The research of self-duality equations on a Riemann surface

HU Zhilin

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Introduction

We check some details of Hitchin's paper[1] in this article. In the first chapter of Hitchin's paper, firstly he defines that a principal connection over $\mathbb{R}^4$ is said to satisfy the *self-dual Yang-Mills equations* if its curvature form is invariant under the Hodge star operator. Then he restrict the principal connection to $\mathbb{R}^2$ and defines *Higgs field*. Thus the self-duality equation becomes coordinate invariant and conformally invariant. So this equation can be generalized to Riemann surface. Then he give two examples satisfying the self-duality equation.

In the second chapter of Hitchin's paper, he discuss two theorems: first is *vanishing theorem* which states some conditions that the solutions of self-duality equation should satisfy. And first condition is the notion of *stability*. Next theorem discuss the condition that make two solutions gauge-equivalent.

In the third chapter, he study this notion of stability from an algebro-geometric point. Later he use Chapter 4 to construct a moduli space for solutions of the self-duality equations and analyse its differential geometric structure.

In my note, I study and introduce some results for understanding this paper. In Preliminary, I recall some results of Lie group then introduce principal connection from two definitions and express locally principal connection form. Then I focus on covariant derivative and principal connections on the frame bundle associated to a vector bundle. In fact there is correspondence between linear connections on vector bundles and principal connections on the associated frame bundle. One can induce the other. Later I write some resluts about characteristic classes and classifying spaces, which is for second chapter of Hitchin's paper.

CHAPTER 0. Preliminary

1. Lie group and Lie algebra

0.1.1 exponential mapping and Maurer-Cartan form

Let G be a Lie group. We denote the left multiplication (resp. right multiplication) by L_a (resp.R_a). Then all left-invariant (or right-invariant) vector fields consist of its Lie algebra $\mathfrak{g}$. And $X \mapsto X_e$, where e is identity of G, is a linear isomorphism from $\mathfrak{g}$ onto the tangent space T_eG, Hence $\mathfrak{g}$ can be regarded as T_eG. We know that for any manifold M, a vector field X can generate a local one-parameter subgroup of local transformations of M and $A \in \mathfrak{g}$ can generate a global one-parameter subgroup a_t of G. $A \mapsto a_1$ is called the *exponential mapping* and denoted by $A \mapsto \exp A$. And we have $a_t = \exp tA$. In this article, we only use the matrix Lie groups, i.e. the closed subgroup of $\mathrm{GL}(\mathbb{R}, n)$ or $\mathrm{GL}(\mathbb{C}, n)$[2], so let us introduce the exponential mapping of matrix. For a matrix A, we define

$$\exp(A) = I + \frac{A}{1} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

We can calculate Jordan form of A, then calculate $\exp(A)$.

A differential form ω on G is called left-invariant if $(L_a)^*\omega = \omega$ for any $a \in G$.

The *canonical 1-form* or *left-invariant Maurer-Cartan form* θ on G is the $\mathfrak{g}$-valued 1-form defined by[3, Ch.3, Maurer-Cartan Form]

$$\theta_g(v) = (L_{g^{-1}})_*(v)$$

for $v \in T_gG, g \in G$. θ is left-invariant: for any $h \in G$,

$$(L_h^*\theta)_g(v) = \theta_{hg}(L_{h*}(v)) = L_{(hg)^{-1}*}L_{h*}(v) = L_{g^{-1}*}(v) = \theta_g(v)$$

Since in this article we only consider the matrix Lie groups, we introduce the Maurer-Cartan form of Lie groups[3, Ch.3, Example 1.7]: If $g : G \longrightarrow GL(n)$ is embedding map into the general linear group, then its Maurer-Cartan form is $g^{-1}dg$.

0.1.2 Fundamental vector field

If G acts on manifold M on the right, then for $A \in \mathfrak{g}$, the action of one-parameter subgroup e^{tA} on M induces a vector field on M, i.e. $p \mapsto \frac{d}{dt}\big|_{t=0} pe^{tA}$, which will be denoted by A^* and called the fundamental vector field corresponding to A.

Now if $\pi : P \longrightarrow M$ is a principal G-bundle, we call $VP := \ker \pi$ consisting of $\ker d_x\pi$ for any $x \in P$ by vertical bundle of P. We have the following proposition[4, Prop.27.18]:

PROPOSITION 0.1.1. *For any $p \in P, A \in \mathfrak{g}$, the mapping $A \mapsto A_p^*$ is an isomorphism of $\mathfrak{g}$ onto the vertical tangent space V_pP*

Proof. We know that $A_p^* = \frac{d}{dt}\big|_{t=0} pe^{tA}$; then we have

$$d_p\pi(A_p^*) = \frac{d}{dt}\bigg|_{t=0} \pi(pe^{tA}) = 0$$

since pa is in the same fiber as p for any $a \in G$, i.e. pe^{tA} is constant. Hence $A_p^* \in V_pP$.

If $A_p^* = 0$, i.e. $\frac{d}{dt}\big|_{t=0} pe^{tA} = 0$ or pe^{tA} is constant around $t = 0$ then A must be zero. Thus $A \mapsto A_p^*$ is injective.

Around p, there is a local trivialization onto $U \times G$ where U is a neighborhood of $\pi(p)$. If under this local trivialization p is $(\pi(p), g) \in U \times G$, then $d_p\pi$ maps $(a, b) \in T_{\pi(p)}M \oplus T_gG$ to $a \in T_{\pi(p)}M$, so $\ker d_p\pi = T_gG$ i.e. $V_pP \cong T_gG$ and they have same dimension with $\mathfrak{g} \cong T_eG$. Thus $A \mapsto A_p^*$ is isomorphism. ∎

0.1.3 Adjoint representation

For $g \in G$, the map Ψ_g defined by $h \mapsto ghg^{-1}$ is an inner automorphism of G and also a Lie group homomorphism. Then define ad_g to be the derivative of Ψ_g at the identity:

$$\mathrm{ad}_g = d_e\Psi_g : T_eG \cong \mathfrak{g} \longrightarrow T_eG \cong \mathfrak{g}$$

So $\mathrm{ad}_g \in \mathrm{GL}(\mathfrak{g})$. Now we get a representation:

$$\mathrm{ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$$
$$g \mapsto \mathrm{ad}_g$$

which is called the adjoint representation of Lie group. For the case of matrix Lie groups, we have[2, Ch.3]

$$\mathrm{ad}_g(X) = gXg^{-1}$$

for $g \in G, X \in \mathfrak{g}$. $ad : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ can induce a Lie algebra homomorphism, also denoted by ad, from $\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, which is called the adjoint representation of Lie algebra. For the case of matrix Lie groups, we have [2, Ch.3]

$$\mathrm{ad}_X(Y) = [X, Y]$$

for $X, Y \in \mathfrak{g}$. Now we can get another useful proposition from [5, Ch.I, Prop.5.1]:

PROPOSITION 0.1.2. *Let A^* be the fundamental vector field corresponding to $A \in \mathfrak{g}$. For each $a \in G$, $R_{a*}(A^*)$ is the fundamental vector field corresponding to $\mathrm{ad}_{a^{-1}}(A) \in \mathfrak{g}$.*

0.1.4 Compact real form

We assume here that G is the compact real form of a complex of a complex Lie group:

DEFINITION 0.1.3. *[2, p.170][6, p.348] A complex Lie algebra $\mathfrak{g}$ is* reductive *if there exists a compact Lie group K such that $\mathfrak{g} \cong \mathfrak{k}_{\mathbb{C}}$ where $\mathfrak{k}$ is the Lie algebra of K. A complex Lie algebra $\mathfrak{g}$ is* semisimple *if it is reductive and the center of $\mathfrak{g}$ is trivial.*

If $\mathfrak{g}$ is a semisimple Lie algebra, a real subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is a compact real form *of $\mathfrak{g}$ if $\mathfrak{k}$ is isomorphic to the Lie algebra of some compact Lie group and every element Z of $\mathfrak{g}$ can be expressed uniquely as $Z = X + iY$ with $X, Y \in \mathfrak{k}$.*

Let G_c be a complex connected Lie group with Lie algebra $\mathfrak{g}_c$, G a real connected Lie subgroup of G_c with a real Lie algebra $\mathfrak{g} \subset \mathfrak{g}_c$. G is said to be a compact real form *of G_c if $\mathfrak{g}$ is a compact real form of $\mathfrak{g}_c$.*

and $*$ is the corresponding anti-involution on the complex Lie algebra[2, p.171]: let $\mathfrak{g} := \mathfrak{k}_{\mathbb{C}}$ be a reductive Lie algebra, then the operator $*$ on $\mathfrak{g}$ is defined by the formula

$$(X_1 + iX_2)^* = -X_1 + iX_2$$

for $X_1, X_2 \in \mathfrak{k}$.

