$ is contractible.

The homotopy given in $[GR]$ p.219 and in $[F]$ p.294 does not work. Instead, we will use the following homotopy:

We define h_q by the following formula:
$(h_q c)(g_0, \dots, g_{q-2})(g) := (-1)^q c(g_0, \dots, g_{q-2}, (g_0 g_1 \dots g_{q-2})^{-1} g)(g)$ for all $c \in C^\infty(G^q, C^\infty(G))$.

Let us verify that $\delta \circ h_q + h_{q+1} \circ \delta = id$.

Let $c \in C^\infty(G^q, C^\infty(G))$.

On one hand we have:

$$\begin{aligned}
\delta(h_q c)(g_0, g_1, \dots, g_{q-1})(g) &= g_0 \cdot (h_q c)(g_1, \dots, g_{q-1})(g) \\
&\quad + \sum_{i=0}^{q-2} (-1)^{i+1} (h_q c)(g_0, \dots, g_i g_{i+1}, \dots g_{q-1})(g) \\
&\quad + (-1)^q (h_q c)(g_0, \dots, g_{q-2})(g) \\
&= (-1)^q c(g_1, \dots, g_{q-1}, (g_1 g_2 \dots g_{q-1})^{-1} g_0^{-1} g)(g_0^{-1} g) \\
&\quad + (-1)^q \sum_{i=0}^{q-2} (-1)^{i+1} c(g_0, \dots, g_i g_{i+1}, \dots, g_{q-1}, (g_0 g_1 \dots g_{q-1})^{-1} g)(g) \\
&\quad + c(g_0, \dots, (g_0 g_1 \dots g_{q-2})^{-1} g)(g)
\end{aligned}$$

On the other hand:

$$\begin{aligned}
(h_{q+1})(\delta c)(g_0, g_1, \dots, g_{q-1})(g) &= (-1)^{q+1} (\delta c)(g_0, \dots, (g_0 g_1 \dots g_{q-1})^{-1} g)(g) \\
&= (-1)^{q+1} c(g_1, \dots, g_{q-1}, (g_0 g_1 \dots g_{q-1})^{-1} g)(g_0^{-1} g) \\
&\quad + (-1)^{q+1} \sum_{i=0}^{q-2} (-1)^{i+1} c(g_0, \dots, g_i g_{i+1}, \dots, g_{q-1}, (g_0 g_1 \dots g_{q-1})^{-1} g)(g) \\
&\quad - c(g_0, \dots, (g_0 g_1 \dots g_{q-2})^{-1} g)(g) \\
&\quad + c(g_0, g_1, \dots, g_{q-1})(g)
\end{aligned}$$

Adding both terms we obtain:

$$(\delta \circ h_q + h_{q+1} \circ \delta)(c)(g_0, \dots, g_{q-1})(g) = c(g_0, \dots, g_{q-1})(g)$$

Thus, $\delta \circ h_q + h_{q+1} \circ \delta$ is the identity map.

From this homotopy, we deduce that the complex is contractible, so it has a trivial cohomology.

Next, for any $p \geq 1$ it is possible to prove that $\Omega^p(G) = C^\infty(G)\hat{\bigotimes}\Lambda^p_{diff}(\mathfrak{g}^*)$ as a G-module. In fact $\Omega^p(G)$ is isomorphic to $C^\infty(G, \Lambda^p_{diff}(\mathfrak{g}^*))$. From this, we can conclude that the previous homotopy can be generalized for the case of differential forms.

$\square$

Remark. The computations of this proof become easier by working in the Alexander-Spanier cochain complex. We can define the following homotopy on the Alexander-Spanier complex, with a more natural formula:

Let $\widetilde{h}_q : \widetilde{C}^q \longrightarrow \widetilde{C}^{q-1}$ given by:

$(\widetilde{h}_q c)(g_0, \dots, g_{q-1})(g) := (-1)^q c(g_0, \dots, g_{q-1}, g)(g)$

The homotopy h_q of the previous proof is obtained from $\widetilde{h}_q$ by the formula $h_q := \Phi^{-1}_{q-1} \circ \widetilde{h}_q \circ \Phi_q$ according to the following diagram:

$$\begin{array}{ccc} \widetilde{C}^q & \xrightarrow{\widetilde{h}_q} & \widetilde{C}^{q-1} \\ \Phi_q \uparrow & & \downarrow \Phi^{-1}_{q-1} \\ C^q & \xrightarrow{h_q} & C^{q-1} \end{array}$$

(See part 1.2.2 for the definitions of the maps Φ_q.)

