Estimates on the lower bound of Schrödinger operators

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Introduction

This is a Mémoire written for the internship for the Master 2 - MFA at Nantes' university.

Our objective is to establish estimates on the lower bound of Schrödinger operators of the form $H = -\Delta - V$, with V non-negative and locally integrable. Through these estimate, we will be able to deduce some sufficient conditions for the operator to be positive.

We will use methods of harmonic analysis, especially a good-lambda inequality, to prove that the Riesz potentials $(-\Delta)^{-s/2}$ and Bessel potentials $\left(\lambda^2 - \Delta\right)^{-s/2}$ are bounded by an appropriate fractional maximal function. This will then allow us to find the desired estimate in Corollary 3.1.

In the first part, we will gives the various Harmonic analysis and Spectral analysis we will need in the following parts.

The second part is dedicated to the study of Riesz Potentials, and we will establish a necessary and sufficient condition for a weighted version of the classical Hardy-Littlewood-Sobolev inequality to hold. One of the result proved here will be important for the proof of the main result.

In the third and last part, we will study the Schrödinger operator, and establish some results on the lower bound of its spectrum that will allow us to get the desired estimates.

Contents

1. Preliminaries results

1.1 Covering lemmas

We will first clarify several terms and notations that will be used throughout this document.

Definition 1.1. *The term* cube *is used to refer to an hypercube with sides parallel to the coordinate axis. That is to say, a cube of length $l > 0$ is a cartesian product :*

$$Q = [x_1, x_1 + l) \times \cdots \times [x_n, x_n + l)$$

With $x = (x_1, \dots, x_n) \in \mathbf{R}^n$. The intervals in the definition might also be taken to be closed or open. In the later case, then Q will be a ball $B(c, r)$ for the norm $|x|_\infty = \sup\{|x_1|, \dots, |x_n|\}$. The center and radius of Q will refer to the centers and radius of this ball. We also have $r = l/2$.

Definition 1.2. *The characteristic function of a set E is :*

$$\mathbb{1}_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

Definition 1.3. *The cardinal of a set E is :*

$$\#E = \sum_{x \in E} \mathbb{1}_E(x)$$

If E is countable, and ∞ if E is uncountable.

Definition 1.4. *The diameter of a set E is :*

$$\mathrm{diam}(E) = \sup\{\mathrm{d}(x, y);\, x, y \in E\}$$

1.1.1 A Besicovitch type lemma

Theorem 1.1. *[2] Let A be a bounded subset of $\mathbf{R}^n$. For each $x \in A$ a closed cube $Q(x)$ with positive radius, centered at x is given,. Then, one can choose, from among the givens cubes $\{Q(x)\}_{x \in A}$, a sequence $\{Q_k\}_k$ covering A :*

$$A \subset \bigcup_k Q_k \tag{1.1.1}$$

And such that there is a constant θ_n depending only on the dimension, such that any point of $\mathbf{R}^n$ is in at most θ_n cubes. That is to say :

$$\sum_k \mathbb{1}_{Q_k} \leq \theta_n \tag{1.1.2}$$

Proof. We note r_x the radius of $Q(x)$, and define a_0 by :

$$a_0 = \sup\{r_x : x \in A\} \tag{1.1.3}$$

If $a_0 = \infty$ then there is a cube that will cover A entirely, and there's nothing left to do. If $a_0 < \infty$, then we choose a cube Q_1 such that :

$$Q_1 = Q(x_1) \in \{Q(x) : x \in A\}, \quad r_1 = r_{x_1} > \frac{a_0}{2} \tag{1.1.4}$$

We now construct a sequence $\{Q_k\}$ such that :

$$a_n = \sup\left\{ r_x : x \in A \setminus \bigcup_{k=1}^{n} Q_k \right\} \tag{1.1.5}$$

$$Q_{n+1} = Q(x_{n+1}), \quad x_{n+1} \in A \setminus \bigcup_{k=1}^{n} Q_k, \quad r_{n+1} = r_{x_{n+1}} > \frac{a_n}{2} \tag{1.1.6}$$

With the Q_i thus defined, we have, if $i \neq j$:

$$\frac{1}{3}Q_i \cap \frac{1}{3}Q_j = \varnothing \tag{1.1.7}$$

Indeed, if $i > j$, then $x_i \notin Q_j$ and, $r_j \leq a_i < 2r_i$. Then let $y \in \frac{1}{3}Q_i$. We have :

$$r_i < |x_i - x_j|_\infty \leq \frac{1}{3}r_i + |y - x_j|_\infty$$

Then since $r_i \geq 2r_j$, we get :

$$|y - x_j| > \frac{1}{3}r_j$$

And so $y \notin \frac{1}{3}Q_j$.

Now let's prove the first part of the theorem. First, if the sequence $\{Q_k\}$ is finite, i.e. if at some step n, there's no possible cube we can chose. Then $A \subset \bigcup_k Q_k$ is trivial.

If the sequence of cubes is infinite, then we necessarily must have $r_k \to 0$. Indeed, let's look at the set :

$$\bigcup_{k \geq 1} Q_k$$

Since A is bounded, then it must be bounded, and so of finite measure. But, by (1.1.7), its measure is more than :

$$m\left(\bigcup_{k \geq 1} \frac{1}{3}Q_k\right) = \sum_{k \geq 1} \left(\frac{2r_k}{3}\right)^n$$

For this to be finite, we must have $r_k \to 0$. But then take :

$$x \in A \setminus \bigcup_{k \geq 1} Q_k$$

We have $r_x \leq a_k \leq 2r_k$ for all $k \geq 1$. But then this must mean that $r_x = 0$. But we require $r_x > 0$ for all $x \in A$, so we have a contradiction, and we must have :

$$A \subset \bigcup_{k \geq 1} Q_k \tag{1.1.8}$$

We have proved the first part. For the second, let $x \in \mathbf{R}^n$. By doing a translation if necessary, we can consider x to be the origin. Then the coordinates hyperplanes split the space into 2^n quadrants. We will show that for each quadrant the number of cubes with center in this quadrant is bounded by a constant that depends only on the dimension.

By changing coordinates if necessary, we can assume we work in the following quadrant :

$$P = \{y \in \mathbf{R}^n : \forall k,\, 1 \leq k \leq n,\, y_k \geq 0\} \tag{1.1.9}$$

Then let i_0 be an integer such that Q_{i_0} is the first cube with center in P containing x. Then if we consider the cube of center x and radius r_i, its intersection with P is contained in Q_i.

Now, let Q_j be another cube with center in P and containing x. Necessarily, $j > i$ so $x_j \notin Q_i$. Then $r_j > r_i$ since Q_j contains the origin 0. But we also have $r_j < 2r_i$. Moreover, $\frac{1}{3}Q_k \cap \frac{1}{3}Q_j = \varnothing$ whenever $k \neq j$. Notice that the region of P with $|y|_\infty \leq 2r_i$ is a cube of radius r_i. Then, the following lemma gives us the desired upper bound on the number of cubes.

Lemma 1.1. *Let Q be a cube of radius r. $\mathcal{Q}$ a collection of disjoint cubes with center in Q and radius greater than δr, with $\delta > 0$.*

Then the cardinal of $\mathcal{Q}$ is bounded by a constant depending only on the dimension n and the parameter δ.

Proof. We have :

$$\bigcup_k Q_k \subseteq (1+\delta)Q$$

Then since the cubes are disjoints, taking the lebesgue measure of those sets we get :

$$\delta^n \#\mathcal{Q} \leq (1+\delta)^n$$

Thus $\#\mathcal{Q} \leq (1 + 1/\delta)^n$. □

We apply the lemma to the $\frac{1}{3}Q_k$ with center in P.

Then x is in at most 4^n cubes in each of the 2^n quadrants. Thus :

$$\sum_{k \geq 1} \mathbb{1}_{Q_k} \leq 8^n \tag{1.1.10}$$

□

1.1.2 Whitney decomposition

Theorem 1.2. *[8] Let F be a non empty, proper closed subset of $\mathbf{R}^n$. $\Omega = F^c$. Then there's a sequence of cubes $\mathcal{Q} = \{Q_k\}$ such that :*

1. *$\Omega = \bigcup_k Q_k$*
2. *$Q_k \cap Q_l = \varnothing$ if $k \neq l$.*
3. *There exists constants c_1, c_2 such that : $c_1 \mathrm{diam}(Q_k) \leq \mathrm{d}(Q_k, F) \leq c_2 \mathrm{diam}(Q_k)$.*

Proof. We let $\mathcal{M}_k$ be the collection of dyadic cubes of length 2^{-k}.

$$\Omega_k = \left\{x \in \Omega : c2^{-k} < \mathrm{d}(x, F) \leq c2^{-k+1}\right\} \tag{1.1.11}$$

Where c is a positive constant to be fixed later. We have $\Omega = \bigcup_k \Omega_k$. We take an initial collection of cube :

$$\mathcal{Q}_0 = \bigcup_k \{Q \in \mathcal{M}_k : Q \cap \Omega_k \neq \varnothing\} \tag{1.1.12}$$

We take a cube $Q \in \mathcal{Q}_0$, let k be such that $Q \in \mathcal{M}_k$. Then there is a $x \in Q \cap \Omega_k$ and so :

$$\mathrm{d}(Q, F) \leq c2^{-k+1} \tag{1.1.13}$$

Take $x \in Q \cap \Omega_k$, $y \in Q$, $z \in F$. We have

$$\mathrm{d}(y, z) \geq \mathrm{d}(x, z) - \mathrm{d}(x, y) \tag{1.1.14}$$

This holds for all $z \in F$, and all $y \in Q$, thus

$$\mathrm{d}(Q, F) \geq \mathrm{d}(x, F) - \mathrm{diam}(Q) > c2^{-k} - \mathrm{diam}(Q) \tag{1.1.15}$$

Then :

$$c2^{-k} - \mathrm{diam}(Q) < \mathrm{d}(Q, F) \leq c2^{-k+1} \tag{1.1.16}$$

Since $\mathrm{diam}(Q) = \sqrt{n}2^{-k}$, if we take $c = 2\sqrt{n}$, then

$$\mathrm{diam}(Q) < \mathrm{d}(Q, F) \leq 4\,\mathrm{diam}(Q) \tag{1.1.17}$$

Thus all the cubes in $\mathcal{Q}_0$ satisfies the third condition with constants $c_1 = 1$, $c_2 = 4$. But the second condition is not satisfied.

For a cube $Q \in \mathcal{Q}_0$, let $Q' \in \mathcal{Q}_0$ such that $Q \subseteq Q'$. Then by 1.1.17 $\mathrm{diam}(Q') < 4\,\mathrm{diam}(Q)$. Thus there exists a maximal dyadic cube in $\mathcal{Q}_0$ containing Q.

Thus $\mathcal{Q}$, the subset of $\mathcal{Q}_0$ comprised of maximal dyadic cubes satisfying 1.1.17, satifies all three conditions. □

Remark 1.1. *Taking $c = (1 + \delta)\sqrt{n}$, with $\delta > 0$, we can get*

$$\delta \mathrm{diam}(Q) < \mathrm{d}(Q, F) < 2(1 + \delta)\mathrm{diam}(Q) \tag{1.1.18}$$

1.2 Interpolation

Definition 1.5. *An operator T is quasilinear if there exist $\kappa > 0$ such that, whenever Tf_1 and Tf_2 are defined, so is $T(f_1 + f_2)$ and :*

$$|T(f_1 + f_2)| \leq \kappa\left(|Tf_1| + |Tf_2|\right) \tag{1.2.1}$$

We let (X, μ) and (Y, ν) be measure spaces. f a measurable function defined over X, T and operator such that Tf is defined over Y.