2. Connections, curvatures and covariant derivative

0.2.1 Linear connections in associated vector bundles

This section cames from [7, Sec.1.5]. Let E be a complex vector bundle over M. Let E^* be the dual vector bundle of E. The dual pairing

$$< , >: E_x \times E_x^* \longrightarrow \mathbb{C}$$

induces a dual pairing

$$< , >: \Omega^0(M, E^*) \times \Omega^0(M, E) \longrightarrow \Omega^0(M)$$

Given a linear connection E on E, we can define the dual connection D^* on E^* by the following formula:

$$d < \xi, \eta > = < D\xi, \eta > + < \xi, D\eta >$$

for $\xi \in \Omega^0(M, E), \eta \in \Omega^0(M, E^*)$. Given a local frame $e = (e_1, \ldots, e_r)$ of E, let t be the dual local frame of E^*. We consider e as a row vector and t as a column vector. If ω denote the matrix of connection 1-forms of D relative to e, i.e. $Ds = s\omega$, then we have

$$D^*t = -\omega t$$

If Θ is the curvature form of D relative to e so that

$$D^2 e = e\Theta$$

then relative to t we have

$$D^{*2} t = -\Theta t$$

We shall now consider two complex vector bundles E and F over the same base M. Let D_E and D_F be connections in E and F. Then we can naturally define a connection $D_E \oplus D_F$ in $E \oplus F$. We also naturally define $D_{E\otimes F}$ in $E \otimes F$ by

$$D_{E\otimes F} = D_E \otimes I_F + I_E \otimes D_F$$

where I_E and I_F denote the identity transformation of E and F. If we denote the curvatures of D_E and D_F by R_E and R_F, then $R_E \oplus R_F$ is the curvature of $D_E \oplus D_F$ and

$$R_E \otimes I_F + I_E \otimes R_F$$

is the curvature of $D_{E\otimes F}$. If $\omega_E, \omega_F, \Theta_E, \Theta_F$ are the connection and curvature forms, then the connection and curvature forms of $D_E \oplus D_F$ are given by

$$\begin{pmatrix} \omega_E & 0 \\ 0 & \omega_F \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \Theta_E & 0 \\ 0 & \Theta_F \end{pmatrix}$$

Those of $D_{E\otimes F}$ are given by

$$\omega_E \otimes I_p + I_r \otimes \omega_F \qquad \text{and} \qquad \Theta_E \otimes I_p + I_r \otimes \Theta_F$$

where I_p and I_r denote the identity matrices of rank p and r.

0.2.2 Principal connections, curvatures and covariant derivative on principal bundle

This section is from [4, Part 3] and [5, Ch.II]. Let (P, π, M, G) be a G-principal bundle P over base M. We have two definitions of connections on P, one is by a kind of subbundle of TP and another is by $\mathfrak{g}$-valued 1-forms.

DEFINITION 0.2.1. *A distribution on P is a smooth subbundle of TP. We call a distribution HP by* horizontal *if $TP = VP \oplus HP$. We call a distribution Q by* right-invariant *if $Q_{pa} = (R_a)_* Q_p$ for any $p \in P, a \in G$ where Q_p is the fiber at p of Q and R_a is the transformation of P by right multiplication of a. A right-invariant horizontal distribution HP is called a connection on P.*

If ω is the bundle projection onto VP, then $HP = \ker \omega$ and of course ω can be regarded as a VP-valued 1-form on P. From 0.1.1 we know that Lie algebra $\mathfrak{g}$ is isomorphic to the standard fiber of VP via the morphism of Lie algebra $A \mapsto A^*$ for $A \in \mathfrak{g}$ where A^* is the fundamental vector field, hence ω also can be regarded as a $\mathfrak{g}$-valued 1-form on P, called the *connection form* of HP. We have the following theorem[5, Ch.II, Sec.1]

THEOREM 0.2.2. *The connection form ω satisfies the following conditions:*

(a) $\omega_p(A^*_p) = A$ for any $A \in \mathfrak{g}$ and $p \in P$;

(b) $(R_a)^*\omega = \mathrm{ad}_{a^{-1}}\omega$, for every $a \in G$, where R_a is the transformation of P by a on the right and ad *is the adjoint representation of $\mathfrak{g}$.*

Conversely, given a $\mathfrak{g}$-valued 1-form ω on P satisfying the two above conditions, there is a unique principal connection in P whose connection form is ω.

Proof. Let ω be the connection form. The condition (a) follows immediately from the definition of ω. Since every vector field of P can be decomposed as a sum of a horizontal vector field and a vertical vector field, it is sufficient to verify (b) in the following two special cases: (1) X is horizontal and (2) X is vertical.

If X is horizontal, so is $(R_a)_* X$ for any $a \in G$ since HP is right-invariant. Then $\omega((R_a)_* X) = 0 = \mathrm{ad}_{a^{-1}}\omega(X)$.

If X is vertical, we can assume that X is a fundamental vector field A^* for $A \in \mathfrak{g}$. Then $(R_a)_* X$ is also fundamental vector field corresponding to $\mathrm{ad}_{a^{-1}} A$ by 0.1.2. Thus we have

$$(R^*_a\omega)(X) = \omega((R_a)_* X) = \omega((\mathrm{ad}_{a^{-1}} A)^*) = \mathrm{ad}_{a^{-1}} A = \mathrm{ad}_{a^{-1}}\omega(X)$$

Conversely, given a $\mathfrak{g}$-valued 1-form ω satisfying (a) and (b), we define $HP = \ker \omega$. Then we have

$$(R_a)_* Q_p = \{(R_a)_* X : X \in T_p P, \omega_p(X) = 0\} = \{X \in T_{pa}P : \omega_p((R_a)_*^{-1}X) = 0\}$$

From condition (b) we have $\omega_p((R_a)_*^{-1}X) = (R_a^* \omega)_{pa}(X) = \mathrm{ad}_{a^{-1}}\omega_{pa}(X) = 0$ which is equivalent to $\omega_{pa}(X) = 0$ hence $(R_a)_* Q_p = Q_{pa}$ i.e. HP is right-invariant. Since for any $p \in P$, $\mathrm{Im}\, \omega_p = \mathfrak{g} = (VP)_p$ and $T_pP = \ker \omega_p \oplus \mathrm{Im}\omega_p = (HP)_p \oplus (VP)_p$, we know that HP is horizontal. And obviously the connection form of HP is ω. □

We shall express a principal connection on P by a family of forms each defined in an open subset of the base manifold M. Let $\{U_\alpha\}$ be an open covering of M with a family of local trivialization $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$, then the corresponding family of transition functions are $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$. For each α, let $\sigma_\alpha : U_\alpha \longrightarrow P$ be the cross section on U_α defined by $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e), x \in U_\alpha$ where e is the identity of G. Let θ be the (left invariant) canonical 1-form on G.

For each non-empty $U_\alpha \cap U_\beta$, we define a $\mathfrak{g}$-valued 1-form $\theta_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ by

$$\theta_{\alpha\beta} = \psi_{\alpha\beta}^* \theta$$

For each α we define a $\mathfrak{g}$-valued 1-form ω_α on U_α by

$$\omega_\alpha = \sigma_\alpha^* \omega$$

Then we have third manner of describing a principal connection on P[5, Ch.II, Sec.1]:

THEOREM 0.2.3. *The forms $\theta_{\alpha\beta}$ and ω_α satisfy the following condition:*

$$\omega_\beta = \mathrm{ad}_{\psi_{\alpha\beta}^{-1}} \omega_\alpha + \theta_{\alpha\beta}$$

Conversely, for every family of $\mathfrak{g}$-valued 1-forms $\{\omega_\alpha\}$ each defined on U_α and satisfying the above condition, there is a unique connection on P which gives such family of $\mathfrak{g}$-valued 1-forms $\{\omega_\alpha\}$.