We can see that $\widetilde{h}_q$ is a contracting homotopy on $\widetilde{C}^q$ if and only if h_q is a contracting homotopy on C^q, and the verification is simpler with $\widetilde{h}_q$.

Remark. There is yet another way to find such a homotopy using the bar resolution of free A-modules (see [W] for more details about the bar resolution).

In this construction, we can obtain the group cohomology by taking $A := \mathbb{R}[G]$ (the algebra of the group G). Then for any G-module M there is a canonical contraction of the bar resolution of A that gives the Eilenberg-MacLane complex of G with coefficients in M.

Then, from this canonical contraction we obtain another contracting homotopy on the Eilenberg-Mac Lane complex, which is given by: $(h_q c)(g_0, \dots, g_{q-2})(g) := c(g^{-1}, g_0, \dots, g_{q-2})(1)$ for all $c \in C^\infty(G^q, C^\infty(G))$ and for all $(g, g_0, \dots, g_{q-2}) \in G^q$.

This lemma allows us to compute the spectral sequence $({}^I E_r^{p,q})_r$.

Because of the inclusion $\Lambda^p_{diff}(\mathfrak{g}^*) \subset \Omega^p(G)$, corresponding to the subcomplex of invariant forms, we can see the De Rham differential as the Chevalley-Eilenberg differential on $\Lambda^p_{diff}(\mathfrak{g}^*)$.

Thus, the lemma gives:

$$\begin{cases} H^p_d(H^q_\delta(C^{\cdot,\cdot})) = 0 \; ; \; q > 0 \\ H^p_d(H^0_\delta(C^{\cdot,\cdot})) = H^p_{diff}(\mathfrak{g}, \mathbb{R}) \end{cases}$$

So, we obtain:

$$\begin{cases} {}^{I}E_2^{p,q} = 0 \ ; \ q > 0 \\ {}^{I}E_2^{p,0} = H^p_{diff}(\mathfrak{g}, \mathbb{R}) \end{cases}$$

To go further, we will need the following result:

Proposition 3.2. *For all integer p and q, the cohomology of the double complex relative to d is given by $H^p_d(C^{q,p}) = C^\infty(G^q, H^p_{dR}(G))$, where $H^p_{dR}(G)$ denotes the De Rham cohomology of G.*

Proof. This comes from the fact that the differential d only acts in the coefficients. □

3.2.2 About the Earle-Eells theorem

To study the spectral sequence $({}^{II}E_r^{p,q})_r$ we will have to apply the following result from C.J.Earle and J.Eells in 1967 (see $[EE]$):

Theorem 3.3. *(Earle - Eells)*
Let $x_1, \ldots, x_n$ be n distinct points on Σ_g and let $D_0(x_1, \ldots, x_n)$ denote the subgroup of $Diff^+(\Sigma_g)$ of all diffeomorphisms f which are homotopic to the identity and fixing the points $x_1, \ldots, x_n$. Then:
(a) If $g = 0$, $D_0(x_1, x_2, x_3)$ is contractible and G is homeomorphic to $A \times D_0(x_1, x_2, x_3)$, where A denotes the group of conformal automorphisms of the Riemann sphere.
(b) If $g = 1$, $D_0(x_1)$ is contractible and G is homeomorphic to $B \times D_0(x_1)$, where B denotes the group of conformal automorphisms of the genus 1 torus $\mathbb{T}^2$.
(c) If $g \geq 2$, G is contractible.

We deduce from it:

Corollary 3.4.
(a) If $g = 0$, G is homotopicaly equivalent to $SO_3(\mathbb{R})$.
(b) If $g = 1$, G is homotopicaly equivalent to $\mathbb{T}^2$.
(c) If $g \geq 2$, G is homotopicaly equivalent to a point.

Proof. Case (a) is proved by S.Smale in $[Sm]$, p.621-626, showing that $SO_3(\mathbb{R})$ is a strong deformation retract of G.