We let :

$$\mu_f(\lambda) = \mu\{x \in X : |f(x)| > \lambda\}, \quad \nu_h(\lambda) = \nu\{y \in Y : |h(y)| > \lambda\} \tag{1.2.2}$$

Definition 1.6. *Let $1 \leq r, s \leq \infty$. An operator T is of type (r,s) or of strong type (r,s) if Tf is defined in $L^r(\mu)$ and if :*

$$\|Tf\|_{L^s(\nu)} \leq M\|f\|_{L^r(\mu)} \tag{1.2.3}$$

The least M such that the estimate holds is the (r,s) norm of T.
For $s < \infty$, T is of weak type (r,s), if :

$$\nu_{Tf}(\lambda) \leq \left(\frac{M}{\lambda}\|f\|_r\right)^s \tag{1.2.4}$$

The least M such that the estimate holds is the weak (r,s) norm of T.
If $s = \infty$, weak type (r,s) is defined as equivalent to strong type (r,s).

Proposition 1.1. *Let f be a measurable function. Then :*

$$\int_X |f(x)|^p \, \mathrm{d}\mu(x) = \int_0^\infty p\lambda^{p-1}\mu_f(\lambda) \, \mathrm{d}\lambda \tag{1.2.5}$$

Theorem 1.3 (Marcinkiewicz). *[9] $1 \leq p_1, q_1, p_2, q_2 \leq \infty$, with $p_i \leq q_i$ and $q_1 \neq q_2$. Let T be a quasilinear operator that is simultaneously of weak types (p_1, q_1) and (p_2, q_2), with norms M_1 and M_2 respectively. Then for any (p,q) with :*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}; \quad \theta \in (0,1) \tag{1.2.6}$$

T is of strong type (p,q) and we have :

$$\|Tf\|_q \leq KM_1^{1-\theta}M_2^\theta\|f\|_p \tag{1.2.7}$$

Where $K = K(\theta, \kappa, p_1, q_1, p_2, q_2)$ is independant of f, and stays bounded if p_1, q_1, p_2, q_2 are fixed and θ stays away from 0 and 1.

Proof. We can suppose without loss of generality that $p_2 \geq p_1$.

Let $f \in L^p(X,\mu)$, $f = f' + f''$ with $f'(x) = f(x)$ if $|f(x)| < 1$ and $f'(x) = 0$ if $|f(x)| > 1$. Then $f' \in L^{p_2}$ and $f'' \in L^{p_1}$. thus Tf' and Tf'' exists, by hypothesis, and then so does $Tf = T(f' + f'')$.

We first consider the case when $q_1, q_2 < \infty$.

$$\|Tf\|^q_{L^q(\nu)} = \int_0^\infty q\lambda^{q-1}\nu_{Tf}(\lambda) \, \mathrm{d}\lambda = (2\kappa)^q \int_0^\infty q\lambda^{q-1}\nu_{Tf}(2\kappa\lambda) \, \mathrm{d}\lambda \tag{1.2.8}$$

Now let $z > 0$, $f = f_1 + f_2$, with :

$$f_1(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq z \\ \mathrm{e}^{i \arg f} z & \text{if } |f(x)| > z \end{cases} \tag{1.2.9}$$

We have

$$|f_1| = \min(|f|, z), \quad |f| = |f_1| + |f_2| \tag{1.2.10}$$

We write $\mu_i = \mu_{f_i}$ and $\nu_i = \nu_{Tf_i}$. We have :

$$\begin{aligned}\nu_{Tf}(2\kappa\lambda) &\leq \nu_1(\lambda) + \nu_2(\lambda) \\ &\leq \left(\frac{M_1}{\lambda}\|f_1\|_{p_1}\right)^{q_1} + \left(\frac{M_2}{\lambda}\|f_2\|_{p_2}\right)^{q_2}\end{aligned}$$

By (1.2.10), we have :

$$\mu_1(\lambda) = \begin{cases}\mu_f(\lambda) & \text{if } \lambda \leq z \\ 0 & \text{if } \lambda > z\end{cases}, \quad \mu_2(\lambda) = \mu_f(\lambda + z) \tag{1.2.11}$$

Thus

$$\|f_1\|_{p_1}^{p_1} = \int_0^z p_1 t^{p_1-1}\mu_f(t)\ \mathrm{d}t, \quad \|f_2\|_{p_2}^{p_2} = \int_z^\infty p_2 (t-z)^{p_2-1}\mu_f(t)\ \mathrm{d}t$$

Then the integral in (1.2.8) is bounded by :

$$\begin{aligned}&M_1^{q_1} p_1^{k_1}\int_0^\infty \lambda^{q-q_1-1}\left(\int_0^z t^{p_1-1}\mu_f(t)\ \mathrm{d}t\right)^{k_1}\ \mathrm{d}\lambda \\ &\qquad + M_2^{q_2} p_2^{k_2}\int_0^\infty \lambda^{q-q_2-1}\left(\int_z^\infty t^{p_2-1}\mu_f(t)\ \mathrm{d}t\right)^{k_2}\ \mathrm{d}\lambda\end{aligned} \tag{1.2.12}$$

With $k_i = \frac{q_i}{p_i} \geq 1$. The idea is then to take for z a monotone function of λ and then choose the right parameters. We note P the first integral in (1.2.12), Q the second. We have :

$$\begin{aligned}P^{\frac{1}{k_1}} &= \sup_\chi \int_0^\infty \lambda^{q-q_1-1}\int_0^{z(\lambda)} t^{p_1-1}\mu_f(t)\ \mathrm{d}t\,\chi(\lambda)\ \mathrm{d}\lambda \\ Q^{\frac{1}{k_2}} &= \sup_\omega \int_0^\infty \lambda^{q-q_2-1}\int_{z(\lambda)}^\infty (t-z)^{p_2-1}\mu_f(t)\ \mathrm{d}t\,\omega(\lambda)\ \mathrm{d}\lambda\end{aligned} \tag{1.2.13}$$

Where χ and ω are taken among nonegative, measurable functions satisfying respectively :

$$\begin{aligned}&\int_0^\infty \chi(\lambda)^{k_1'}\lambda^{q-q_1-1}\ \mathrm{d}\lambda \leq 1 \\ &\int_0^\infty \omega(\lambda)^{k_2'}\lambda^{q-q_2-1}\ \mathrm{d}\lambda \leq 1\end{aligned} \tag{1.2.14}$$

Indeed, by Hölder's inequality, $P^{\frac{1}{k_1}}$ is larger than the integral inside the supremum for all such χ. There's equality if and only if :

$$\chi(\lambda)^{k_1'} = c\left(\int_0^z t^{p_1-1}\mu_f(t)\ \mathrm{d}t\right)^{k_1}, \quad \int_0^\infty \chi(\lambda)^{k_1'}\lambda^{q-q_1-1}\ \mathrm{d}\lambda = 1$$

And since c in the first equation is arbitrary, we can choose it so that the second is satisfied. Now take $p_2 > p_1$ and $q_1 > q_2$. We let :

$$z = \left(\frac{\lambda}{A}\right)^\xi \tag{1.2.15}$$

With $A, \xi > 0$ to be determined. We have

$$\begin{aligned}\int_0^\infty \lambda^{q-q_1-1} \int_0^z t^{p_1-1}\mu_f(t)\ \mathrm{d}t\chi(\lambda)\ \mathrm{d}\lambda &= \int_0^\infty t^{p_1-1}\mu_f(t) \int_{At^{\frac{1}{\xi}}}^\infty \chi(\lambda)\lambda^{q-q_1-1}\ \mathrm{d}\lambda\ \mathrm{d}t \\ &\leq \int_0^\infty t^{p_1-1}\mu_f(t) \left(\int_{At^{\frac{1}{\xi}}}^\infty \lambda^{q-q_1-1}\ \mathrm{d}\lambda\right)^{\frac{1}{k_1}} \mathrm{d}t \qquad (1.2.16) \\ &\leq \left(\frac{A^{q-q_1}}{q_1-q}\right)^{\frac{1}{k_1}} \int_0^\infty t^{p_1-1-\frac{q-q_1}{k_1\xi}}\mu_f(t)\ \mathrm{d}t\end{aligned}$$

Then :

$$P \leq \frac{A^{q-q_1}}{q_1-q}\left(\int_0^\infty t^{p_1-1-\frac{q-q_1}{k_1\xi}}\mu_f(t)\ \mathrm{d}t\right)^{k_1} \qquad (1.2.17)$$

We do for Q, and we get. The integral in the sup in (1.2.13) is :

$$\begin{aligned}&\int_0^\infty (t-z)^{p_2-1}\mu_f(t)\int_0^{At^{\frac{1}{\xi}}} \omega(\lambda)\lambda^{q-q_2-1}\ \mathrm{d}\lambda\ \mathrm{d}t \\ &\qquad\leq \int_0^\infty t^{p_2-1}\mu_f(t)\left(\int_0^{At^{\frac{1}{\xi}}} \lambda^{q-q_2-1}\ \mathrm{d}\lambda\right)^{\frac{1}{k_2}} \mathrm{d}t \qquad (1.2.18) \\ &\qquad\leq \left(\frac{A^{q-q_2}}{q-q_2}\right)^{\frac{1}{k_2}} \int_0^\infty t^{p_2-1-\frac{q-q_2}{k_2\xi}}\mu_f(t)\ \mathrm{d}t\end{aligned}$$

And thus we have :

$$\begin{aligned}\|Tf\|_q^q \leq (2\kappa)^q q \Bigg(&M_1^{q_1}p_1^{k_1}\frac{A^{q-q_1}}{q_1-q}\left(\int_0^\infty t^{p_1-1-\frac{q-q_1}{k_1\xi}}\mu_f(t)\ \mathrm{d}t\right)^{k_1} \\ &+ M_2^{q_2}p_2^{k_2}\frac{A^{q-q_2}}{q-q_2}\left(\int_0^\infty t^{p_2-1-\frac{q-q_2}{k_2\xi}}\mu_f(t)\ \mathrm{d}t\right)^{k_2}\Bigg) \qquad (1.2.19)\end{aligned}$$

Now we choose ξ so that the power of t in both integral is equal to $p-1$. For it to be true in the first integral, we need :

$$\xi = \frac{(q-q_1)p_1}{(p-p_1)q_1} = \frac{\frac{1}{q_1}-\frac{1}{q}\frac{1}{p}}{\frac{1}{p_1}-\frac{1}{p}\frac{1}{q}}$$

But :

$$\frac{1}{q_1} = \frac{1}{1-\theta}\left(\frac{1}{q}-\frac{\theta}{q_2}\right), \text{ and } \frac{1}{q_1}-\frac{1}{q} = \frac{-\theta}{1-\theta}\left(\frac{1}{q_2}-\frac{1}{q}\right)$$

The same holds for p, so we have :

$$\xi = \frac{\frac{1}{q_2}-\frac{1}{q}\frac{1}{p}}{\frac{1}{p_2}-\frac{1}{p}\frac{1}{q}}$$