Proof. If $U_\alpha \cap U_\beta$ is non-empty, $\psi_\alpha \circ \psi_\beta^{-1}(x, e) = (x, \psi_{\alpha\beta}(x))$ for all $x \in U_\alpha \cap U_\beta$ then $\psi_\beta^{-1}(x, e) = \psi_\alpha^{-1}(x, \psi_{\alpha\beta}(x)) = \psi_\alpha^{-1}(x, e)\psi_{\alpha\beta}(x)$ since ψ_α is G-homeomorphism and then $\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$. For every vector $X \in T_xM$, the vector $\sigma_{\beta*}(X) \in T_uP$ where $u = \sigma_\beta(x)$, is the image of $(\sigma_{\alpha*}(X), \psi_{\alpha\beta*}(X)) \in T_{u'}P \oplus T_aG$ where $u' = \sigma_\alpha(x)$ and $a = \psi_{\alpha\beta}(x)$, under the differential of the mapping $P \times G \longrightarrow P$. By Leibniz's formula [5, Ch.I, Prop.1.4] we have

$$\sigma_{\beta*}(X) = (R_a)_*(\sigma_{\alpha*}(X)) + u'_*(\psi_{\alpha\beta*}(X))$$

where u'_* is the differential of the mapping $g \mapsto u'g$ from G into P. Taking the values of ω_u on both sides of the equality, we obtain

$$\omega_u(\sigma_{\beta*}(X)) = \omega_u((R_a)_*\sigma_{\alpha*}(X)) + \omega_u(u'_*\psi_{\alpha\beta*}(X)) \tag{0.2.1}$$

We have

$$\omega_u(\sigma_{\beta*}(X)) = (\sigma_\beta^*\omega)_x(X) = (\omega_\beta)_x(X)$$

and

$$\omega_u((R_a)_*\sigma_{\alpha*}(X)) = (R_a^*\omega)_{u'}(\sigma_{\alpha*}(X)) = \mathrm{ad}_{a^{-1}}\omega_{u'}(\sigma_{\alpha*}(X)) = \mathrm{ad}_{a^{-1}}(\omega_\alpha)_x(X)$$

by condition (b) of Theorem 0.2.2. Let $A \in \mathfrak{g}$ be the vector field such that $A_a = \psi_{\alpha\beta*}(X) = \frac{d}{dt}\big|_{t=0} ae^{tA}$;then from the definition of Maurer-Cartan form we know that $\theta_a(\psi_{\alpha\beta*}(X)) = L_{a^{-1}*}A_a = A_e$ or A. From the definition of u'_* we have

$$u'_*(\psi_{\alpha\beta*}(X)) = \frac{d}{dt}\bigg|_{t=0} u'ae^{tA} = \frac{d}{dt}\bigg|_{t=0} \sigma_\alpha(x)\psi_{\alpha\beta}(x)e^{tA}$$

that is the value of the fundamental vector field A^* at $u = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$. From the condition (a) of Theorem 0.2.2 we have

$$\omega_u(u'_*\psi_{\alpha\beta*}(X)) = \omega_u(A_u^*) = A = \theta_a(\psi_{\alpha\beta*}(X)) = (\psi_{\alpha\beta}^*\theta)_x(X) = (\theta_{\alpha\beta})_x(X)$$

Finally we have for any $x \in M, X \in T_xM$

$$(\omega_\beta)_x(X) = \mathrm{ad}_{a^{-1}}(\omega_\alpha)_x(X) + (\theta_{\alpha\beta})_x(X)$$

For the converse case, we can define ω on $\pi^{-1}(U_\alpha)$ by the pullback

$$\pi^{-1}(U_\alpha) \xrightarrow{\psi_\alpha} U_\alpha \times G \xrightarrow{\text{proj.}} U_\alpha$$

Then we can verify that such ω is well-defined $\mathfrak{g}$-valued 1-form on P and satisfies the two conditions of Theorem 0.2.2 and gives $\{\omega_\alpha\}$ on U_α. □

Later we call the family of $\{\omega_\alpha\}$ by the connection form in local expression.

REMARK 0.2.4. *For matrix Lie group, $\theta_{\alpha\beta} = \psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta}$ from Section 0.1. Then the formula (0.2.3) is*

$$\omega_\beta = \psi_{\alpha\beta}^{-1}\omega_\alpha\psi_{\alpha\beta} + \psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta}$$

Now consider Lie bracket of Lie algebra-valued forms. Let $\alpha = \sum_i \alpha^i e_i, \beta = \sum_j \beta^j e_j$, where α_i, β_j are ordinary differential forms, e_i are elements of Lie algebra $\mathfrak{g}$. Then we define

$$[\alpha, \beta] = \sum_{i,j} \alpha^i \wedge \beta^j [e_i, e_j]$$

Then we can define the curvature:

DEFINITION 0.2.5. *The* curvature *of the connection given by 1-form ω is the $\mathfrak{g}$-valued 2-form*

$$\Xi = d\omega + \frac{1}{2}[\omega, \omega]$$

For matrix Lie group, the above formula can[4, Prop.21.7] be writen as $\Xi = d\omega + \omega \wedge \omega$.

Now we introduce covariant derivative in local expression from[4, Th.31.19] Before that, we introduce the associated bundle. Let $\pi : P \longrightarrow M$ be a principal G-bundle and F a manifold on which G acts on the left. Then we can replace the fiber G of P by F, the transition functions are same. On the product manifold $P \times F$, let G act on the right as follows: an element $a \in G$ maps $(u, \xi) \in P \times F$ into $(ua, a^{-1}\xi) \in P \times F$. The quotient space of $P \times F$ by this group action is denoted by $E = P \times_G F$. Since $\pi(ua) = \pi(u)$ for any $u \in P, a \in G$, the mapping which maps (u, ξ) into $\pi(u)$ induces a mapping π_E of E into M. Every point $x \in M$ has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$. Identifying $\pi^{-1}(U)$ with $U \times G$, we see that the action of G on $\pi^{-1}(U) \times F \subset P \times F$ on the right is given by

$$(x, a, \xi)b \mapsto (x, ab, b^{-1}\xi) \qquad \text{for} \quad (x, a, \xi) \in U \times G \times F \quad \text{and} \quad b \in G$$

We use $[x, a, \xi]$ to represent the conjugacy class, then $\tau([x, a, \xi]) := (x, a\xi)$ is well-defined mapping of $\pi_E^{-1}(U)$ into $U \times F$. It is bijection whose inverse mapping is $\tau^{-1}(x, \xi) := [x, e, \xi]$ where e is identity of G. Then from these bijections can given E a differentiable structure such that $\pi_E^{-1}(U)$ is diffeomorphic to $U \times F$. So E is a G-bundle over M with standard fiber F, which is called the associated bundle. For two open subsets U, V of M, if $\psi_{UV} : U \cap V \longrightarrow G$ is transition function, then for $\tau_U^{-1}(x, \xi) = [x, e, \xi]$, where $(x, e) \in U \times G$ corresponds to $(x, \psi_{UV}(x)) \in V \times G$, we have $\tau_V \circ \tau_U^{-1}(x, \xi) = (x, \psi_{UV}(x)\xi)$, so the transition functions of E are same as of P. And then we also can use the transition functions of P and F to construct the associated bundle E by setting the transition functions of E as the left action of $\psi_{UV}(x)$[3, Ch.1, Sec.3, Construction of Bundles].

THEOREM 0.2.6. *Let $\pi : P \longrightarrow M$ be a principal G-bundle with the famiily of local expressions of connection $\{\omega_\alpha\}$ on $\{U_\alpha\}$, $\rho : G \longrightarrow \mathrm{GL}(V)$ be a finite-dimensional complex representation of G, and $E = P \times_\rho V$ the associated vector bundle. If $\varphi \in \Omega^k(M, E)$ whose expression on U_α under local frame $(e_1, \ldots, e_r)$ is $\sum \varphi^i e_i$, then its covariant derivative is given by*

$$D\varphi = d\varphi + \rho(\omega_\alpha)\varphi$$

where $d\varphi = \sum(d\varphi^i)e_i$, $\rho(\omega_\alpha)$ acts on the value of ϕ.