For case (b), we can prove that every conformal automorphism of the torus is of the form: $x \mapsto \frac{ax+b}{cx+d}$. Then, because these automorphisms come from $\mathbb{R}^2$ (by seeing the torus as the quotient $\mathbb{R}^2/\mathbb{Z}^2$), it is possible to show that $c = 0$ and that $d = \pm a$.

Case (c) is clear. □

3.2.3 Links between Gelfand-Fuchs and differential cohomology

The last important result that will be used for all genus says that there is an isomorphism between the differential cohomology of $\mathfrak{g}$ and its Gelfand-Fuchs cohomology. This can be obtain by adapting the work of Fuchs in $[F]$, p.104 concerning continuous cochains. Every argument still work with differential cochains. Thus, we can obtain:

Proposition 3.5.
$H^q_{diff}(\mathfrak{g},\mathbb{R}) \cong H^q_{GF}(\mathfrak{g},\mathbb{R})$ for all $q \geq 0$

Thanks to this proposition, knowing the Gelfand-Fuchs cohomology of $\mathfrak{g}$, we obtain:

Corollary 3.6. *The 3 first cohomology spaces of $\mathfrak{g}$ are:*

$$\begin{cases} H^0_{diff}(\mathfrak{g},\mathbb{R}) \cong \mathbb{R} \\ H^1_{diff}(\mathfrak{g},\mathbb{R}) = H^2_{diff}(\mathfrak{g},\mathbb{R}) = \{0\} \end{cases}$$

Proof. Let W_2 be the Lie Algebra of formal vector fields on the plane.
Then:

$$\begin{cases} H^0_{GF}(W_2,\mathbb{R}) \cong \mathbb{R} \\ H^k_{GF}(W_2,\mathbb{R}) = \{0\} \text{ for } k = 1,2,3,4 \\ H^5_{GF}(W_2,\mathbb{R}) \neq \{0\} \end{cases}$$

(see $[F]$, p.89)

By using the Gelfand-Fuchs spectral sequence we obtain:

$$\begin{cases} H^0_{GF}(\mathfrak{g},\mathbb{R}) \cong \mathbb{R} \\ H^1_{GF}(\mathfrak{g},\mathbb{R}) = H^2_{GF}(\mathfrak{g},\mathbb{R}) = \{0\} \end{cases}$$

Finally, this corollary is given by the previous proposition. □

We will now study each of these 3 cases separately to try to understand the spectral sequence $({}^{II}E_r^{q,p})_r$. Let us start by the simplest case: $g \geq 2$.

3.3 Case $g \geq 2$

For $g \geq 2$, thanks to corollary 3.4, the De Rham cohomology of G is easy to compute:

$$\begin{cases} H^p_{dR}(G) = \{0\} \ ; \ p \geq 1 \\ H^0_{dR}(G) \cong \mathbb{R} \end{cases}$$

Then, using proposition 3.2 we obtain, for all $q \geq 0$:

$$\begin{cases} H^p_d(C^{q,p}) = \{0\} \ ; \ p \geq 1 \\ H^0_d(C^{q,0}) \cong C^\infty(G^q,\mathbb{R}) \end{cases}$$

Thus, taking the cohomology relative to δ, we get:

$$\begin{cases} {}^{II}E_2^{q,p} = \{0\} \ ; \ p \geq 1 \\ {}^{II}E_2^{q,0} \cong H^q_\delta(C^\infty(G^q,\mathbb{R})) \end{cases}$$

Finally, we deduce, for all $q \geq 0$:

$$\begin{cases} {}^{II}E_2^{q,p} = \{0\} \ ; \ p \geq 1 \\ {}^{II}E_2^{q,0} \cong H^q_{diff}(G, \mathbb{R}) \end{cases}$$

The most important result that we can deduce from it is the following theorem:

Theorem 3.7.
For all $q \geq 0$, we have the following isomorphism:
$H^q_{diff}(G, \mathbb{R}) \cong H^q_{diff}(\mathfrak{g}, \mathbb{R})$

Proof. This comes from the fact that both $({}^{I}E_r^{p,q})$ and $({}^{II}E_r^{q,p})$ converge to the same space (the total cohomology of the double complex) and the fact that for both spectral sequences, all differentials are trivial past the second pages. Hence, we have, for all p and q:

$${}^{I}E_2^{q,p} \cong {}^{I}E_\infty^{q,p} \cong {}^{II}E_\infty^{p,q} \cong {}^{II}E_2^{p,q}$$