And so we can write ξ the two following ways :

$$\xi = \frac{p_1(q-q_1)}{q_1(p-p_1)} = \frac{p_2(q-q_2)}{q_2(p-p_2)} \tag{1.2.20}$$

But the term on the right is the one such that $p_2 - (q-q_2)/k_2\xi = p$. And so we get :

$$\|Tf\|_q^q \leq (2\kappa)^q q \left(M_1^{q_1} \left(\frac{p_1}{p}\right)^{k_1} \frac{A^{q-q_1}}{q_1-q} \|f\|_p^{pk_1} + M_2^{q_2} \left(\frac{p_2}{p}\right)^{k_2} \frac{A^{q-q_2}}{q-q_2} \|f\|_p^{pk_2} \right) \tag{1.2.21}$$

Now we choose A so that in both terms of the sum, M_1, M_2 and $\|f\|_p$ have the same power. Or more precisely such that :

$$A^{q-q_1} M_1^{q_1} \|f\|_p^{pk_1} = A^{q-q_2} M_2^{q_2} \|f\|_p^{pk_2}$$

We get :

$$A = M_1^{\frac{-q_1}{q_2-q_1}} M_2^{\frac{q_2}{q_2-q_1}} \|f\|_p^{\frac{k_2-k_1}{q_2-q_1}}$$

We now verify that we get the desired result when we plug this back in (1.2.21). For this, note that :

$$\frac{q-q_1}{q_2-q_1} = \frac{q}{q_2} \frac{\frac{1}{q_1} - \frac{1}{q}}{\frac{1}{q_1} - \frac{1}{q_2}} = \frac{\theta}{q_2} q = 1 - \frac{1-\theta}{q_1} q$$

Then we have :

$$q_1 - q_1 \frac{q-q_1}{q_2-q_1} = (1-\theta)q, \quad q_2 \frac{q-q_1}{q_2-q_1} = \theta q$$

And :

$$\begin{aligned}
pk_1 + p(k_2-k_1)\frac{q-q_1}{q_2-q_1} &= p\left(k_2 - q\frac{1-\theta}{q_1}(k_2-k_1)\right) \\
&= p\left(\frac{q_2}{p_2} - q\left(\frac{1}{q} - \frac{\theta}{q_2}\right)\frac{q_2}{p_2} + \frac{1-\theta}{p_1} q\right) \\
&= pq\left(\frac{\theta}{p_2} + \frac{1-\theta}{p_1}\right) \\
&= q
\end{aligned}$$

Thus we finally get :

$$\|Tf\|_q \leq (2\kappa) \left(\left(\frac{p_1}{p}\right)^{k_1} \frac{q}{q_1-q} + \left(\frac{p_2}{p}\right)^{k_2} \frac{q}{q-q_2} \right)^{\frac{1}{q}} M_1^{1-\theta} M_2^{\theta} \|f\|_p \tag{1.2.22}$$

If $q_1 < q_2$, then, by taking $z = \left(\frac{\lambda}{A}\right)^{\xi}$ but with $\xi < 0$, we get in the same way (1.2.22), except with $q-q_1$ and q_2-q instead of q_1-q and $q-q_2$.

The proofs of the cases $q_1 = q_2$ and $q_1 = \infty$ are similar. $\square$

1.3 Maximal function

We define the Hardy-Littlewood maximal function by :

$$Mf(x) = \sup_{Q \in \mathcal{Q}(x)} ⨍_Q |f(y)| \, \mathrm{d}y \tag{1.3.1}$$

Where $\mathcal{Q}(x)$ refers to the collection of all cubes of $\mathbf{R}^n$ containing x. We can also define the centered maximal function where we instead take the cubes with center x. There are constants c, C such that, for all real x :

$$cM_c f(x) \le Mf(x) \le CM_c f(x) \tag{1.3.2}$$

It is also possible to take the sups over balls rather than cubes. The resulting functions are also equivalent to M.

We also define M_d the dyadic maximal functions where the supremum is taken over dyadic cubes containing x. The dyadic maximal function is interesting because of the following result : if $f \in L^1$ and $\lambda > 0$, then

$$\{x \in \mathbf{R}^n : M_d f(x) > \lambda\} = \bigcup_k Q_k$$

Where the Q_k are maximal dyadic cubes such that $⨍_{Q_k} f(x) \, \mathrm{d}x > \lambda$.

We have, as a consequence of Theorems 1.1 and 1.3 :

Proposition 1.2. *M is of type (p, p) for all p with $1 < p \le \infty$, and of weak type (1,1).*

M is clearly bounded on L^∞, and the weak L^1 estimate follows from the following slightly more general result and the equivalence of centered and uncentered maximal functions :

Proposition 1.3. *Let μ be a positive Borel measure. We let M_μ be the maximal function defined by :*

$$M_\mu f(x) = \sup_{Q \in \mathcal{Q}(x)} \frac{1}{\mu(Q)} \int_Q |f(x)| \, \mathrm{d}\mu(x)$$

With $\mathcal{Q}(x)$ being the collection of cubes with center x. Then there is a $\theta_n > 0$ depending only on the dimension n such that :

$$\mu\{x \in \mathbf{R}^n : M_\mu f(x) > \lambda\} \le \frac{\theta_n}{\lambda} \int_{\mathbf{R}^n} |f(x)| \, \mathrm{d}\mu(x) \tag{1.3.3}$$

Proof. We let $E_\lambda = \{M_\mu f > \lambda\}$. Then for any $x \in E_\lambda$ there is a cube with center x, such that :

$$\frac{1}{\mu(Q_x)} \int_{Q_x} |f(x)| \, \mathrm{d}\mu(x) > \lambda$$

Thus by Theorem 1.1 there is a subsequences $\{Q_k\}$ of the $\{Q_x : x \in E_\lambda\}$, and a constant θ_n depending only on the dimension n, such that any point of $\mathbf{R}^n$ is in at most θ_n of the Q_k, and such that the Q_k cover E_λ. Then :

$$\mu(E_\lambda) \le \sum_k \mu(Q_k) \le \sum_k \frac{1}{\lambda} \int_{Q_k} |f(x)| \, \mathrm{d}\mu(x) \le \frac{\theta_n}{\lambda} \int_{\mathbf{R}^n} |f(x)| \, \mathrm{d}\mu(x)$$

Which is what we wanted to show. □

1.4 Calderón-Zygmund decomposition

We let Q_0 be a cube of $\mathbf{R}^n$, and $f \in L^1(Q_0)$. We define, for $\lambda > 0$,

$$E_\lambda = \{x \in Q_0 : M_{d,0} f(x) > \lambda\}$$

$M_{d,0}$ refers to the dyadic maximal functions of Q_0, where the supremum is taken over the dyadic cubes of Q_0, i.e. if we have :

$$Q_0 = \prod_{i=1}^{n} [x_i, x_i + l)$$

Then the dyadic cubes of Q_0 are those cubes Q of the form :

$$Q = \prod_{i=1}^{n} \left[x_i + \frac{k_i}{2^m} l, x_i + \frac{k_i + 1}{2^m} l\right)$$

Where $k_1, \ldots, k_n, m$ are non-negative integers with $0 \leq k_i < 2^m$, $1 \leq i \leq n$.
Now we let :

$$\lambda_0 = \fint_{Q_0} |f(x)| \, \mathrm{d}x$$

Then, for $\lambda > \lambda_0$, $E_\lambda = \bigcup_k Q_k$, with Q_k maximal dyadic such that $\fint_{Q_k} |f(x)| \, \mathrm{d}x > \lambda$. Then $Q_k \subsetneq Q_0$ and so, with Q_k^* being the dyadic parent of Q_k :

$$\lambda \leq \fint_{Q_k} |f(x)| \, \mathrm{d}x \leq 2^n \fint_{Q_k^*} |f(x)| \, \mathrm{d}x \leq 2^n \lambda \tag{1.4.1}$$

Now if $\kappa > 1$, then $E_{\kappa\lambda} \cap Q_k = \bigcup_l Q_{k,l}$, with $Q_{k,l}$ maximal dyadic cube in Q_k such that $\fint_{Q_{k,l}} |f(x)| \, \mathrm{d}x > \kappa\lambda$, and we have :

$$\kappa\lambda \leq \fint_{Q_{k,l}} |f(x)| \, \mathrm{d}x \leq 2^n \kappa\lambda \tag{1.4.2}$$

Indeed, either $Q_{k,l} \subsetneq Q_k$ and we do as previously, or $Q_{k,l} = Q_k$ and then we use (1.4.1) and $\lambda \leq \kappa\lambda$. To summarize :

Proposition 1.4. *With the same notations, we have :*

$$E_\lambda = \bigcup_k Q_k, \quad E_{\kappa\lambda} = \bigcup_{k,l} Q_{k,l}$$

With $Q_{k,l} \subset Q_k$ for all k, l, $Q_k \cap Q_{k'} = \varnothing$ if $k \neq k'$ and $Q_{k,l} \cap Q_{k,l'} = \varnothing$ if $l \neq l'$. Moreover :

$$\lambda \leq \fint_{Q_k} |f(x)| \, \mathrm{d}x \leq 2^n \lambda \tag{1.4.3}$$

$$\kappa\lambda \leq \fint_{Q_{k,l}} |f(x)| \, \mathrm{d}x \leq 2^n \kappa\lambda \tag{1.4.4}$$

1.5 Weights

In all that follows, w is a locally integrable positive function, and $\mathrm{d}\mu = w(x)\mathrm{d}x$

Definition 1.7. *Let $1 < p < \infty$. We says that w satisfies the A_p condition, or that $w \in A_p$ if, there exists a constant C_p such that for all cubes $Q \subset \mathbf{R}^n$, we have :*

$$\fint_Q w \left(\fint_Q w^{-\frac{1}{p-1}} \right)^{p-1} \leq C_p \tag{1.5.1}$$

If $p = 1$ we says that $w \in A_1$ if, there is a constant C such that for all cubes $Q \subset \mathbf{R}^n$:

$$\fint_Q w \leq \operatorname*{ess\,inf}_Q w \tag{1.5.2}$$

We also define A_∞ to be the union of the A_p :

$$A_\infty = \bigcup_{p \geq 1} A_p \tag{1.5.3}$$

Proposition 1.5. *Let $1 \leq p < \infty$, then $w \in A_p$ if and only if the Hardy Littlewood maximal function M is of weak type (p, p) for the measure μ.*

Proof. First if $p > 1$. Suppose that the maximal function is of weak type (p, p) Then for $\lambda > 0$, $f \in L^p(\mu)$, we have :

$$\mu\{Mf(x) > \lambda\} \leq C \frac{1}{\lambda^p} \int_{\mathbf{R}^n} |f(x)|^p \; w(x)\mathrm{d}x$$

Let Q be a cube of $\mathbf{R}^n$, and $\lambda = \fint_Q |f(x)| \; \mathrm{d}x$. Then for all $x \in Q$ and for $\varepsilon > 0$, we have $Mf(x) > \lambda - \varepsilon$. If f is not 0 almost everywhere on Q, then for ε small enough, then $\lambda - \varepsilon > 0$ and :

$$\mu(Q) \leq C \frac{1}{(\lambda - \varepsilon)^p} \int_{\mathbf{R}^n} |f(x)|^p \; w(x)\mathrm{d}x$$

This for all ε with $\lambda > \varepsilon > 0$, thus, using the given value of λ :

$$\left(\fint_Q |f(x)| \; \mathrm{d}x \right)^p \leq C \frac{1}{\mu(Q)} \int_{\mathbf{R}^n} |f(x)|^p \; w(x)\mathrm{d}x \tag{1.5.4}$$

Taking $f = (\varepsilon + w)^{-\frac{1}{p-1}} \mathbb{1}_Q$ for $\varepsilon > 0$, $f \in L^p$, and so applying (1.5.4) and taking $\varepsilon \to 0$ with the monotone convergence theorem, we get :

$$\fint_Q w \left(\fint_Q w^{-\frac{1}{p-1}} \right)^p \leq C \int_Q w^{-\frac{1}{p-1}}$$

Now conversely, if (1.5.1) is true. First we will shows that (1.5.4) holds. Indeed, let $Q \subset \mathbf{R}^n$ and $f \in L^p(Q, \mu)$. Then by Hölder's inequality :

$$\fint_Q |f(x)| \; \mathrm{d}x \leq \left(\frac{1}{m(Q)} \int_{\mathbf{R}^n} |f(x)|^p \; w(x)\mathrm{d}x \right)^{\frac{1}{p}} \left(\fint_Q w(x)^{-\frac{p'}{p}} \right)^{\frac{1}{p'}}$$

But $p'/p = 1/(p-1)$, and so by (1.5.1), we have :

$$\fint_Q |f(x)|\ \mathrm{d}x \leq \left(\frac{1}{m(Q)}\int_{\mathbf{R}^n} |f(x)|^p\ w(x)\mathrm{d}x\right)^{\frac{1}{p}} \left(\frac{m(Q)}{\mu(Q)}\right)^{\frac{1}{p}}$$

Which reduces to (1.5.4).