Of course, if ρ is the adjoint representation of G, then $D\varphi = d\varphi + [\omega, \varphi]$. If X is a vector field on M, we call $(D\varphi)(X)$ by the covariant derivative in the direction of X.

0.2.3 Connections on a frame bundle

Now we need to consider an important example of principal bundles and principal connections: frame bundle associated to a vector bundle and principal connection determined by a linear connection on this vector bundle. We recall that for a vector bundle $E \longrightarrow M$, a frame on $x \in M$ is an ordered basis e_x of E_x, equivalent to a linear isomorphism $p : \mathbb{R}^r \longrightarrow E_x$. Let F_x be the set of all frames on x; $g \in \mathrm{GL}(\mathbb{R}, r)$ rightly acts F_x by $p \mapsto p \circ g$, which is obviously free and transitive. Then we have

$$\mathrm{Fr}(E) := \bigsqcup_{x \in M} F_x$$

and a natural projection $\pi : \mathrm{Fr}(E) \longrightarrow M$. If (U, φ) is a local trivialization of E, then $\varphi_x : E_x \longrightarrow R^r$ is linear isomorphism, so we have a bijection $\psi : \pi^{-1}(U) \longrightarrow U \times \mathrm{GL}(\mathbb{R}, r)$ given by

$$\psi_(x, p) = (x, \varphi_x \circ p)$$

Thus every $\pi^{-1}(U)$ and then $\mathrm{Fr}(E)$ are given a topology such that $\mathrm{Fr}(E)$ is a principal $\mathrm{GL}(\mathbb{R}, r)$-bundle. Clearly, the transition functions of $\mathrm{Fr}(E)$ are same as E. And then each vector bundle corresponds bijectively to the associated frame bundle, so we can regard vector bundle as the associated frame bundle.

If E is given a Riemann metric, we also focus on the orthonormal frame bundles, i.e. the set of all orthonormal frames(ordered orthonormal basis) or equivalently all distance-preserved linear isomorphism $p : \mathbb{R}^r \longrightarrow E_x$. In this case, for (U, φ) the isomorphism $\varphi_x : E_x \longrightarrow \mathbb{R}^r$ is supposed as a distance-preserved map, then the transition functions go into the orthogonal group $O(r)$, in other words, $O(r)$ is a reduction of $\mathrm{GL}(\mathbb{R}, r)$. Moreover, if E is oriented vector bundle, then the transition functions go into the special orthogonal group $SO(r)$, in other words $SO(r)$ is a reduction of $O(r)$ and then $\mathrm{GL}(\mathbb{R}, r)$.

Note that for the complex vector bundles E, if E is given an hermitian metric, then $O(r)$ wil become $U(r)$, which is a reduction of $\mathrm{GL}(\mathbb{C}, r)$. And complex vector bundles are oriented for the underlying real vector bundle, so $U(r)$ also is reducible to $SU(r)$.

Of course, for the representation $i : \mathrm{GL}(\mathbb{C}, r) \longrightarrow \mathrm{GL}(\mathbb{C}, r)$, the associated bundle $\mathrm{Fr}(E) \times_i \mathbb{C}^r = E$ if E is complex vector bundle. For the case of real vector bundle, it is similar.

Recall that for a linear connection ∇ on E, if a local frame $e : U \longrightarrow \mathrm{Fr}(E)$ where U is an open subset of M, is given, then ∇ can be expressed as a matrix of 1-forms ω_U, which is also an element in $\mathfrak{gl}(\mathbb{R}, r)$ or $\mathfrak{gl}(\mathbb{C}, r)$, i.e. ω_U can be viewed as a Lie algebra-valued 1-form over U. And these ω_U satisfy the following condition[7, Sec.1.1]:

$$\omega_U = g_{VU}^{-1} \omega_V g_{VU} + g_{VU}^{-1} dg_{VU} \qquad \text{on} \quad U \cap V$$

where $\{U\}$ is an open cover of M with a local frame $e : U \longrightarrow \mathrm{Fr}(E)$. Via Theorem 0.2.3 and Remark 0.2.4, these Lie algebra-valued 1-forms define a unique principal connection on the frame bundle $\mathrm{Fr}(E)$. And from Theorem 0.2.6, this principal connection induces a same covariant derivative on the vector bundle E with ∇. Another description of principal connection induced by a linear connection is following method:

Let $\eta : E \longrightarrow M$ be a vector bundle, $\nabla : \Omega(M, E) \longrightarrow \Omega(M, E)$ be a linear connection on E. for a section $s \in \Omega^0(M, E)$, ∇s is called *covariant derivative* of s corresponding to ∇. For $X \in T_pM$, the eval of ∇s at X, which is an element of E_x, denoted by $\nabla_X s$, is called the *covariant derivative* of s in the direction X. And if X is a section of M, then $\nabla_X s$ is also section.

DEFINITION 0.2.7. *A section $s \in \Omega(M, E)$ is* parallel along a curve *$c : [a, b] \longrightarrow M$ if $\nabla_{c'(t)} s = 0$ for $a \leq t \leq b$.*

Given a local frame $(e_1, \ldots, e_r)$ on U, section $s = \sum s^i e_i$ and let connection form of ∇ be $\omega = (\omega_i^j)$ then s is parallel along c if and only if $(s_1, \ldots s_r)$ satisfies the following ODE:

$$\frac{ds^i}{dt} + \sum_j \omega_j^i(c'(t)) s^j = 0$$

If s_0 is an element of $E_{c(a)}$, by the existence and uniqueness of ODE, there is a unique curve $s : [a, b] \longrightarrow E$ such that $s(a) = s_0$ and $s(t) \in E_{c(t)}$ and s is parallel along a curve c (here we can only consider a section along c). Then $s(b) \in E_{c(b)}$ is called the *parallel transport* of s_0 along c. The map $s_0 \mapsto s(b)$ of $E_{c(a)}$ into $E_{c(b)}$ is called *parallel translation* from $E_{c(a)}$ to $E_{c(b)}$. We have the following theorem[4, Theorem 29.2]:

THEOREM 0.2.8. *Let $\eta : E \longrightarrow M$ be a vector bundle with a connection ∇ and let $c : [a, b] \longrightarrow M$ be a smooth curve in M. There is a unique parallel translation $\varphi_{a,b}$ from $E_{c(a)}$ to $E_{c(b)}$ along c. This parallel translation is a linear isomorphism.*

A *parallel frame along* c is a collection $(e_1(t), \ldots, e_r(t)), t \in [a, b]$ of parallel sections such that for each t, the elements $e_1(t), \ldots, e_r(t)$ form a basis of $E_{c(t)}$.

Let $\pi : \mathrm{Fr}(E) \longrightarrow M$ be the frame bundle associated to E. A curve $\tilde{c}(t)$ in $\mathrm{Fr}(E)$ is called a *lift* of c if $c(t) = \pi(\tilde{c}(t))$. It is called *horizontal lift* if in addition $\tilde{c}(t)$ is a parallel frame along c.

By Theorem 0.2.8, if a collection of parallel sections $(s_1(t, \ldots, s_r(t))$ forms a basis at one time t, then it forms a basis at every time $t \in [a, b]$. For every smooth curve $c : [a, b] \longrightarrow M$ and ordered basis $(s_{1,0}, \ldots, s_{r,0})$ for $E_{c(a)}$, there is a unique parallel frame along c whose value at a is $(s_{1,0}, \ldots, s_{r,0})$. In terms of the frame bundle $\mathrm{Fr}(E)$, this shows the existence and uniqueness of a horizontal lift with a specified initial point in $\mathrm{Fr}(E)$.