And the theorem comes by comparing the second pages of both spectral sequences. □

Remark. The isomorphisms $H^1_{diff}(G, \mathbb{R}) \cong H^1_{diff}(\mathfrak{g}, \mathbb{R})$ and $H^2_{diff}(G, \mathbb{R}) \cong H^2_{diff}(\mathfrak{g}, \mathbb{R})$ can also be directly deduced from the work of K.H. Neeb. See $[N]$.

Remark. We could also try to study $H^{\cdot}_{diff}(G, \mathbb{R})$ by giving it an algebra structure using the multiplicativity of $({}^{I}E_r^{p,q})_r$ and $({}^{II}E_r^{q,p})_r$ but the isomorphism given by this theorem shows that this algebra is not finitely generated. This comes from the fact that $H^{\cdot}_{diff}(\mathfrak{g}, \mathbb{R})$ is not finitely generated because of Shibata's theorem.

Now, let us conclude our study for $g \geq 2$. Using this theorem and corollary 3.6, we get the following theorem that summarizes what we obtained about the differential cohomology of G and $\mathfrak{g}$ for $g \geq 2$:

Theorem 3.8. *We have:*

$$\begin{cases} H^q_{diff}(\mathfrak{g}, \mathbb{R}) \cong H^q_{diff}(G, \mathbb{R}) \cong \mathbb{R} \ ; \forall q \geq 0 \\ H^1_{diff}(\mathfrak{g}, \mathbb{R}) = H^2_{diff}(\mathfrak{g}, \mathbb{R}) = H^1_{diff}(G, \mathbb{R}) = H^2_{diff}(G, \mathbb{R}) = \{0\} \end{cases}$$

3.4 Case $g = 0$

In this case we are going to obtain a Gysin-like exact sequence that will give us some informations about the cohomology of $G = Diff_0^+(\Sigma_g)$ and $\mathfrak{g} = Vect(\Sigma_g)$.

There are some details about the Gysin sequence in $[BT]$ p.177 but we do not really need it.

First, we know how to compute the de Rham cohomology of G:

Proposition 3.9.
$H^p_{dR}(G) \cong \mathbb{R}$ *if* $p = 0$ *or* $p = 3$ *and* $H^p_{dR}(G) = \{0\}$ *otherwise.*

Proof. Because of corollary 3.4, we have $H_{dR}(G) \cong H_{dR}(SO_3(\mathbb{R}))$. The universal covering of $SO_3(\mathbb{R})$ is $SU_2(\mathbb{C})$, which is homeomorphic to the 3-dimensional sphere $\mathbb{S}^3$. We deduce that for all $p \geq 2$, $H^p_{dR}(G) \cong H^p_{dR}(\mathbb{S}^3)$.

Also, we have $\pi_1(SO_3(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$. As it is a finite group, we can conclude that $H^1_{dR}(G) = \{0\}$. □

Now, we can use proposition 3.2 to obtain, for all $q \geq 0$:

$$\begin{cases} H^p_d(C^{q,p}) \cong C^\infty(G^q, \mathbb{R}) \text{ ; } p = 0 \text{ or } p = 3 \\ H^p_d(C^{q,p}) = \{0\} \text{ otherwise} \end{cases}$$

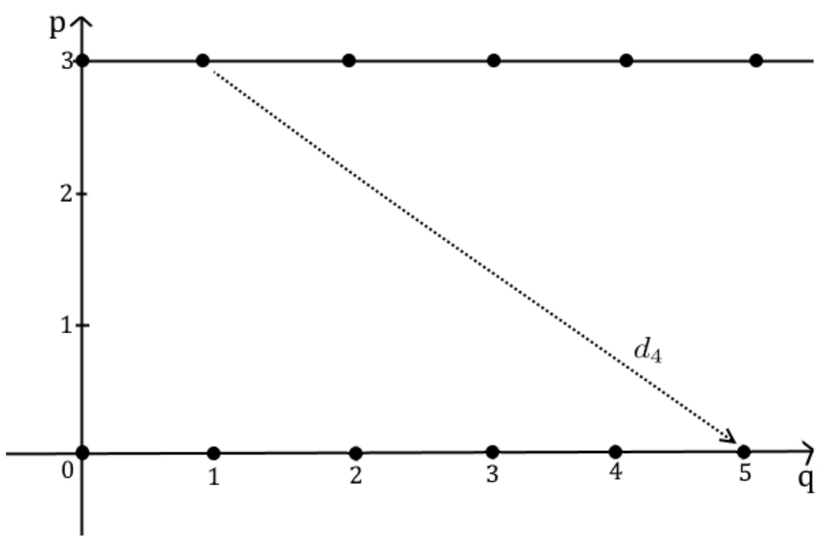
Then:

$$\begin{cases} {}^{II}E_2^{q,p} \cong H^q_\delta(C^\infty(G^q, \mathbb{R})) \text{ ; } p = 0 \text{ or } p = 3 \\ {}^{II}E_2^{q,p} = \{0\} \text{ otherwise} \end{cases}$$

And we finally obtain, for all $q \geq 0$:

$$\begin{cases} {}^{II}E_2^{q,p} \cong H^q_{diff}(G, \mathbb{R}) \text{ ; } p = 0 \text{ or } p = 3 \\ {}^{II}E_2^{q,p} = \{0\} \text{ otherwise} \end{cases}$$

From that, we deduce that for the spectral sequence $({}^{II}E_r^{q,p})_r$, the only non trivial differential other than d_0 and d_1 is $d_4 : {}^{II}E_4^{q,p} \longrightarrow {}^{II}E_4^{q+4,p-3}$
Here is a representation of the page ${}^{II}E_4$. There are two non trivial rows: the row for $p = 0$ and the row for $p = 3$. One of the maps d_4 is also represented.



This kind of diagram is typical of the Gysin sequence.
We can see that this situation gives the following exact sequence, for all $p \geq 0$:

$$0 \longrightarrow {}^{II}E_\infty^{p,3} \longrightarrow {}^{II}E_4^{p,3} \xrightarrow{d_4} {}^{II}E_4^{p+4,0} \longrightarrow {}^{II}E_\infty^{p+4,0} \longrightarrow 0$$

Because $({}^{II}E_r^{q,p})_r$ converges to the total cohomology $H_D(C^{\cdot,\cdot})$, we have for all $p \geq 0$:

$$0 \longrightarrow {}^{II}E_{\infty}^{0,p} \longrightarrow H_D^p(C^{\cdot,\cdot}) \longrightarrow {}^{II}E_{\infty}^{1,p-1} \longrightarrow 0$$

And we also know that $H_D^p(C^{\cdot,\cdot}) = H_{diff}^p(\mathfrak{g}, \mathbb{R})$

This gives us, for all $p \geq 0$, the following long Gysin exact sequence:

$$\ldots \longrightarrow H_{diff}^p(\mathfrak{g}, \mathbb{R}) \longrightarrow H_{diff}^{p-3}(G, \mathbb{R}) \xrightarrow{d_4} H_{diff}^{p+1}(G, \mathbb{R}) \xrightarrow{D} H_{diff}^{p+1}(\mathfrak{g}, \mathbb{R}) \longrightarrow H_{diff}^{p-2}(G, \mathbb{R}) \xrightarrow{d_4} \ldots$$

Finally, this long exact sequence gives the following informations about the first cohomology spaces:

Theorem 3.10.
$H_{diff}^1(G, \mathbb{R}) = H_{diff}^1(\mathfrak{g}, \mathbb{R}) = H_{diff}^2(G, \mathbb{R}) = H_{diff}^2(\mathfrak{g}, \mathbb{R}) = \{0\}$
Moreover, there is an isomorphism: $H_{diff}^5(G, \mathbb{R}) \cong H_{diff}^5(\mathfrak{g}, \mathbb{R})$

Proof. Firstly, we have $H_{diff}^1(\mathfrak{g}, \mathbb{R}) = H_{diff}^2(\mathfrak{g}, \mathbb{R}) = \{0\}$ thanks to corollary 3.6.

Then, using some parts of the exact sequence, we will obtain isomorphisms between some cohomology spaces of G and $\mathfrak{g}$.