Now, take :

$$M_\mu f(x) = \sup_{Q\in\mathcal{Q}(x)} \frac{1}{\mu(Q)}\int_Q |f(y)|\ w(x)\mathrm{d}x \tag{1.5.5}$$

Where $\mathcal{Q}(x)$ is the collection of all cubes with center x. Then by (1.5.4), $Mf(x)^p \leq M_\mu f(x)$. But M_μ is of weak type (1, 1) for μ, so M is of weak type (p, p) for μ.

Now suppose that $p = 1$, and M is of weak type (1, 1). Then by (1.5.4) :

$$\fint_Q w \leq C\frac{1}{\int_Q |f|\ \mathrm{d}x}\int_{\mathbf{R}^n} |f|\ \mathrm{d}\mu$$

Let $x \in Q$, and $\varepsilon > 0$ such that $B(x, \varepsilon) \subset Q$, where $B(x, \varepsilon)$ refers to the euclidian ball of center x and with radius ε. Then taking $f = \mathbb{1}_{B(x,\varepsilon)}$, we have

$$\fint_Q w \leq C\fint_{B(x,\varepsilon)} w(x)\mathrm{d}x$$

Then by Lebesgue's differentiation theorem, for almost every $x \in Q$,

$$\fint_Q w \leq Cw(x)$$

And so w is an A_1 weight. Conversely, if $w \in A_1$, then :

$$C\int_Q |f(x)|\ w(x)\mathrm{d}x \geq \fint_Q w\int_Q |f(x)|\ \mathrm{d}x = \mu(Q)\fint |f(x)|\ \mathrm{d}x$$

And so (1.5.4) holds, and we prove M is of weak type (1, 1) as when $p > 1$. □

Corollary 1.1. *Let $1 \leq p \leq q \leq \infty$, then $A_p \subset A_q$.*

Proof. Let $w \in A_p$.

We will first prove $L^\infty(\mathrm{d}\mu) = L^\infty(\mathrm{d}x)$. This is equivalent to say that a set is negligible for μ if and only if it is negligible for the Lebesgue measure. Naturally, since $\mathrm{d}\mu = w(x)\mathrm{d}x$, then if a set is negligble for the Lebesgue measure, it is negligible for μ. Moreover since $w \in A_p$, then $w^{-1/(p-1)}$ is locally integrable and so is finite almost everywhere. Then w^{-1} is also finite almost everywhere. $\mathrm{d}x = w(x)^{-1}\mathrm{d}\mu$, and so if a set is negligible for μ, it is negligible for the Lebesgue measure.

Thus, for the measure μ, the maximal function is of weak type (p, p) and of type (∞, ∞), and by the Marcinkiewicz interpolation theorem, it is of type (q, q), and so $w \in A_q$ □

Proposition 1.6. *Let w be in A_∞. Then μ is a doubling measure. There is a constant $C > 0$ such that if Q is a cube in $\mathbf{R}^n$, then*

$$\mu(2Q) \leq C\mu(Q) \tag{1.5.6}$$

Proof. $w \in A_\infty$, then $w \in A_p$ for some $p > 1$, and, in (1.5.4), taking $f = \mathbb{1}_{\kappa^{-1}Q}$, with $\kappa > 1$, then :

$$\kappa^{-np} = \left(\frac{m(\kappa^{-1}Q)}{m(Q)}\right)^p \leq C\frac{\mu(\kappa^{-1}Q)}{\mu(Q)}$$

And so, for $\kappa > 1$, and Q a cube of $\mathbf{R}^n$:

$$\mu(\kappa Q) \leq C\kappa^{np}\mu(Q) \tag{1.5.7}$$

And so μ is a doubling measure. □

We also have the following characterizations of A_∞ weight :

Proposition 1.7. *A weight w is in A_∞ if and only if one of the following equivalent condition is satisfied :*

1. *There exist δ, $\varepsilon \in (0,1)$ such that, for all cubes $Q \subset \mathbf{R}^n$ and $E \subset Q$*

$$(m(E) < \delta m(Q)) \Rightarrow (\mu(E) < \varepsilon\mu(Q)) \tag{1.5.8}$$

2. *The weight w is a A_∞ weight if and only if, there exist a $r > 1$, and a constant C such that for all cubes $Q \subset \mathbf{R}^n$,*

$$\left(\fint_Q w^r\right)^{\frac{1}{r}} \leq c\fint_Q w \tag{1.5.9}$$

3. *A weight w is in A_∞ if and only if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for every cube $Q \subset \mathbf{R}^n$ and every $E \subset Q$*

$$(m(E) < \delta m(Q)) \Rightarrow (\mu(E) < \varepsilon\mu(Q)) \tag{1.5.10}$$

The second property is called the Reverse-Hölder. We will only prove that the last property follows from it. The same results are also true if we replace cubes with euclidian balls.

Proof. Indeed, we have, for f measurable, non-negative :

$$\begin{aligned}\fint_Q f(x)\ w(x)\mathrm{d}x &\leq \left(\fint_Q f(x)^{r'}\ \mathrm{d}x\right)^{\frac{1}{r'}}\left(\fint_Q w(x)^r\ \mathrm{d}x\right)^{\frac{1}{r}}\\ &\leq c\left(\fint_Q f(x)^{r'}\ \mathrm{d}x\right)^{\frac{1}{r'}}\fint_Q w(x)\mathrm{d}x\end{aligned}$$

Taking $f = \mathbb{1}_E$, we then have :

$$\mu(E) \leq c\left(\frac{m(E)}{m(Q)}\right)^{\frac{1}{r'}}\mu(Q) \leq c\delta^{\frac{1}{r'}}\mu(Q)$$

Then for $\delta = \left(\frac{\varepsilon}{c}\right)^{r'}$, (1.5.10) holds. □

Another consequence of reverse Hölder is the following theorem :

Theorem 1.4 (Muckhenhoupt)**.** *Let w be an A_p weight, for $1 < p < \infty$. Then, there is some $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$.*

1.6 Spectral Analysis

1.6.1 Operators on Hilbert space

We let H be a Hilbert space, and T a linear operator on H with domain $D(T)$. We will be interested in operators for which $D(T)$ is a dense subspace of H. The *graph* of T is the set $\Gamma(T) = \{(\psi, T\psi)\,;\, \psi \in D(T)\}$. T is *closed* if its graph is a closed subspace of $H \times H$.

An operator on H T' is an *extension* of T if $\Gamma(T) \subset \Gamma(T')$. T is *closable* if it has a closed extension. We write $T \subset T'$ to say that T' is an extension of T. Every closable operator T has a smallest closed extension, called its *closure*, and denoted by $\overline{T}$.

We denote by $I : H \to H$ the identity operator $I\phi = \phi$.

Definition 1.8. *Let T be a densely defined linear operator on H. Define $D(T^*)$ by :*

$$D(T^*) = \{\phi \in H;\, \exists \eta \in H, \forall \psi \in D(T),\, \langle T\psi,\, \phi\rangle = \langle \psi,\, \eta\rangle\} \tag{1.6.1}$$

When $D(T)$ is dense, then η is uniquely determined, and we define, for any $\phi \in D(T^*)$, $T^*\phi = \eta$. By the Riesz lemma, $\phi \in D(T^*)$ if and only if $|\langle T\psi,\, \phi\rangle| \leq C\|\psi\|$ for all $\psi \in D(T)$.

T^* is called the adjoint *of T.*

Theorem 1.5. *Let T be a densely defined operator on a Hilbert space H, then :*

1. *T^* is closed.*
2. *T is closable if and only if $D(T^*)$ is dense. If so, then $\overline{T} = T^{**}$*
3. *If T is closable then $\overline{T}^* = T^*$.*

Definition 1.9. *A densely defined operator T is called* symmetric *if $T \subset T^*$. Equivalently, T is symmetric if and only if :*

$$\forall \phi, \psi \in D(T),\ \langle T\phi,\, \psi\rangle = \langle \phi,\, T\psi\rangle \tag{1.6.2}$$

T is called self-adjoint *if $T = T^*$, i.e. if and only if T is symmetric and $D(T) = D(T^*)$.*

A symmetric operator T is essentially self-adjoint *if its closure is self-adjoint.*

Theorem 1.6 (Basic criterion for self-adjointness)**.** *Let T be a symmetric operator on H. The following statements are equivalent :*

1. *T is self-adjoint.*
2. *T is closed and* $\mathrm{Ker}(T^* \pm i) = \{0\}$.
3. $\mathrm{Ran}(T \pm i) = H$

Where $\mathrm{Ker}(T) = \{\phi \in D(T);\, T\phi = 0\}$ and $\mathrm{Ran}(T) = \{T\phi;\, \phi \in D(T)\}$.

Corollary 1.2. *Let T be a symmetric operator on H. The following statements are equivalent :*

1. *T is essentially self-adjoint.*
2. $\mathrm{Ker}(T^* \pm i) = \{0\}$.
3. $\mathrm{Ran}(T \pm i)$ *are dense.*

1.6.2 The spectral theorem

Let T be a closed operator on a Hilbert space H. The *resolvent set* of T is the subset of the $\lambda \in \mathbf{C}$ such that $\lambda I - T$ is a bijection of $D(T)$ onto H with a bounded inverse. If $\lambda \in \rho(T)$, then $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the *resolvent* of T at λ.

The *spectrum* $\sigma(T)$ is the complement of the resolvent. The *point spectrum* of T is the set of eigenvalues of T, i.e. the λ such that $\mathrm{Ker}(\lambda I - T) \neq \{0\}$. The *discrete spectrum* $\sigma_{disc}(A)$ is the set of eigenvalues of T of finite multiplicity, which are isolated points of the spectrum. The *essential spectrum* $\sigma_{ess}(A)$ is the complement of the discrete spectrum. In other words, it contains the element of the spectrum which are not eigenvalues, as well as eigenvalues of infinite multiplicities and limites points of the point spectrum.