Now we define a principal connection on the frame bundle from a linear connection. For $x \in M$ and $e_x \in \mathrm{Fr}(E)_x$, a tangent vector $v \in T_{e_x}(\mathrm{Fr}(E))$ is said to be *horiozntal* if there is a curve c through x such that $v = \tilde{c}'(0)$ where $\tilde{c}$ is the unique horizontal lift of c to $\mathrm{Fr}(E)$ starting at e_x. we have the following proposition[4, Prop.29.6]:

PROPOSITION 0.2.9. *Let $\pi : E \longrightarrow M$ be a smooth vector bundle with a connection over a manifold M of dimension n. For $x \in M$ and e_x an ordered basis for the fiber E_x, the subset H_{e_x} of horizontal vectors in $T_{e_x}(\mathrm{Fr}(E))$ is a vector space of dimension n, and $\pi_* : H_{e_x} \longrightarrow T_x M$ is a linear isomorphism.*

From the result of the standard linear algebra, we know that $T_{e_x}\mathrm{Fr}(E) = \ker \pi_* \oplus \mathrm{Im}\pi_* \cong V_{e_x}\mathrm{Fr}(E) \oplus H_{e_x}$. And these vector subspaces H_{e_x} form a distribution on the frame bundle[4, Th.29.9]:

THEOREM 0.2.10. *A connection ∇ on a smooth vector bundle $E \longrightarrow M$ defines a distribution on the frame bundle $\pi : P = \mathrm{Fr}(E) \longrightarrow M$ such that at any $p \in P$,*

(i) $T_pP = V_p \oplus H_p$;

(ii) $(R_g)_* H_p = H_{pg}$ *for any* $g \in G = \mathrm{GL}(\mathbb{R}, r)$, *where* $R_g : P \longrightarrow P$ *is the right action of* G *on* P.

i.e. a linear connection on E defines a principal connection on the frame bundle $\mathrm{Fr}(E)$.

Recall that a connection ∇ on a vector bundle E can be represented on a local frame $(U, e_1, \ldots, e_r)$ by a connection 1-forms matrix ω_e. Such a frame $e = (e_1, \ldots, e_r)$ is in fact a section $e : U \longrightarrow \mathrm{Fr}(E)$ of the frame bundle. Now we have the following theorem[4, Th.29.10]:

THEOREM 0.2.11. *Let ∇ be a connection on a vector bundle $E \longrightarrow M$ and let ω be the principal connection on the frame bundle $\mathrm{Fr}(E)$ determined by ∇ (Theorem 0.2.10). If $e = (e_1, \ldots, e_r)$ is a frame for E over an open set U of M, viewed as a section $e : U \longrightarrow \mathrm{Fr}(E)$, and ω_e is the connection matrix of ∇ relative to the frame e, then $\omega_e = e^*\omega$.*

3. Characteristic classes and classifying spaces

0.3.1 Axiomatic descriptions of characteristic classes

Let E be a real vector bundle of rank r on base space B; $H^i(B; G)$ be the i-th singular cohomology group of B with coefficients in G. The *Stiefel-Whitney classes* of E consist of a sequence of $w_i(E) \in H^i(B; \mathbb{Z}/2\mathbb{Z})$ which satisfies the following 4 axioms[8, Sec.4 and Sec.8]:

AXIOM 1. **RANK**. $w_0(E) = 1 \in H^0(B; \mathbb{Z}/2\mathbb{Z})$ and $w_i(E) = 0$ for $i > r$.

AXIOM 2. **NATURALITY**. If $f : B' \longrightarrow B$ is a map and f^*E is the pullback bundle then $w_i(f^*E) = f^*w_i(E)$.

AXIOM 3. **WHITNEY PRODUCT FORMULA**. If E' is another vector bundle over B then

$$w_k(E \oplus E') = \sum_{i=0}^{k} w_i(E) \smile w_{k-i}(E')$$

where $\smile$ means cup product.

AXIOM 4. **NORMALIZATION**. For the tautological line bundle $\mathscr{O}(-1)$ over $\mathbb{RP}^1$, the first Stiefel-Whitney class $w_1(\mathscr{O}(-1)) \in H^1(\mathbb{RP}^1; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ is non-zero.

Let E be a complex vector bundle of rank r over a base space B. The *Chern classes* of E consist of a sequence of $c_i(E) \in H^{2i}(B; \mathbb{Z})$ which satisfies the following axioms[9, Ch.16, Sec.3]:

AXIOM 1. **RANK**. $c_0(E) = 1 \in H^0(B; \mathbb{Z})$ and $c_i(E) = 0$ for $i > r$.

AXIOM 2. **NATURALITY**. If $f : B' \longrightarrow B$ is a map then $c_i(f^*E) = f^*c_i(E)$.

AXIOM 3. **PRODUCT FORMULA**. If E' is another complex vector bundle over B then

$$c_k(E \oplus E') = \sum_{i=0}^{k} c_i(E) \smile c_{k-i}(E')$$

AXIOM 4. **NORMALIZATION**. For the tautological line bundle $\mathscr{O}(-1)$ over $\mathbb{CP}^1 = S^2$, the first Chern class $c_1(\mathscr{O}(-1))$ is the generator of $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$

Their existences and uniqueness lie in Chapter 16, Section 6 of [9].

0.3.2 Classifying spaces

Grassmann Manioflds or Grassmannians and tautological vector bundles

Grassmann Manifold is a generalization of projective space. A *real Grassmann manifold* $G(n, k)$ is defined as the space of all k-dimensional subspaces of $\mathbb{R}^n$. Now we define a chart in $G(n, k)$ in the following way[10, Ch.1, Sec.5]: Choose a base $(v_1, \dots, v_k)$ of $P \in G(n, k)$, then P can be represented by the $k \times n$ matrix:

$$\begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{k1} & \cdots & v_{kn} \end{pmatrix}$$

of rank k. Clearly any tow such matrices A, A' represent the same element of $G(n, k)$ if and only if $A = gA'$ for some $g \in \mathrm{GL}(\mathbb{R}, k)$. For every multi-index $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, let $V_{I^\circ} \subset \mathbb{R}^n$ be the $(n-k)$-dimensional subspace spanned by the standard basis vectors $\{e_j : j \notin I\}$ and let

$$U_I = \{P \in G(n, k) : P \cap V_{I^\circ} = \{0\}\}$$

that is the set of $P \in G(n, k)$ such that the minor consisting of $i_1, \dots, i_k$-th columns of one, and hence for any, matrix representation for P is non-singular. By elementary matrix transformations, any $P \in U_I$ can be represented uniquely by a matrix of the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & * & \cdots & * \\ 0 & 1 & & \vdots & * & & \\ \vdots & & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & & \cdots & 1 & * & \cdots & * \end{pmatrix}$$

The right $k \times (n-k)$ minor can be regarded as a coordinate in $\mathbb{R}^{k(n-k)}$. Now we get a bijection:

$$\varphi_I : U_I \longrightarrow \mathbb{R}^{k(n-k)}$$

Note that $\varphi_I(U_I \cap U_{I'})$ is open in $\mathbb{R}^{k(n-k)}$ for all I, I' and $\varphi_I \circ \varphi_{I'}^{-1}$ is smooth so $G(n, k)$ is smooth manifold. Clearly this local coordinate is a generalization of homogeneous coordinates of $\mathbb{RP}^n$.

For complex case, the definition is similar, in this case, $G(n, k)$ is a complex manifold, denoted by $G_{\mathbb{C}}(n, k)$. And we denote by $G_+(n, k)$ the set of all *oriented* k-dimensional subspaces of $\mathbb{R}^n$. Finally, the infinite-dimensional version of Grassmannians is provided by the *Grassmann space* $G(\infty, k)$, which is the union of the chain $G(k+1, k) \subset G(k+2, k) \subset G(k+3, k) \subset \cdots$. There are also spaces $G_{\mathbb{C}}(\infty, k), G_+(\infty, k)$.

Like tautological line bundle over $\mathbb{RP}^n$ or $\mathbb{CP}^n$, we also have tautological vector bundle $\Omega_{n,k}$ over the Grassmannians $G(n, k), G_+(n, k), G_{\mathbb{C}}(n, k)$: The total space is the set

$$\{(P, v) : P \in G(n, k), v \in P\}$$

The projection is $(P, v) \mapsto P$. The tautological vector bundle Ω_n over the infinite Grassmannians $G(\infty, n)$ is similar.

The classification theorem for vector bundles

This theorem comes from [11, Ch.2, Lec.19.4.A]

THEOREM 0.3.1. *Let X be a manifold. Then*

(i) For every n-dimensional vector bundle $E \longrightarrow X$, there exists a continuous map $f : X \longrightarrow G(\infty, n)$ such that $f^*\Omega_n = E$.

(ii) This map f is unique up to a homotopy; i.e. if $f_1^*\Omega_n \cong f_2^*\Omega_n$, then $f_1 \sim f_2$.

(iii) Conversely if $f_1 \sim f_2$ then $f_1^*\Omega_n \cong f_2^*\Omega_n$.