To begin, let us write this segment of the previous exact sequence:

$$\ldots \longrightarrow H_{diff}^{-3}(G, \mathbb{R}) \xrightarrow{d_4} H_{diff}^1(G, \mathbb{R}) \xrightarrow{D} H_{diff}^1(\mathfrak{g}, \mathbb{R}) \longrightarrow H_{diff}^{-2}(G, \mathbb{R}) \longrightarrow \ldots$$

By using the fact that $({}^{II}E_r^{p,q})$ is a first quadrant spectral sequence, we know that the terms $H_{diff}^{-3}(G, \mathbb{R})$ and $H_{diff}^{-2}(G, \mathbb{R})$ are trivial.

Thus, we obtain $H_{diff}^1(G, \mathbb{R}) = H_{diff}^1(\mathfrak{g}, \mathbb{R}) = \{0\}$.

Similarly, we get $H_{diff}^2(G, \mathbb{R}) = H_{diff}^2(\mathfrak{g}, \mathbb{R}) = \{0\}$ by using the fact that $H_{diff}^{-2}(G, \mathbb{R}) = H_{diff}^{-1}(G, \mathbb{R}) = \{0\}$ and by writing:

$$\ldots \longrightarrow H_{diff}^{-2}(G, \mathbb{R}) \xrightarrow{d_4} H_{diff}^2(G, \mathbb{R}) \xrightarrow{D} H_{diff}^2(\mathfrak{g}, \mathbb{R}) \longrightarrow H_{diff}^{-1}(G, \mathbb{R}) \longrightarrow \ldots$$

Finally, we obtain the isomorphism $H_{diff}^5(G, \mathbb{R}) \cong H_{diff}^5(\mathfrak{g}, \mathbb{R})$ using the fact that $H_{diff}^1(G, \mathbb{R}) = H_{diff}^2(G, \mathbb{R}) = \{0\}$ and the following part of the exact sequence:

$$\ldots \longrightarrow H_{diff}^1(G, \mathbb{R}) \xrightarrow{d_4} H_{diff}^5(G, \mathbb{R}) \xrightarrow{D} H_{diff}^5(\mathfrak{g}, \mathbb{R}) \longrightarrow H_{diff}^2(G, \mathbb{R}) \longrightarrow \ldots$$

□

3.5 Case $g = 1$

This is by far the hardest case to deal with. We will start by using the same method than for the two other cases but this will not be very successful. In a second time we see that thanks to some results established by K.H. Neeb in [N] we will still be able to obtain some interesting results.

3.5.1 Using the double complex

Here, we start again with corollary 3.4, to compute the De Rham cohomology of G, knowing the De Rham cohomology of $\mathbb{T}^2$

$$\begin{cases} H^0_{dR}(G) \cong H^2_{dR}(G) \cong \mathbb{R} \\ H^1_{dR}(G) \cong \mathbb{R}^2 \\ H^p_{dR}(G) = \{0\} \ ; \ p \geq 3 \end{cases}$$

Thus, proposition 3.2 gives, for all $q \geq 0$:

$$\begin{cases} H^0_d(C^{q,0}) \cong H^2_d(C^{q,0}) \cong C^\infty(G^q, \mathbb{R}) \\ H^1_d(C^{q,0}) \cong C^\infty(G^q, \mathbb{R}^2) \\ H^p_d(C^{q,p}) = \{0\} \ ; \ p \geq 3 \end{cases}$$

We obtain:

$$\begin{cases} {}^{II}E_2^{q,0} \cong {}^{II}E_2^{q,2} \cong H^q_\delta(C^\infty(G^q, \mathbb{R})) \\ E_2^{q,1} \cong H^q_\delta(C^\infty(G^q, \mathbb{R}^2)) \\ {}^{II}E_2^{q,p} = \{0\} \ ; \ p \geq 3 \end{cases}$$

Hence, for all $q \geq 0$, we have:

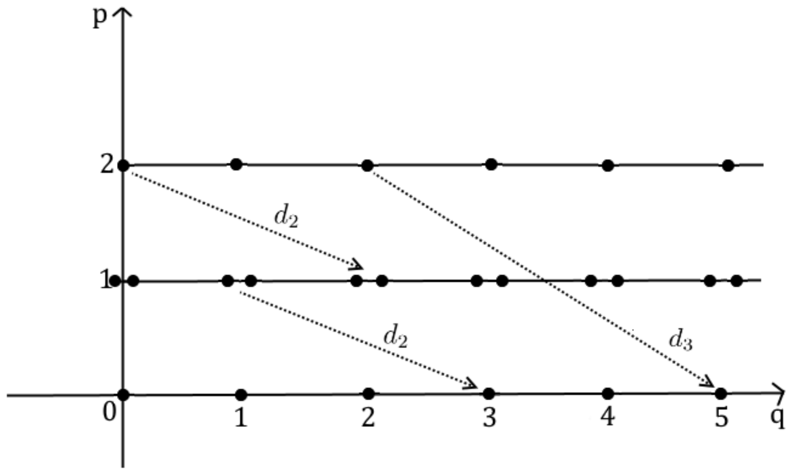
$$\begin{cases} {}^{II}E_2^{q,0} \cong {}^{II}E_2^{q,2} \cong H^q_{diff}(G, \mathbb{R}) \\ {}^{II}E_2^{q,1} \cong H^q_{diff}(G, \mathbb{R}^2) \\ {}^{II}E_2^{q,p} = \{0\} \ ; \ p \geq 3 \end{cases}$$

This spectral sequence is much harder to study than the previous ones because on its page 2 there are three non trivial rows.

It is required to understand the map $d_2 : {}^{II}E_2^{q,p} \longrightarrow {}^{II}E_2^{q+2,p-1}$ which is a priori non trivial from the row given by $p = 2$ to the row given by $p = 1$ and also from the row given by $p = 1$ to the row given by $p = 0$.

Then it is necessary to study the map $d_3 : {}^{II}E_3^{q,p} \longrightarrow {}^{II}E_3^{q+3,p-2}$, which is probably non trivial from the row given by $p = 2$ to the row given by $p = 0$.

Here is a representation of the second page of ${}^{II}E$ with two of the non trivial differential maps d_2.



One of the a priori non trivial map d_3 of the third page is also represented. The dots on the three rows represent generators of the spaces $E_2^{q,p}$. There are one generator for $p = 0$ and $p = 2$ and two generators for $p = 1$.

To understand this spectral sequence and to study its convergence, we need to study more deeply the maps d_2 and d_3. In particular it is needed to see how the generators are mapped to each other via d_2 and then study d_3.

Also, it should be a good thing to understand how G acts on $\mathbb{R}^2$. This action is probably trivial and this can prove that $H^q_{diff}(G, \mathbb{R}^2) \cong (H^q_{diff}(G, \mathbb{R}))^2$, which can simplify the problem.

3.5.2 Using Neeb's diagram

Here we use the following result, established by K.H. Neeb. See [N], p.3.

Theorem 3.11. *Let $\widetilde{G}$ be the universal cover of G.*

The following diagram has exact columns and an exact second row:

$$
\begin{array}{ccccccccc}
\{0\} & & \{0\} & & & & Hom(\pi_1(G),\mathbb{R}) & & \{0\} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
H^1_{diff}(G,\mathbb{R}) & \to & H^1_{diff}(\widetilde{G},\mathbb{R}) & \to & Hom(\pi_1(G),\mathbb{R}) & \to & H^2_{diff}(G,\mathbb{R}) & \to & H^2_{diff}(\widetilde{G},\mathbb{R}) \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
H^1_{GF}(\mathfrak{g},\mathbb{R}) & = & H^1_{GF}(\mathfrak{g},\mathbb{R}) & \to & Hom(\pi_1(G),\mathbb{R}) & & H^2_{GF}(\mathfrak{g},\mathbb{R}) & = & H^2_{GF}(\mathfrak{g},\mathbb{R}) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
Hom(\pi_1(G),\mathbb{R}) & & \{0\} & & & & \begin{array}{c} Hom(\pi_2(G),\mathbb{R}) \bigoplus \\ Hom(\pi_1(G), H^1_{GF}(\mathfrak{g},\mathbb{R})) \end{array} & & Hom(\pi_2(G),\mathbb{R})
\end{array}
$$

Remark. This theorem stands for any Lie group G and $\mathbb{R}$ can be replaced by a more general module. See [N], p.3 for a more general statement.