The spectrum is a closed subset of the complex plane. If T is bounded, then it is a compact set. If T is symmetric, then $\sigma(T) \subset \mathbf{R}$.

Theorem 1.7 (Spectral theorem, multiplication operator form)**.** *[5] Let A be a self-adjoint operator on a separable Hilbert space H with domain $D(A)$. Then there is a measure space (M, μ), with μ a finite measure, an unitary operator $U : H \to L^2(M, \mathrm{d}\mu)$, and a real-valued function $a : M \to \mathbf{R}$, which is finite almost everywhere, such that :*

1. *$\psi \in D(A)$ if and only if $a(\cdot)(U\psi)(\cdot) \in L^2(M, \mathrm{d}\mu)$.*
2. *If $\phi \in U(D(A))$, then $(UAU^{-1}\phi)(x) = a(x)\phi(x)$.*

Idea of the proof. We first prove the spectral theorem for bounded self-adjoint operators. Using the basic criterion of self-adjointness, we can show that $(A \pm i)^{-1}$ are bounded operators, and use the spectral theorem for them. □

One of the interest of the spectral theorem is that it allow us to define functional calculus on self-adjoint operators. If h is a bounded Borel function on $\mathbf{R}$ we define $h(A) = U^{-1} T_{h(a)} U$, where T_m is the operator on L^2 defined by $T_m\psi(x) = m(x)\psi(x)$. In this way we get :

Theorem 1.8 (Spectral theorem, functional calculus form)**.** *Let A be a self-adjoint operator on H. Then there is a unique map Φ from the bounded Borel functions on $\mathbf{R}$ into the bounded linear operators on H so that :*

1. *Φ is an algebraic *-homomorphism, i.e. it is an algebra homomorphism and $\Phi\left(\widehat{f}\right) = \Phi(f)^*$.*
2. *Φ is norm-continuous, that is $\|\Phi(h)\|_{\mathcal{L}(H)} \leq \|h\|_\infty$.*
3. *Let h_n be a sequence of bounded Borel functions with $h_n(x) \to x$ for each x and $|h_n(x)| \leq |x|$ for all x and n. Then, for any $\psi \in D(A)$, $\lim_{n\to\infty} \Phi(h_n)\psi = A\psi$.*
4. *If $h_n(x) \to h(x)$ pointwise and if the sequence $\|h_n\|_\infty$ is bounded, then $\Phi(h_n) \to \Phi(h)$ strongly, i.e. for all ψ, $\|\Phi(h_n)\psi - \Phi(h)\psi\| \to 0$.*
5. *If $A\psi = \lambda\psi$, then $\Phi(h)\psi = h(\lambda)\psi$.*
6. *If $h \geq 0$, then $\Phi(h) \geq 0$.*

Example 1.1. *If we take the Fourier transform $\mathcal{F}$ for the operator $A = -\Delta$ on $L^2(\mathbf{R}^n)$, with domain $D(A) = \{\psi \in L^2; \Delta\psi \in L^2\}$, then we have $\mathcal{F}(-\Delta\psi)(\xi) = 4\pi^2|\xi|^2\mathcal{F}\psi(\xi)$. We have $M = \mathbf{R}^n$, $\mathrm{d}\mu = \mathrm{d}x$, $U = \mathcal{F}$, $a(\xi) = 4\pi^2|\xi|^2$. Though in this case, μ isn't a finite measure.*

We can now define $h(-\Delta)$ by $\mathcal{F}(h(-\Delta)\psi)(\xi) = h\left(4\pi^2|\xi|^2\right)\mathcal{F}\psi(\xi)$.

This representation also let us study the spectrum of $-\Delta$. λ is in the resolvent set if and only if there is a constant $c > 0$ such that, for almost every $\xi \in \mathbf{R}^n$, $\left|4\pi^2|\xi|^2 - \lambda\right| \geq c$. This happen if and only if λ is not a non-negative real number. Thus $\sigma(-\Delta) = [0, +\infty)$. Since the spectrum has no isolated point, then $\sigma_{ess}(-\Delta) = \sigma(-\Delta) = [0, +\infty)$.

The following criterion is useful to determine the spectrum of an operator :

Theorem 1.9 (Weyl's criterion)**.** *Let A be a self-adjoint operator. Then $\lambda \in \sigma(A)$ if and only if there exists $\{\psi_n\}_{n=1}^{\infty}$ in $D(A)$ so that for all $n \geq 1$, $\|\psi_n\| = 1$ and $\lim_{n\to\infty} \|(A-\lambda)\psi_n\| = 0$. $\lambda \in \sigma_{ess}(A)$ if and only if the $\{\psi_n\}$ can be chosen to be orthogonal.*

Proposition 1.8. *Let (M, μ) be a measure space, with μ a finite measure. Let a be a measurable, real-valued function on M, which is finite almost everywhere. We define the operator A on $L^2(M, \mu)$ by $D(A) = \left\{\psi \in L^2(M, \mu);\, a\psi \in L^2(M, \mu)\right\}$, and $A\psi = a\psi$. Then A is self-adjoint and its spectrum is the essential range of A :*

$$\sigma(A) = \left\{\lambda \in \mathbf{R};\, \forall \varepsilon > 0, \mu\left(a^{-1}(\lambda - \varepsilon, \lambda + \varepsilon\right) > 0\right\} \tag{1.6.3}$$

Proof. That A is symmetric is clear. Let $\psi \in D(A^*)$, and $\chi_N = \mathbb{1}_{\{|f(x)| \leq N\}}$. Then by the monotone convergence theorem,

$$\begin{aligned}
\|A^*\psi\| &= \lim_{N\to\infty} \|\chi_N A^*\psi\| \\
&= \lim_{N\to\infty} \left(\sup_{\|\phi\|=1} |\langle \phi, \chi_N A^*\psi\rangle|\right) \\
&= \lim_{N\to\infty} \left(\sup_{\|\phi\|=1} |\langle A\chi_N \phi, \psi\rangle|\right) \\
&= \lim_{N\to\infty} \left(\sup_{\|\phi\|=1} |\langle \phi, \chi_N a\psi\rangle|\right) \\
&= \lim_{N\to\infty} \|\chi_N a\psi\|
\end{aligned} \tag{1.6.4}$$

Thus $a\psi \in L^2(M, \mu)$, so $\psi \in D(A)$, and A is self-adjoint.

Now, let $\lambda \in \mathbf{R}$. $(A-\lambda)\psi(x) = (a(x)-\lambda)\psi(x)$. $\lambda \in \rho(A)$ if and only if $(A-\lambda)$ has a bounded inverse. When this inverse exist, then

$$(A-\lambda)^{-1}\phi(x) = \frac{1}{a(x)-\lambda}\phi(x) \tag{1.6.5}$$

And conversely, if the right hand side define a bounded operator on $L^2(M)$, then the inverse of $A - \lambda$ exists and is bounded. A multiplication operator on L^2 is bounded if and only if the multiplier is in L^∞.

Thus λ is in the resolvent set of A if and only if $(a-\lambda)^{-1}$ is essentially bounded. That is equivalent to say that there is a constant $C > 0$ such that for almost every $x \in M$, we have $(a(x)-\lambda)^{-1} \leq C$, or equivalently, $(a(x)-\lambda) \geq 1/C > 0$, i.e. there is a constant $\varepsilon > 0$ such that $\mu\left(a^{-1}(\lambda-\varepsilon, \lambda+\varepsilon)\right) = 0$.

And so λ is in the resolvent set if and only if λ is not in the essential range of A. □

Proposition 1.9. *Let A be a self-adjoint operator, then we have :*

$$\inf_{\|\psi\|=1} \langle A\psi,\, \psi\rangle = \inf \sigma(A) \tag{1.6.6}$$

Proof. By the spectral theorem, we can see A as a multiplication operator on a $L^2(M,\mu)$ space with μ a finite measure. Then $\langle A\psi,\, \psi\rangle = \int_M a(x)|\psi(x)|^2 \, \mathrm{d}x \geq \operatorname{ess\,inf}_x a(x)\|\psi\|_2 = \operatorname{ess\,inf}_x a(x)$, if $\|\psi\| = 1$.

Now assume $\operatorname{ess\,inf}_x a(x) = c \in \mathbf{R}$. Then for all $\varepsilon > 0$, there is a non-negligible set E on which $c \leq a(x) < c+\varepsilon$. Taking $\psi = \frac{1}{\mu(E)^{1/2}}\mathbb{1}_E$, we have $c \leq \langle A\psi,\, \psi\rangle \leq (c+\varepsilon)\|\psi\|^2 = c+\varepsilon$. And so :

$$\inf_{\|\psi\|=1} \langle A\psi,\, \psi\rangle = \operatorname*{ess\,inf}_x a(x) = \inf \sigma(A) \tag{1.6.7}$$

If $\operatorname{ess\,inf}_x a(x) = -\infty$, then for all $C > 0$, the measure of the set $E = \{a(x) < -C\}$ is non-zero. Taking again $\psi = \frac{1}{\mu(E)^{1/2}}\mathbb{1}_E$, $\langle A\psi,\, \psi\rangle \leq -C$. Thus $\inf\langle A\psi,\, \psi\rangle = -\infty$. □

1.6.3 Quadratic forms

Definition 1.10. *A* quadratic form *is a map $q : Q(q) \times Q(q) \to \mathbf{C}$, where $Q(q)$ is a dense linear subspace of H called the* form domain, *such that $q(\cdot,\psi)$ is conjugate linear and $q(\phi,\cdot)$ is linear for $\phi,\psi \in Q(q)$. If $q(\phi,\psi) = \overline{q(\psi,\phi)}$ we say that q is* symmetric. *If $q(\phi,\phi) \geq 0$ for all $\phi \in Q(q)$, q is called* positive, *and if $q(\phi,\phi) \geq -M\|\phi\|^2$ for some M we say that q is* semibounded.

Definition 1.11. *Let q be a semibounded quadratic form, $q(\phi,\, \phi) \geq -M\|\phi\|^2$. q is called* closed *if $Q(q)$ is complete under the norm :*

$$\|\phi\|_{+1} = \sqrt{q(\phi,\phi) + (M+1)\|\phi\|^2} \tag{1.6.8}$$

If q is closed and $D \subseteq Q(q)$ is dense in $Q(q)$ in the $\|\cdot\|_{+1}$ norm, then D is called a form core *for q.*

The $\|\cdot\|_{+1}$ norm comes from the inner product $\langle\psi,\, \phi\rangle_{+1} = q(\psi,\, \phi) + (M+1)\langle\psi,\, \phi\rangle$.