COROLLARY 0.3.2. *The correspondence $f \mapsto f^*\Omega_n$ establishes a bijection between the set of homotopy classes of maps $X \longrightarrow G(\infty, n)$ and equivalence classes of n-dimensional vector bundles over X.*

Or by the language of category theory, $\Omega_n \longrightarrow G(\infty, n)$ is terminal object. $\Omega_n \longrightarrow G(\infty, n)$ is called universal vector bundle. From the naturality of characteristic classes, we only need to know the characteristic classes of the universal vector bundles. But now we need more general constructions.

Classifying spaces of topological groups and homotopy classification of principal bundles

This section comes from [12]. We need some results about principal bundles.

PROPOSITION 0.3.3. *Let P and P' be principal G-bundles over B. If $\phi : P \longrightarrow P'$ is a principal bundle morphism lying over $id : B \longrightarrow B$, then ϕ is an isomorphism.*

Proof. To see that ϕ is injective, suppose $\phi(p) = \phi(q)$ for $p, q \in P$. Since ϕ lies over the identity of B, it follows that p, q must lie in the same fiber $\pi(p) = \pi(q) \in P$. Then there is a unique $g \in G$ such that $p \cdot g = q$ and $\phi(p \cdot g) = \phi(p) \cdot g = \phi(q)$. Since G-action is free, we have $g = e$ and $p = q$.

To see that ϕ is surjective, let $p' \in P'$ and $b = \pi'(p') \in B$. Choose any $p \in \pi^{-1}(b) \subset P$. Then $\pi' \circ \phi(p) = id_B \circ \pi(p)$ i.e. $\phi(p)$ and p' lie in the same fiber, therefore $p' = \phi(p) \cdot g = \phi(p \cdot g)$ for some $g \in G$.

To see that ϕ^{-1} is continuous, it suffices to consider locally. Thus suppose $\pi^{-1}(U) \cong U \times G$ and $\pi'^{-1}(U) \cong U \times G$. Then ϕ locally express as

$$\phi : (b, g) \mapsto (b, \phi'(b, g)) = (b, \phi'(b, e)g)$$

for some $\phi' : U \times G \longrightarrow G$ satisfying $\phi'(b, gh) = \phi'(b, g)h$. Thus ϕ^{-1} has local form

$$\phi^{-1} : (b, g) \mapsto (b, \phi'(b, e)^{-1}g)$$

which is clearly continuous. □

PROPOSITION 0.3.4. *Let $\pi : P \longrightarrow B$ and $\pi' : Q \longrightarrow B'$ be two principal G-bundles respectively. There is a bijective correspondence between morphisms of bundles $\phi : (\pi, P, B) \longrightarrow (\pi', Q, B')$ and global sections of the associated bundle $P \times_G Q$. Here we regard Q as a left G-space with the action $g \cdot q := q \cdot g^{-1}$.*

PROPOSITION 0.3.5. *If $\pi : P \longrightarrow B'$ is a principal G-bundle and if $f_0 \sim f_1 : B \longrightarrow B'$ are homotopic maps, then the pullback bundles $f_0^*(P)$ and $f_1^*(P)$ over B are isomorphic.*

DEFINITION 0.3.6. *A principal G-bundle $\pi : EG \longrightarrow BG$ is said to be* universal *if the total space EG is contractible.*

A topological space is said to be weakly contractible if all of its homotopy groups are trivial. For CW-complex, since we have Whitehead's Theorem[11, Ch.1, Sec.11.5]:

THEOREM 0.3.7. *Let X and Y be CW-complexes, and let $f : X \longrightarrow Y$ be a continuous map. If*

$$f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

is an isomorphism for all n and x_0, then f is a homotopy equivalence.

then if X is weakly contractible and Y is one-point space, then f_* is clearly is isomorphism and then f is homotopy equivalence. Hence CW-complex is contractible if and only if it is weakly contractible.

We denote the homotopy classes of continuous maps between two topological spaces X, Y by $[X, Y]$, i.e. $\mathrm{Hom}_{\mathbf{hTop}}(X, Y)$. For any space B, $\mathcal{G}(B)$ denote the set of isomorphism classes of principal G-bundles over B. Then if $f : A \longrightarrow B$ is continuous map, $\mathcal{G}(B) \ni P \mapsto f^*(P) \in \mathcal{G}(A)$ is a function from $\mathcal{G}(B)$ to $\mathcal{G}(A)$. From Proposition 0.3.5, we can say that $\mathcal{G}$ is a contravariant functor from **hTop** to the set of isomorphism classes of principal G-bundles. The following theorem says that $\mathcal{G}$ is *representable functor.*

THEOREM 0.3.8. *Let $\pi : EG \longrightarrow BG$ be a universal G-bundle. Then for any CW-complex B, the functors $[-, BG]$ and $\mathcal{G}$ are naturally isomorphic by $[f] \mapsto [f^*EG]$.*

LEMMA 0.3.9. *If (B, A) is a CW-pair and F is a space such that $\pi_k(F) = 0$ for each k such that $B \backslash A$ has cells of dimension $k + 1$, then every map $f : A \longrightarrow F$ extends to a map $\tilde{f} : B \longrightarrow F$ such that $\tilde{f}|_A = f$.*

COROLLARY 0.3.10. *Let (B, A) be a CW-pair and (π, E, B) a fiber bundle with fiber F. If $\pi_k(F) = 0$ for each k such that $B \backslash A$ has cells of dimension $k + 1$, then every sections $s : A \longrightarrow E$ can be extended a global section $\tilde{s} : B \longrightarrow E$. In particular, taking $A = \emptyset$, it follows that (π, E, B) admits global sections if F is k-connected where $k = \dim B$.*

Proof of Theorem 0.3.8. Let $\pi' : Q \longrightarrow B$ be a principal G-bundle; then associated bundle $Q \times_G EG$ has a global section since EG is contractible then k-connected for any k and by Corollary 0.3.10, which corresponds by Proposition 0.3.4 to a morphism of bundle $(\pi', Q, B) \longrightarrow (\pi, EG, BG)$ lying over some map $f : B \longrightarrow BG$ of the base spaces. From the universal property of pullback of fiber bundle[12, Proposition 1.4], there is a morphism $Q \longrightarrow f^*(EG)$ over the identity map of B. Then by Proposition 0.3.3, $Q \cong f^*(EG)$. Thus $[f] \longrightarrow [f^*EG]$ is surjective.

To see injectivity, suppose that $f_0, f_1 : B \longrightarrow BG$ are two maps such that the pullbacks of EG are isomorphic:$\phi : f_0^*(EG) \cong f_1^*(EG)$. We claim that $f_1 \sim f_2$. Indeed, consider the principal G-bundle

$$\pi' : P := f_0^*(EG) \times I \longrightarrow B \times I$$

where $I = [0, 1]$. Since $P|_{B\times 0} \cong f_0^*(EG)$ and $P|_{B\times 1} \cong f_1^*(EG)$, we have the G-bundle morphism:

$$\begin{array}{ccccc} P|_{B\times 0} & \xrightarrow{\cong} & f_0^*(EG) & \longrightarrow & EG \\ & \searrow & \downarrow & & \downarrow \\ & & B \times 0 & \xrightarrow{f_0} & BG \end{array}$$

Then by Proposition 0.3.4, this morphism corresponds to a section $s_0 : B \times 0 \longrightarrow P \times_G EG$. Similarly, we have the G-bundle morphism

$$\begin{array}{ccccc} P|_{B\times 1} & \xrightarrow{\phi} & f_1^*(EG) & \longrightarrow & EG \\ & \searrow & \downarrow & & \downarrow \\ & & B \times 1 & \xrightarrow{f_1} & BG \end{array}$$

which corresponds to a section $s_1 : B \times 1 \longrightarrow P \times_G EG$. Now we have the section $s_0 \cup s_1 : B \times 0 \cup B \times 1 \longrightarrow P \times_G EG$. Since EG is contractible, from Corollary 0.3.10 $s_1 \cup s_2$ extends to a global section $s : B \times I \longrightarrow P \times_G EG$, which therefore corresponds to a bundle morphism $(\pi', P, B \times I) \longrightarrow (\pi, EG, BG)$ and the map $B \times I \longrightarrow BG$ is a homotopy between f_0 and f_1. □

Now we will see that B is a functor.