We can use this diagram in our case, for $G = Diff_0^+(\mathbb{T}^2)$, to establish the following result:

Theorem 3.12. *For $g = 1$, we have:*

$$\begin{cases} H^1_{diff}(G, \mathbb{R}) = \{0\} \\ H^2_{diff}(G, \mathbb{R}) \cong \mathbb{R}^2 \end{cases}$$

Proof. We know, thanks to corollary 3.4 that G is homotopically equivalent to $\mathbb{T}^2$ itself. Thus, we have $\pi_1(G) \cong \mathbb{Z}^2$ and we get $Hom(\pi_1(G), \mathbb{R}) \cong \mathbb{R}^2$. We also have $\pi_2(G) = \{0\}$, so $Hom(\pi_2(G), \mathbb{R}) = \{0\}$.

Also, remember that $H^1_{GF}(\mathfrak{g}, \mathbb{R}) = H^2_{GF}(\mathfrak{g}, \mathbb{R}) = \{0\}$ (see the proof of corollary 3.6).

Using these informations, in our case, the previous diagram becomes:

$$\begin{array}{ccccccccc}
\{0\} & & \{0\} & & & & \mathbb{R}^2 & & \{0\} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
H^1_{diff}(G, \mathbb{R}) & \to & H^1_{diff}(\widetilde{G}, \mathbb{R}) & \to & \mathbb{R}^2 & \to & H^2_{diff}(G, \mathbb{R}) & \to & H^2_{diff}(\widetilde{G}, \mathbb{R}) \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
\{0\} & = & \{0\} & \to & \mathbb{R}^2 & & \{0\} & = & \{0\} \\
\downarrow & & \downarrow & & & & \| & & \| \\
\mathbb{R}^2 & & \{0\} & & & & \{0\} & & \{0\}
\end{array}$$

Using the first and the second columns, we obtain $H^1_{diff}(G, \mathbb{R}) = H^1_{diff}(\widetilde{G}, \mathbb{R}) = \{0\}$ and the last column also gives $H^2_{diff}(\widetilde{G}, \mathbb{R}) = \{0\}$.

Now, the second row is:

$$\{0\} \longrightarrow \mathbb{R}^2 \longrightarrow H^2_{diff}(G, \mathbb{R}) \longrightarrow \{0\}$$

And because this row is exact, we get $H^2_{diff}(G, \mathbb{R}) \cong \mathbb{R}^2$. □

Remark. This show that theorem 3.7 is false for $g = 1$, but most importantly, the fact that $H^2_{diff}(G, \mathbb{R}) \neq \{0\}$ shows that there exists non trivial abelian extensions for G. More precisely, because $dim(H^2_{diff}(G, \mathbb{R})) = 2$, there are two different non trivial abelian extensions. Theses extensions can be seen like "Virasoro groups" for the torus. See $[N]$ for more details about abelian extensions and $[GR]$ for a complete study of the Virasoro group for the circle.

4 Conclusion

With the spectral sequence methods we managed to obtain informations about the first cohomology spaces of G and $\mathfrak{g}$. The study of the spectral sequences for $g = 1$ is much harder but may be achieved by writing in detail the expressions of the generators of the cohomology and the differentials for the spectral sequence.

Concerning minimal models, even if the cohomology of $\mathfrak{g}$ is not finitely generated, it may be still possible to use a model to obtain additional information about it, but it would be pretty laborious.

There are maybe two other methods to study the cohomology of G and $\mathfrak{g}$. The first way is to try to build cocycles on $Diff^+(\Sigma_g)$ by integration on the fiber. This method is used by L.Guieu and C.Roger in the case of the circle. See $[GR]$, p.222.

The other possible method uses diagonal cohomology. This cohomology is for example introduced in $[G]$, p.197. It may be possible to define a diagonal subcomplex of $C^\infty(G^q)$ or $C^\infty(G^q, \Omega^p(G))$ such that a link could be established between the diagonal cohomology of G and the diagonal cohomology of $\mathfrak{g}$. This can give extra informations about the spectral sequences of the double complex.

Looking at the non trivial abelian extensions of $Diff_0^+(\mathbb{T}^2)$ can also be an interesting continuation to this study.

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