Theorem 1.10. *If q is a closed semibounded quadratic form, then q is the quadatic form of a unique self-adjoint operator.*

Theorem 1.11 (Friedrichs extension)**.** *[6] Let A be a positive symmetric operator, and let $q(\phi,\, \psi) = \langle\phi,\, A\psi\rangle$ for $\phi,\psi \in D(A)$. Then q is a closable quadratic form and its closure $\hat{q}$ is the quadratic form of a unique self adjoint operator $\hat{A}$. $\hat{A}$ is a positive extension of A, and the lowere bound of its spectrum is the lower bound of q. Further, $\hat{A}$ is the only self-adjoint extension of A whose domain is contained in the form domain of $\hat{q}$. Then q is a closable quadratic form and its closure $\hat{q}$ is the quadratic form of a unique self adjoint operator $\hat{A}$. $\hat{A}$ is a positive extension of A, and the lower bound of its spectrum is the lower bound of q. Further, $\hat{A}$ is the only self-adjoint extension of A whose domain is contained in the form domain of $\hat{q}$.*

Example 1.2. *We define the Schrödinger operator $H = -\Delta - V$, $V \in L^1_{loc}$, with domain $D(H) = \left\{\psi \in L^2;\, \Delta\psi \in L^2,\, V\psi \in L^2\right\}$. If H is densely defined and semibounded, then the Friedrichs extension $\hat{H}$ exists.*

The quadratic form $\langle\nabla\phi,\, \nabla\psi\rangle + \langle\phi,\, V\psi\rangle$ actually always is well defined at least on $\mathcal{C}^\infty_c$. If it is semibounded, and if it is closable, then its closure is associated with a self-adjoint operator. It allows us to give a sense to $-\Delta - V$ even when its domain wouldn't be dense.

2. Fractional integrals

2.1 Riesz Potentials

In the following chapter, we define the *Riesz Potentials* I_α by :

$$I_\alpha f(x) = c_{\alpha,n} \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \,\mathrm{d}y, \quad 0 < \alpha < n \tag{2.1.1}$$

Defining K_α by :

$$K_\alpha(x) = c_{\alpha,n}|x|^{\alpha-n} \tag{2.1.2}$$

Then :

$$I_\alpha f = K_\alpha * f \tag{2.1.3}$$

K is locally integrable, and bounded on $\{|x| > 1\}$, so I_α is well defined at least for $f \in \mathcal{S}(\mathbf{R}^n)$. We choose $c_{\alpha,n}$ such that the following is true :

Proposition 2.1.

$$\mathcal{F}(K_\alpha)(\xi) = |2\pi\xi|^{-\alpha} \tag{2.1.4}$$

Where we use for the Fourier transform :

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\mathrm{e}^{-2i\pi x\cdot\xi} \,\mathrm{d}x$$

Thus $\mathcal{F}\left(I_\alpha f\right)(\xi) = |2\pi\xi|^{-\alpha}\mathcal{F}f(\xi)$ and, $I_\alpha = (-\Delta)^{-\alpha/2}$.

Proof. For $t > 0$ and $x \in \mathbf{R}^n$, we define

$$g_t(x) = \mathrm{e}^{-4\pi^2 t|x|^2} \tag{2.1.5}$$

We have :

$$\widehat{g_t}(\xi) = \frac{1}{(4t\pi)^{n/2}}\mathrm{e}^{-\frac{|\xi|^2}{4t}}$$

Notice that we have, for $\gamma > 0$:

$$\begin{aligned}\int_0^\infty t^\gamma \mathrm{e}^{-4\pi^2 t|x|^2} \frac{\mathrm{d}t}{t} &= \left(\frac{1}{2\pi|x|}\right)^{2\gamma} \int_0^\infty s^\gamma \mathrm{e}^{-s} \frac{\mathrm{d}s}{s} \\ &= \frac{\Gamma(\gamma)}{(2\pi)^{2\gamma}} \frac{1}{|x|^{2\gamma}}\end{aligned}$$

On the other hand, we have :

$$\left(\frac{1}{2\sqrt{\pi}}\right)^n \int_0^\infty t^{\gamma-\frac{n}{2}} \mathrm{e}^{-\frac{|\xi|^2}{4t}} \frac{\mathrm{d}t}{t} = \frac{\Gamma\left(\frac{n}{2}-\gamma\right)}{2^{2\gamma}\pi^{n/2}} \frac{1}{|\xi|^{n-2\gamma}}$$

And we just need to justify that :

$$\mathcal{F}\left(\int_0^\infty g_t(\cdot)t^\gamma \frac{\mathrm{d}t}{t}\right)(\xi) = \int_0^\infty \widehat{g_t}(\xi)t^\gamma \frac{\mathrm{d}t}{t} \tag{2.1.6}$$

We let $G_\gamma(x)$ refers to :

$$G_\gamma(x) = \int_0^\infty g_t(x)t^\gamma \frac{\mathrm{d}t}{t} = C_{\gamma,n}|x|^{-2\gamma}$$

For $\gamma < n/2$, $G_\gamma \in L^1 + L^\infty$, and so G_γ is a tempered distribution and its Fourier transform is well defined. We let

$$G_{\gamma,N}(x) = \int_{\frac{1}{N}}^N g_t(x)t^\gamma \frac{\mathrm{d}t}{t}$$

Then for $\phi \in \mathcal{S}(\mathbf{R}^n)$:

$$\langle G_{\gamma,N}, \phi\rangle = \int_{\mathbf{R}^n}\int_{1/N}^N g_t(x)t^\gamma\phi(x)\frac{\mathrm{d}t}{t} \tag{2.1.7}$$

We have $|G_{\gamma,N}(x)\phi(x)| \leq G_\gamma(x)\phi(x)$ which is integrable since $G_\gamma \in L^1 + L^\infty$ and ϕ is rapidly decreasing, so $G_{\gamma,N} \to G_\gamma$ in the sense of tempered distributions, and so $\mathcal{F}G_{\gamma,N} \to \mathcal{F}G_\gamma$ in the sense of tempered distributions.

$$\begin{aligned}\langle \mathcal{F}G_{\gamma,N}, \phi\rangle &= \int_{R^n}\int_{R^n}\left(\int_{1/N}^N g_t(x)t^\gamma\frac{\mathrm{d}t}{t}\right)\phi(\xi)\mathrm{e}^{-2i\pi x\cdot\xi}\ \mathrm{d}\xi\ \mathrm{d}x \\ &= \int_{R^n}\phi(\xi)\int_{1/N}^N t^\gamma\int_{R^n} g_t(x)\mathrm{e}^{-2i\pi x\cdot\xi}\ \mathrm{d}x\frac{\mathrm{d}t}{t}\ \mathrm{d}\xi \\ &= \int_{R^n}\phi(\xi)\int_{1/N}^N \widehat{g_t}(\xi)t^\gamma\frac{\mathrm{d}t}{t}\ \mathrm{d}\xi\end{aligned} \tag{2.1.8}$$

The changes in order of integration is justified as $G_{\gamma,N}$ is integrable as we have :

$$|G_{\gamma,N}(x)| \leq N^{\gamma+1}\left(N - \frac{1}{N}\right)\mathrm{e}^{-4\pi^2|x|^2/N}$$

And $(t,x) \mapsto g_t(x)t^{\gamma-1}\mathrm{e}^{-2i\pi x\cdot\xi}$ is integrable on $(1/N, N)\times\mathbf{R}^n$.
And so :

$$\mathcal{F}G_{\gamma,N}(\xi) = \int_{1/N}^N \widehat{g_t}(\xi)t^\gamma\frac{\mathrm{d}t}{t} \to \int_0^\infty \widehat{g_t}(\xi)t^\gamma\frac{\mathrm{d}t}{t} \tag{2.1.9}$$

Then for $\gamma = \frac{n-\alpha}{2}$

$$\mathcal{F}\left(\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{1}{2^\alpha\pi^{n/2}}|x|^{-(n-\alpha)}\right) = |2\pi\xi|^{-\alpha} \tag{2.1.10}$$

□

Theorem 2.1 (Hardy-Littlewood-Sobolev). *Let $\alpha \in (0,n)$, $p \in (1, n/\alpha)$. Let $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, i.e. $q = np/(n-p\alpha)$. Then*

$$\|I_\alpha f\|_{L^q} \leq C(n,p,\alpha)\|f\|_{L^p} \tag{2.1.11}$$

For $p=1$, we instead have the following, for $\frac{1}{q} = 1 - \frac{\alpha}{n}$:

$$m\{x \in \mathbf{R}^n : |I_\alpha f(x)| > \lambda\} \leq C(n,\alpha)\left(\frac{\|f\|_{L^1}}{\lambda}\right)^q \tag{2.1.12}$$

In the following, we will take $I_\alpha f(x) = \int_{\mathbf{R}^n} f(y)|x-y|^{\alpha-n}\,\mathrm{d}y$, since the constant does not meaningfully impact the results.

Proof. For $K(x) = |x|^{\alpha-n}$, we let $K = K_1 + K_\infty$, with :

$$K_1(x) = \begin{cases} K(x) & x \leq \mu \\ 0 & x > \mu \end{cases} \qquad K_\infty(x) = \begin{cases} 0 & x \leq \mu \\ K(x) & x > \mu \end{cases}$$

Where $\mu > 0$ is a constant. Then $K_1 \in L^1$, thus, for all $f \in L^p$, $K_1 * f \in L^p$. Meanwhile, $K_\infty \in L^{p'}$. Indeed, if $p > 1$, we have $\frac{1}{p} > \frac{\alpha}{n}$, thus $\frac{1}{p'} < 1 - \frac{\alpha}{n}$, i.e. $p'(n-\alpha) > n$ and $K_\infty(x)^{p'}$ is integrable. Thus, for all $f \in L^p$, $K_\infty * f \in L^\infty$. If $p = 1$, then $K_2 \in L^\infty$ is obvious.

And so $I_\alpha f$ is defined for all $f \in L^p$, $1 \leq p < \frac{n}{\alpha}$.

We will prove that the following weak type estimate holds for all $1 < p < \frac{n}{\alpha}$:

$$m\{x \in \mathbf{R}^n : |I_\alpha f(x)| > \lambda\} \leq C_{n,\alpha,p}\left(\frac{\|f\|_p}{\lambda}\right)^q \tag{2.1.13}$$

It is sufficient to show that (2.1.13) for $\|f\|_p = 1$. Then just apply it to $\frac{f}{\|f\|_p}$ with $\frac{\lambda}{\|f\|_p}$. It is also sufficient to prove that (2.1.13) holds but for $\{|I_\alpha f| > 2\lambda\}$ instead.

Then we estimate :

$$m\{|K_1 * f| > \lambda\} \leq \frac{\|K_1 * f\|_p^p}{\lambda^p} \leq \frac{\|K_1\|_1^p}{\lambda^p} = c_1\left(\frac{\mu^\alpha}{\lambda}\right)^p$$

Since :

$$\|K_1\|_1 = c\int_0^\mu r^{\alpha-1}\,\mathrm{d}r = c_1\mu^\alpha$$

But we also have :

$$\|K_\infty * f\|_\infty \leq \|K_\infty\|_{p'} = c_2\mu^{-\frac{n}{q}}$$

Since :

$$\|K_\infty\|_{p'} = c\left(\int_\mu^\infty r^{(\alpha-n)p'+n-1}\,\mathrm{d}r\right)^{\frac{1}{p'}} = c_2\mu^{\alpha-n+\frac{n}{p'}} = c_2\mu^{-\frac{n}{q}}$$

Then take μ such that $c_2\mu^{-\frac{n}{q}} = \lambda$, i.e. $\mu = c_3\lambda^{-\frac{q}{n}}$. Then $\|K_\infty * f\|_\infty \leq \lambda$ and so, since $\frac{\alpha pq}{n} = q - p$:

$$m\{|I_\alpha f| > 2\lambda\} \leq m\{|K_1 * f| > \lambda\} \leq c_4 \lambda^{-\left(\frac{qp\alpha}{n}+p\right)} = c_4 \left(\frac{\|f\|_p}{\lambda}\right)^q$$

□

2.2 Weighted estimates

We now search for the locally integrable functions V such that we have a weighted equivalent to the Hardy-Littlewood Sobolev inequality. Specifically, we want to have :

$$\|I_\alpha f(x)V(x)\|_q \leq C\|I_\alpha f(x)V(x)\|_p \tag{2.2.1}$$

B. Muckenhoupt and R.L. Wheeden established in [4] that this inequality holds if and only V is such that there exist a constant $c > 0$, such that for all cubes $Q \subset \mathbf{R}^n$, we have :

$$\left(\fint_Q V(x)^q \, \mathrm{d}x\right)^{\frac{1}{q}} \left(\fint_Q V(x)^{-p'} \, \mathrm{d}x\right)^{\frac{1}{p'}} \leq c \tag{2.2.2}$$

This is equivalent to $V^q \in A_r$ with $r = 1 + \frac{q}{p'}$.