THEOREM 0.3.11. *Given a topological group G, there exists a universal principal G-bundle (π, EG, BG).*

Sketch of proof. For each, let EG^n be the n-fold join $G * G * \cdots * G$. Then it is possible to show that EG^n is $(n-1)$-connected and it has free action by G given by right multiplication in each factor of G. Thus the colimit

$$EG := \lim_{n\to\infty} EG^n$$

is a weakly contractible G-space, and $BG := EG/G$ is a classifying space. □

PROPOSITION 0.3.12. *For each topological group homomorphism $\phi : G \longrightarrow H$, there is a natural homotopy class $B\phi \in [BG, BH]$ such that $B(\phi \circ \psi) = B\phi \circ B\psi$ and $B(id_G) = id_{BG}$, i.e. B is a functor from the category of topological groups to* **hTop**. *Moreover, B preserves products in the sense that $B(G \times H) = BG \times BH$.*

Proof. The associated bundle $EG \times_{G,\phi} H$ is a principal H-bundle over BG hence there is a map $B\phi \in [BG, BH]$ such that $EG \times_{G,\phi} H \cong (B\phi)^* EG$. Functoriality follows from the evident isomorphism

$$(EG \times_{G,\phi} H) \times_{H,\psi} K \cong EG \times_{G,\psi\circ\phi} K$$

and that $B(id_G) = id_{BG}$ follows from the fact $EG \times_G G \cong EG$.

For the product result, we can see that $EG \times EH$ is contractible space with a $G \times H$ free right action so

$$B(G \times H) = (EG \times EH)/(G \times H) = BG \times BH$$

□

4. Hodge star operator

Let V be a real fintie-dimensional vector space of dimension d with an inner product $< , >$. For each degree p, the vector vector space $\wedge^p V$ has an inner product induced from V:

$$\langle u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p \rangle = \det(\langle u_j, v_k \rangle)_{jk}$$

If $(e_1, \ldots, e_d)$ is an orthonormal basis for V, then clearly $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \le i_1 < i_2 < \cdots < i_p \le d\}$ is an orthonormal basis for $\wedge^p V$. We now define the *Hodge $*$-operator.* The Hodge star operator si a mapping

$$* : \bigwedge^p V \longrightarrow \bigwedge^{d-p} V$$

defined by setting

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \pm e_{j_1} \wedge \cdots \wedge e_{j_{d-p}}$$

where $\{j_1, \ldots, j_{d-p}\}$ is the complement of $\{i_1, \ldots, i_p\}$ in $\{1, \ldots, d\}$, and we assign the plus sign if $\{i_1, \ldots, i_p, j_1, \ldots, j_{d-p}\}$ is an even permutation of $\{1, \ldots, d\}$, and the minus sigh otherwise. Hence we have

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_1 \wedge \cdots \wedge e_d = \text{volume}$$

Extending $*$ by linearity, we can prove[13, Ch.V, Sec.1] for $\alpha, \beta \in \wedge^p V$

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \text{vol}$$

CHAPTER 1. Self-duality

1. self dual Yang-Mills equations

Let A be a connection on a principal G-bundle P over $\mathbb{R}^4$, and $F(A)$ its curvature.

A connection is said to satisfy the *self-dual Yang-Mills* equations, or self-duality equations for short, if $F(A)$ is invariant under the Hodge star operator. In terms of a local trivialization of P over $\mathbb{R}^4$, and the basis coordinates (x_1, x_2, x_3, x_4), $F(A)$ may be written as a Lie algebra-valued 2-form:

$$F(A) = \sum_{i<j} F_{ij} dx_i \wedge dx_j = F_{12}dx_1\wedge dx_2 + F_{13}dx_1\wedge dx_3 + F_{14}dx_1\wedge dx_4 + F_{23}dx_2\wedge dx_3 + F_{24}dx_2\wedge dx_4 + F_{34}dx_3\wedge dx_4$$

Then

$$*F(A) = F_{12}dx_3 \wedge dx_4 - F_{13}dx_2 \wedge dx_4 + F_{14}dx_2 \wedge dx_3 + F_{23}dx_1 \wedge dx_4 - F_{24}dx_1 \wedge dx_3 + F_{34}dx_1 \wedge dx_3$$

Hence the self-duality equations means that

$$\begin{cases} F_{12} &= F_{34} \\ F_{13} &= -F_{24} \\ F_{14} &= F_{23} \end{cases} \tag{1.1.1}$$

With respect to this trivialization, the connection is described by a Lie algebra-valued 1-form

$$A = A_1dx_1 + A_2dx_2 + A_3dx_3 + A_4dx_4$$

where A_i is a matrix-valued function of $\mathbb{R}^4$ with respect to a local frame of $\mathrm{ad}(P)$. Via Section 0.2.2 we introduce the covariant derivative on $\mathrm{ad}(P)$ in the direction of $\frac{\partial}{\partial x_i}$:

$$\nabla_i = \frac{\partial}{\partial x_i} + A_i$$

and since A is matrix valued-form, the curvature is then expressed as

$$F(A) = dA + A \wedge A$$

Then we have:

$$\begin{aligned} dA &= \sum_i \left(\sum_j \frac{\partial A_i}{\partial x_j} dx_j \right) \wedge dx_i \\ &= \sum_{1\leq i,j\leq 4} \frac{\partial A_i}{\partial x_j} dx_j \wedge dx_i \\ &= \sum_{1\leq i<j\leq 4} \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) dx_i \wedge dx_j \end{aligned}$$

Similarly

$$\begin{aligned} A \wedge A &= \sum_{i,j} A_iA_j dx_i \wedge dx_j \\ &= \sum_{1\leq i<j\leq 4} (A_iA_j - A_jA_i) dx_i \wedge dx_j \end{aligned}$$

And we can see that $\frac{\partial}{\partial x_i}$ and A_i are both linear operator on the space of sections. If $(e_1, \ldots, e_r)$ is a local frame, $s = \sum_i s^i e_i$, then $\frac{\partial s}{\partial x_j} = \sum_i \frac{\partial s_i}{\partial x_j} e_i$ and for column vector ${}^t(s^1, \ldots, s^r)$, $A_j s = (e_1, \ldots, e_r) A_j {}^t(s^1, \ldots, s^r)$.

Since by Lebniz's rule,

$$\frac{\partial}{\partial x_i}(A_j\xi) = \frac{\partial A_j}{\partial x_i}\xi + A_j\frac{\partial \xi}{\partial x_i}$$

Hence

$$\frac{\partial}{\partial x_i}A_j = \frac{\partial A_j}{\partial x_i} + A_j\frac{\partial}{\partial x_i}$$

Then

$$\left[\frac{\partial}{\partial x_i}, A_j\right] = \frac{\partial}{\partial x_i}A_j - A_j\frac{\partial}{\partial x_i} = \frac{\partial A_j}{\partial x_i}$$

Clearly $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$. Therefore

$$[\nabla_i, \nabla_j] = \left[\frac{\partial}{\partial x_i} + A_i, \frac{\partial}{\partial x_j} + A_j\right] = \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + A_iA_j - A_jA_i\right) = F_{ij}$$

we now assume that the Lie algebra-valued functions A_i are independent of x_3, x_4. Then we can restrict P to submanifold $\mathbb{R}^2$. Thus A_1 and A_2 define a connection

$$A = A_1dx_1 + A_2dx_2$$

over $\mathbb{R}^2$, and A_3 and A_4 which we relabel as ϕ_1 and ϕ_2 are auxiliary fields over $\mathbb{R}^2$, called *Higgs fields* which are Lie algebra-valued.

Now we consider the restriction of P on $\mathbb{R}^2$, clearly the sections of this restriction are independent of x_3, x_4, so the actions of $\frac{\partial}{\partial x_3}$ and $\frac{\partial}{\partial x_3}$ are zero, we can see that $\nabla_3 = A_3, \nabla_4 = A_4$. The self-duality equations (1.1.1) may now be written as

$$\begin{cases} F_{12} &= [\nabla_1, \nabla_2] = [\phi_1, \phi_2] = F_{34} \\ F_{13} &= [\nabla_1, \phi_1] = [\phi_2, \nabla_2] = -F_{24} \\ F_{14} &= [\nabla_1, \phi_2] = [\nabla_2, \phi_1] = F_{23} \end{cases}$$

Introducing the complex Higgs field $\phi = \phi_1 - i\phi_2$ we can write the above equations as

$$\begin{cases} F_{12} = \frac{1}{2}i[\phi, \phi^*] \\ [\nabla_1 + i\nabla_2, \phi] = 0 \end{cases} \tag{1.1.2}$$

Now we consider the induced connection on the principal bundle P over $\mathbb{R}^2$, and its corresponding curvature form etc:

$$F \in \Omega^2(\mathbb{R}^2, \mathrm{ad}(P)) \quad \text{and} \quad \phi \in \Omega^0(\mathbb{R}^2, \mathrm{ad}(P) \otimes \mathbb{C})$$

The first equation of (1.1.2) is coordinate dependent. But if we write $z = x_1 + ix_2$ and introduce

$$\Phi = \frac{1}{2}\phi dz \in \Omega^{1,0}(\mathbb{R}^2, \mathrm{ad}(P) \otimes \mathbb{C})$$
$$\Phi^* = \frac{1}{2}\phi^* d\bar{z} \in \Omega^{0,1}(\mathbb{R}^2, \mathrm{ad}(P) \otimes \mathbb{C})$$

Then the first equation of (1.1.2) becomes

$$F = -[\Phi, \Phi^*]$$

In fact since P is over $\mathbb{R}^2$, $F = F_{12}dx_1 \wedge dx_2$, then from $dz = dx_1 + idx_2$ and (1.1.2), we have

$$-[\Phi, \Phi^*] = -\frac{1}{4}[\phi, \phi*]dz \wedge d\bar{z} = \frac{1}{2}i[\phi, \phi^*]dx_1 \wedge dx_2 = F$$

We can write

$$A = A_1dx_1 + A_2dx_2 = \frac{1}{2}(A_1 - iA_2)dz + \frac{1}{2}(A_1 + iA_2)d\bar{z} = A'dz + A''d\bar{z}$$

Then we have

$$\begin{aligned}[\nabla_1 + i\nabla_2, \phi] &= \left[\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + A_1 + iA_2, \phi\right] \\ &= 2\left[\frac{\partial}{\partial \bar{z}} + A'', \phi\right]\end{aligned}$$

This formula is zero if and only if

$$0 = \left[\frac{\partial}{\partial \bar{z}} + A'', \frac{1}{2}\phi\right] d\bar{z} \wedge dz = d''\Phi + [A''d\bar{z}, \Phi] = d''_A\Phi$$

by Theorem 0.2.6. where d''_A is $(0,1)$-type connection. Then the equations (1.1.2) become

$$\begin{cases} F = -[\Phi, \Phi^*] \\ d''_A\Phi = 0 \end{cases} \tag{1.1.3}$$

This system of equations also can be writen on a compact Riemann surface M. We consider a connection A on a principal G-bundle P over M where G is compact, and a complex Higgs field $\Phi \in \Omega^{1,0}(M, \mathrm{ad}(P) \otimes \mathbb{C})$. The pair (A, Φ) will be said to satisfy the self-duality equations if it satisfies the system of equations (1.1.3).

Here are two examples:

EXAMPLE 1.1.1. *Let $\Phi = 0$ and A be flat connection, then it is easy to see that (A, Φ) satisfies the self-duality.*

EXAMPLE 1.1.2. *Let M be given a Riemannian metric $g = h^2\, dzd\bar{z}, h > 0$ compatible with the conformal structure, K be the canonical line bundle, which is T'^*M for M. The structure group of the associated frame bundle $\mathrm{Fr}(K)$ is $\mathrm{GL}(\mathbb{C}, 1)$, then since M is given a conformal metric, which gives an hermitian metric on the holomorphic tangent bundle $T'M$ and then K. So K is a principal $U(1)$-bundle from Section 0.2.3. Since M is Khler manifold, the Chern connection coincides with the Levi-Civita connection of the associated Riemannian metric. Thus we consider the Chern connection ∇ on T'^*M.*

Let $K^{1/2}$ be a holomorphic line bundle over M such that

$$K^{1/2} \otimes K^{1/2} \cong K$$

And $K^{1/2}$ is given a connection $\nabla^{1/2}$ such that the tensor product of $\nabla^{1/2}$

$$\nabla^{1/2} \otimes I + I \otimes \nabla^{1/2}$$

where I is identity of $\Omega(M, K^{1/2})$, is the Chern connection of K. For the dual vector bundle $K^{1/2*}$ of $K^{1/2}$, there is also a dual connection, denoted by $\nabla^{1/2*}$. We consider the rank-2 vector bundle $V = K^{1/2} \oplus K^{1/2*}$ with the linear connection $\nabla_V = \nabla^{1/2} \oplus \nabla^{1/2*}$. Note that V is given an hermitian metric so the associated frame bundle is a principal $SU(2)$-bundle, denoted by P. Let A be the $SU(2)$-connection defined by the linear connection ∇_V via Section 0.2.3.

Since $P = \mathrm{Fr}(V)$, its structure group is $\mathrm{GL}(\mathbb{C}, 2) \cong \mathrm{GL}(V_x)$, so the fiber of $\mathrm{ad}(P)$ is $\mathfrak{gl}(\mathbb{C}, 2) \cong \mathfrak{gl}(V_x) = \mathrm{End}(V_x)$, so $\mathrm{ad}(P) \cong \mathrm{End}(V)$. We define

$$\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} hdz \in \Omega^{1,0}(\mathrm{ad}(P))$$

We claim that $d''_A\Phi = 0$. From [10, Ch.0, Sec.5] we know that the connection form of Chern connection of the tangent bundle K^* is

$$\omega = \frac{1}{h}\frac{\partial h}{\partial z}dz - \frac{1}{h}\frac{\partial h}{\partial \bar{z}}d\bar{z}$$

$K^*, K^{1/2*}$ are line bundles, for a local frame e of K^* and a local frame $e^{1/2}$ of $K^{1/2*}$, we identify $e^{1/2} \otimes e^{1/2}$ and e since $K^{1/2*} \otimes K^{1/2*} \cong K^*$. Then we have

$$\begin{aligned} \omega e &= (\omega_{1/2} \otimes I + I \otimes \omega_{1/2})(e^{1/2} \otimes e^{1/2}) \\ &= (\omega_{1/2}e^{1/2}) \otimes e^{1/2} + e^{1/2} \otimes (\omega_{1/2}e^{1/2}) \\ &= 2\omega_{1/2}e^{1/2} \otimes e^{1/2} \end{aligned}$$

where $\omega_{1/2}$ is connection form of $K^{1/2*}$, note that they are complex-valued forms. So $\omega_{1/2} = \frac{1}{2}\omega$. Then we obtain the connection form of $K^{1/2}$ is $-\frac{1}{2}\omega$. Finally we obtain the connection form of $V = K^{1/2} \oplus K^{1/2*}$:

$$\omega_V = \begin{pmatrix} -\frac{1}{2}\omega & 0 \\ 0 & \frac{1}{2}\omega \end{pmatrix}$$

which also can be regarded as the connection form of A which is a connection induced from the connection on V. The part of $(0,1)$-type of ω_V is

$$\omega''_V = -\frac{1}{2h}\frac{\partial h}{\partial \bar{z}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} d\bar{z}$$

Then we can see that

$$\begin{aligned} d''_A\Phi &= \frac{\partial h}{\partial \bar{z}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} d\bar{z}\wedge dz + [\omega''_V, \Phi] \\ &= \frac{\partial h}{\partial \bar{z}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} d\bar{z}\wedge dz + h \times \left(-\frac{1}{2h}\frac{\partial h}{\partial d\bar{z}}\right) \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] d\bar{z}\wedge dz \\ &= 0 \end{aligned}$$

Similar, if F_0 is the curvature form of the holomorphic tangent bundle K^*, then the curvature forms of $K^{1/2*}$ and V is respectively $\frac{1}{2}F_0$ and

$$\begin{pmatrix} -\frac{1}{2}F_0 & 0 \\ 0 & \frac{1}{2}F_0 \end{pmatrix}$$

On the other hand, we have

$$-[\Phi, \Phi^*] = -\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} hdz, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} hd\bar{z} \right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h^2 dz\wedge d\bar{z}$$

Thus the equations becomes

$$F_0 = -2h^2 dz \wedge d\bar{z}$$

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