In order to establish those estimates, we will use the following fractional maximal function :

$$M_\alpha f(x) = \sup_{r>0} m(Q)^{-1+\frac{\alpha}{n}} \int_{Q(x,r)} |f(y)| \, \mathrm{d}y \tag{2.2.3}$$

Where $Q(x,r)$ is the cube of center x and radius r.

2.2.1 Estimates on $M_\alpha f$

In the following, for $\lambda > 0$ we let

$$E_\lambda = \{x \in \mathbf{R}^n : M_\alpha f(x) > \lambda\} \tag{2.2.4}$$

We first show the following weak-type estimate :

Theorem 2.2. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let V be a locally integrable and non-negative function satisfying* (2.2.2)*. Then, there is a constant $C(n, \alpha, p, V)$, independant of f, such that, for all $\lambda > 0$:*

$$\left(\int_{E_\lambda} V(x)^q \, \mathrm{d}x\right)^{\frac{1}{q}} \leq \frac{C(n,\alpha,p,V)}{\lambda} \left(\int_{\mathbf{R}^n} |f(x)V(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \tag{2.2.5}$$

Proof. Let $R > 0$, we let $E_{\lambda,R} = E_\lambda \cap \{|x| < R\}$. By definition, for each $x \in E_{\lambda,R}$, there is a cube Q_x with center x such that :

$$m(Q_x)^{-1+\alpha/n} \int_{Q_x} |f(y)| \, \mathrm{d}y > \lambda$$

Then using Theorem 1.1, we extract a subsequence of cubes $\{Q_k\}_k$ such that any point of $\mathbf{R}^n$ is in at most θ_n of the cubes. Then since $p/q \leq 1$ we have

$$\begin{aligned}\left(\int_{E_{\lambda,R}} V(x)^q \,\mathrm{d}x\right)^{\frac{p}{q}} &\leq \left(\sum_k \int_{Q_k} V(x)^q \,\mathrm{d}x\right)^{\frac{p}{q}} \\ &\leq \sum_k \left(\int_{Q_k} V(x)^q \,\mathrm{d}x\right)^{\frac{p}{q}}\end{aligned} \tag{2.2.6}$$

Moreover we have, for all k :

$$\lambda < m(Q_k)^{-1+\alpha/n} \int_{Q_k} |f(x)| \,\mathrm{d}x \tag{2.2.7}$$

So that :

$$\left(\int_{E_{\lambda,R}} V(x)^q \,\mathrm{d}x\right)^{\frac{p}{q}} \leq \sum_k \left(\frac{m(Q_k)^{-1+\alpha/n}}{\lambda} \int_{Q_k} |f(x)| \,\mathrm{d}x \left(\int_{Q_k} V(x)^q \,\mathrm{d}x\right)^{\frac{1}{q}}\right)^p$$

By Hölder, we have :

$$\int_{Q_k} |f(x)| \,\mathrm{d}x \leq \left(\int_{Q_k} |f(x)V(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{Q_k} V(x)^{-p'} \,\mathrm{d}x\right)^{\frac{1}{p'}}$$

And finally, since $1/p' + 1/q = 1 - \alpha/n$, using (2.2.2) :

$$\left(\int_{E_{\lambda,R}} V(x)^q \,\mathrm{d}x\right)^{\frac{p}{q}} \leq \sum_k \left(\frac{c}{\lambda}\right)^p \int_{Q_k} |f(x)V(x)|^p \,\mathrm{d}x$$

And so, since no $x \in \mathbf{R}^n$ is in more than θ_n of the cubes Q_k, we get :

$$\left(\int_{E_{\lambda,R}} V(x)^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq c\theta_n^{1/p} \frac{1}{\lambda} \left(\int_{\mathbf{R}^n} |f(x)V(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \tag{2.2.8}$$

This, for all $R > 0$, and so taking $R \to \infty$, we get (2.2.5). $\square$

We can now use Theorem 2.2 to prove the following norm inequality :

Theorem 2.3. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. V a locally integrable and non-negative function satisfying (2.2.2). Then there is a constant C independant of f such that :*

$$\left(\int_{\mathbf{R}^n} |M_\alpha f(x)V(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbf{R}^n} |f(x)V(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \tag{2.2.9}$$

Proof. $w = V^q$ satisfies A_r, for $r = 1 + \frac{q}{p'}$. Thus, there is a r_1 with $1 < r_1 < r$ such that w satisfies A_{r_1}. $r_1 = 1 + \frac{q_1}{p_1'}$, $1 < p_1 < p$, and $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$.

Indeed, let p_1, q_1 be defined as such, we simply need to check $1 < p_1 < p$. Notice that $p_1 < p$ if and only if $q_1 < q$. But $q_1(1 - 1/p_1) < q(1 - 1/p)$, but then rewriting p, p_1 in term of q, q_1, we get $q_1 < q$. $p_1 > 1$ simply because otherwise, we would have $r_1 \leq 1$.

Thus, by Theorem 2.2, letting $\mathrm{d}\mu = w(x)\mathrm{d}x$, we have :

$$\mu\{x \in \mathbf{R}^n : M_\alpha f(x) > \lambda\} \leq \frac{C}{\lambda^{q_1}} \left(\int_{\mathbf{R}^n} |f(x)V(x)|^p \, \mathrm{d}x \right)^{\frac{q_1}{p_1}}$$

We define a sublinear operator T by :

$$Tg(x) = M_\alpha \left(g(x) w(x)^{\frac{\alpha}{n}} \right) \tag{2.2.10}$$

And we let $g(x)$ be such that $f(x) = g(x)w(x)^{\alpha/n}$. Then :

$$\mu\{x \in \mathbf{R}^n : Tg(x) > \lambda\} \leq \frac{C}{\lambda^q} \left(\int_{\mathbf{R}^n} |g(x)|^{p_1} w(x) \, \mathrm{d}x \right)^{\frac{q_1}{p_1}} \tag{2.2.11}$$

And so, for the measure μ, T is of weak type (p_1, q_1)

In the same way, $w \in A_{r_2}$ with $r < r_2$, $r_2 = 1 + \frac{q_2}{p_2'}$, $p < p_2 < \frac{n}{\alpha}$, and T is of weak type (p_2, q_2). Since we have $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{n}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For the $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, we have $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

Then by the Marcinciewicz interpolation theorem, T is of strong type (p,q) for the measure μ. That is to say :

$$\left(\int_{\mathbf{R}^n} \left| M_\alpha \left(g w^{\frac{\alpha}{n}} \right)(x) \right|^q w(x) \mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbf{R}^n} |g(x)|^p \, w(x) \mathrm{d}x \right)^{\frac{1}{p}} \tag{2.2.12}$$

Then for $g(x) = f(x)w(x)^{\frac{-\alpha}{n}}$ we get, since $-\frac{\alpha p}{n} = \frac{p}{q} - 1$ we get (2.2.9). $\square$

2.2.2 Comparison of $I_\alpha f$ and $M_\alpha f$

Theorem 2.4. *Let $0 < \alpha < n$, w be an A_∞ weight and $0 < q < \infty$. Then there is a constant C, independant of f, such that we have :*

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^q \, w(x) \mathrm{d}x \leq C \int_{\mathbf{R}^n} |M_\alpha f(x)|^q \, w(x) \mathrm{d}x \tag{2.2.13}$$

As well as :

$$\sup_{\lambda > 0} \lambda^q \mu\{x \in \mathbf{R}^n : \, |I_\alpha f(x)| > \lambda\} \leq C \sup_{\lambda > 0} \lambda^q \mu\{x \in \mathbf{R}^n : \, |M_\alpha f(x)| > \lambda\} \tag{2.2.14}$$

Lemma 2.1. *There exist positive constants C, K, such that, if $\lambda > 0$, $\gamma > 0$ and $\kappa > K$, and if $f \geq 0$ and Q is a cube such that there is a $x \in Q$ with $I_\alpha f(x) \leq \lambda$, then :*

$$m\{x \in Q : \, I_\alpha f(x) > \kappa\lambda, \, M_\alpha f(x) \leq \gamma\lambda\} \leq C \left(\frac{\gamma}{\kappa} \right)^{\frac{n}{n-\alpha}} m(Q) \tag{2.2.15}$$

Proof. We let $g = f \mathbb{1}_{2Q}$, $h = f - g$. By Theorem 2.1 :

$$m\left\{ x \in \mathbf{R}^n : \, |I_\alpha g(x)| > \frac{\kappa\lambda}{2} \right\} \leq C \left(\frac{1}{\kappa\lambda} \int_{\mathbf{R}^n} |g(x)| \, \mathrm{d}x \right)^{\frac{n}{n-\alpha}}$$

Let $t \in Q$ be such that $M_\alpha f(t) \leq \gamma\lambda$. If there's no such t, then the lemma is trivial. Let P be the cube of center t, with sides parallel to the axes and three time as long as Q. Then $2Q \subset P$ and :

$$\int_{\mathbf{R}^n} |g(x)| \,\mathrm{d}x \leq \int_P |f(x)| \,\mathrm{d}x \leq m(P)^{1-\frac{\alpha}{n}} M_\alpha f(t) \leq \gamma\lambda m(3Q)^{1-\frac{\alpha}{n}}$$

Then :

$$m\left\{x \in \mathbf{R}^n : |I_\alpha g(x)| > \frac{\kappa\lambda}{2}\right\} \leq C\left(\frac{\gamma}{\kappa}\right)^{\frac{n}{n-\alpha}} m(3Q) \tag{2.2.16}$$

Now let $s \in Q$ such that $I_\alpha f(s) \leq \lambda$. Then there is a $L \geq 1$, depending only on n such that if $y \notin 2Q$ and $x \in Q$,

$$|s-y| \leq L|x-y|$$

Indeed, $|s-y| \leq |s-x| + |x-y|$. But $x \in Q$, $y \notin 2Q$, so $|x-y| \geq \mathrm{d}\left(Q, (2Q)^c\right)$. But this distance is exactly the radius of Q, and $\mathrm{diam}(Q) \leq 2\sqrt{n} r_Q$. Thus :

$$|s-y| \leq \left(1+2\sqrt{n}\right)|x-y|$$

$$I_\alpha h(x) \leq L^{n-\alpha} \int_{\mathbf{R}^n \setminus 2Q} \frac{f(y)}{|s-y|^{n-\alpha}} \,\mathrm{d}y \leq L^{n-\alpha} I_\alpha f(s) \leq L^{n-\alpha}\lambda \tag{2.2.17}$$

Then take $K = 2L^{n-\alpha}$. If $\kappa \geq K$, then we have $I_\alpha h(x) \leq \frac{\kappa\lambda}{2}$. We thus have :

$$\{x \in Q : I_\alpha f(x) > \kappa\lambda\} \subset \left\{x \in Q : I_\alpha g(x) > \frac{\kappa\lambda}{2}\right\}$$

Then either there is a $t \in Q$ with $I_\alpha f(t) \leq \gamma\lambda$ and we can apply (2.2.16), or there isn't and the measure of the set we're trying to estimate is zero. In both case, (2.2.15) holds. $\square$

proof of the theorem. Let f be locally integrable. We can assume $f \geq 0$: replacing f by $|f|$, we only increase the left sides of (2.2.13) and (2.2.14). We first take f with compact support. $\{I_\alpha f > \lambda\}$ is an open set.

Indeed, if f is essentially bounded and with compact support K, then :

$$|I_\alpha f(x) - I_\alpha f(z)| \leq \|f\|_\infty \int_K \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|z-y|^{n-\alpha}} \right| \mathrm{d}y$$

And by Lebesgue's dominated convergence theorem, then integral goes to 0 as $z \to x$. Now, we let $f_m = f\mathbb{1}_{\{f<m\}} + m\mathbb{1}_{\{f \geq m\}}$. Since f has compact support, so does f_m.

f_m is increasing, and for all $x \in \mathbf{R}^n$, $f_m(x) \to f(x)$. By the dominated convergence theorem, for all $x \in \mathbf{R}^n$, $I_\alpha f_m(x) \to I_\alpha f(x)$, and $I_\alpha f_m$ is also an increasing sequence. Thus :

$$\{I_\alpha f > \lambda\} = \bigcup_{m \in \mathbf{N}} \{I_\alpha f_m > \lambda\}$$

And so $\{I_\alpha f > \lambda\}$ is open. We use Theorem 1.2 :

$$\{x \in \mathbf{R}^n : I_\alpha f(x) > \lambda\} = \bigcup_j Q_j$$

With the Q_j being disjoint cubes such that for each cube Q_j, there is a $x \in Q_j$ with $|I_\alpha f(x)| \leq \lambda$. Then for :

$$E_j = \{x \in Q_j : I_\alpha f(x) > \kappa\lambda,\, M_\alpha f(x) \leq \gamma\lambda\}$$

By the lemma applied to $4Q_j$:

$$m(E_j) \leq C4^n \left(\frac{\gamma}{\kappa}\right)^{\frac{n}{n-\alpha}} m(Q_j)$$

Where we take $\kappa = \min(1, K)$, and for $\delta > 0$ associated, in the A_∞ condition satisfied by w, with $\varepsilon = \frac{1}{2}\kappa^{-q}$. Then, we let Γ be such that $C4^n \left(\frac{\Gamma}{\kappa}\right)^{n/(n-\alpha)} = \delta$. Then, for all $\gamma \leq \Gamma$, we have :

$$\mu(E_j) \leq \frac{1}{2}\kappa^{-q}\mu(Q_j)$$

Then :

$$\mu\{I_\alpha f > \kappa\lambda,\, M_\alpha f \leq \gamma\lambda\} \leq \frac{1}{2}\kappa^{-q}\mu\{I_\alpha f > \lambda\} \tag{2.2.18}$$

And so :

$$\mu\{I_\alpha f > \kappa\lambda\} \leq \mu\{M_\alpha f > \gamma\lambda\} + \frac{1}{2}\kappa^{-q}\mu\{I_\alpha f > \lambda\} \tag{2.2.19}$$

Now we let Q be a cube containing the support of f. Then, if $x \notin 3Q$, if P is the smallest cube with center x containing Q, and u the point of Q closest to x. Then there's is a L, depending only on the dimension n and $L \geq 1$, such that :

$$m(P) \leq L|x-u|^n$$

Indeed. First, since $x \notin 3Q$, then $|x-u| \geq 2r_Q$. Moreover, $r_P \leq |x-u| + 2r_Q$, since the cube with this as radius and centered in x with contain Q : Indeed, let $y \in Q$, then :

$$|x-y|_\infty \leq |u-x|_\infty + |u-y|_\infty \leq |u-x| + 2r_Q$$

Thus $r_P \leq 2|x-u|$. And so :

$$m(P) \leq 4^n|x-u|^n$$

Then :

$$I_\alpha f(x) \leq \frac{1}{|x-u|^{n-\alpha}} \int_P f(y)\, \mathrm{d}y \leq L^n m(P)^{1-\frac{\alpha}{n}} \int_P f(y)\, \mathrm{d}y \leq L^n M_\alpha f(x)$$

Then for $\gamma = \min(\Gamma,\, 1/L^n)$, we have :

$$\{I_\alpha f > \lambda\} \cap (3Q)^c \subset \{M_\alpha f > \gamma\lambda\}$$

And :

$$\mu\{I_\alpha f > \kappa\lambda\} \leq 2\mu\{M_\alpha f > \gamma\lambda\} + \frac{1}{2}\kappa^{-q}\mu(\{I_\alpha f > \lambda\} \cap 3Q) \tag{2.2.20}$$

Then :

$$\kappa^{-q}\int_0^{\kappa N} \lambda^{q-1}\mu\{I_\alpha f > \lambda\}\, \mathrm{d}\lambda \leq 2\gamma^{-q}\int_0^{\gamma N} \lambda^{q-1}\mu\{M_\alpha f > \lambda\}\, \mathrm{d}\lambda + \frac{1}{2}\kappa^{-q}\int_0^N \lambda^{q-1}\mu(\{I_\alpha f > \lambda\} \cap 3Q)\, \mathrm{d}\lambda \tag{2.2.21}$$

Since w is locally integrable, this last integral is finite, and smaller than half of that in the left side. Thus :

$$\frac{1}{2}\kappa^{-q}\int_0^{\kappa N}\lambda^{q-1}\mu\{I_\alpha f>\lambda\}\ \mathrm{d}\lambda \leq 2\gamma^{-q}\int_0^{\gamma N}\lambda^{q-1}\mu\{M_\alpha f h\lambda\}\ \mathrm{d}\lambda$$

And taking $N\to+\infty$

$$\|I_\alpha f\|^q_{L^q(\mu)} \leq 4\left(\frac{\kappa}{\gamma}\right)^q \|M_\alpha f\|^q_{L^q(\mu)}$$

Now, to prove (2.2.14), we start again from (2.2.20), multiply by λ^q, and take the supremum for $0\leq\lambda\leq N$. We have :

$$\sup_{0\leq\lambda\leq N}\lambda^q\mu\{I_\alpha f>\kappa\lambda\} \leq 2\sup_{0\leq\lambda\leq N}\lambda^q\mu\{M_\alpha f>\gamma\lambda\}+\frac{1}{2}\kappa^{-q}\sup_{0\leq\lambda\leq N}\lambda^q\mu\left(\{I_\alpha f>\lambda\}\cap 3Q\right) \tag{2.2.22}$$

Then a change of variables gives :

$$\kappa^{-q}\sup_{0\leq\lambda\leq\kappa N}\lambda^q\mu\{I_\alpha f>\lambda\}\leq 2\gamma^{-q}\sup_{0\leq\lambda\leq\gamma N}\lambda^q\mu\{M_\alpha f>\lambda\}+$$
$$\frac{1}{2}\kappa^{-q}\sup_{0\leq\lambda\leq N}\lambda^q\mu\left(\{I_\alpha f>\lambda\}\cap 3Q\right) \tag{2.2.23}$$

Since the last term is finite, and less than half the left side, we finally get, after taking $N\to\infty$, the desired :

$$\sup_{0\leq\lambda}\lambda^q\mu\{I_\alpha f>\lambda\}\leq 4\left(\frac{\kappa}{\gamma}\right)^q\sup_{0\leq\lambda}\lambda^q\mu\{M_\alpha f>\lambda\} \tag{2.2.24}$$

□

2.2.3 Norm inequality for I_α

Theorem 2.5. *Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Let V be a locally integrable non-negative function satisfying (2.2.2). Then there is a constant C independant of f such that :*

$$\left(\int_{\mathbf{R}^n}|I_\alpha f(x)V(x)|^q\ \mathrm{d}x\right)^{\frac{1}{q}}\leq C\left(\int_{\mathbf{R}^n}|f(x)V(x)|^p\ \mathrm{d}x\right)^{\frac{1}{p}} \tag{2.2.25}$$

If $p=1$, $\frac{1}{q}=1-\frac{\alpha}{n}$, and if V is such that there is a constant c such that for all cubes Q :

$$⨍_Q V(x)^q\ \mathrm{d}x\leq c\operatorname*{ess\,inf}_{x\in Q}V(x)^q \tag{2.2.26}$$

Then for $\lambda>0$:

$$\int_{\{I_\alpha f>\lambda\}}V(x)^q\ \mathrm{d}x\leq C\left(\frac{1}{\lambda}\int_{\mathbf{R}^n}|f(x)V(x)|\ \mathrm{d}x\right)^q \tag{2.2.27}$$

Proof. If V satisfy (2.2.2), then V^q satisfies A_r for some $r > 1$, and if it satisfies (2.2.26) then V^q satisfies A_1. In both case, V^q is an A_∞ weight, and so by Theorem 2.4, we have :

$$\int_{\mathbf{R}^n} |I_\alpha f(x)V(x)|^q \, \mathrm{d}x \le C \int_{\mathbf{R}^n} (M_\alpha f(x)V(x))^q \, \mathrm{d}x$$

And, with $\mathrm{d}\mu = V(x)^q \, \mathrm{d}x$:

$$\sup_{\lambda>0} \lambda^q \mu\{x \in \mathbf{R}^n : |I_\alpha f(x)| > \lambda\} \le C \sup_{\lambda>0} \lambda^q \mu\{x \in \mathbf{R}^n : M_\alpha f(x) > \lambda\}$$

Then using either Theorem 2.3 (for the norm inequality) or Theorem 2.2 (for the weak-type estimate), we get (2.2.25) or (2.2.27) □

Theorem 2.6. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Let* V *be a locally integrable non-negative function, and assume that there is a constant* C *such that for all* f, $\lambda > 0$, *we have :*

$$\int_{\{I_\alpha f > \lambda\}} V(x)^q \, \mathrm{d}x \le C \frac{1}{\lambda^q} \left(\int_{\mathbf{R}^n} |f(x)V(x)|^p \, \mathrm{d}x \right)^{\frac{q}{p}} \tag{2.2.28}$$

Then V *satisfy* (2.2.2) *if* $p > 1$, *and* (2.2.26) *if* $p = 1$.

Proof. First, if $p > 1$. Let Q be a cube of $\mathbf{R}^n$. Let $A = \int_Q V(x)^{-p'} \, \mathrm{d}x$. If $A = 0$ then trivially (2.2.2) is satisfied. If $A = \infty$, then $1/V(x)$ is not in $L^{p'}$. Thus, there exist a $g \in L^p$ such that :

$$\int_Q \frac{g(x)}{V(x)} \, \mathrm{d}x = \infty$$

Let $f = \frac{g}{V} \mathbb{1}_Q$. Then $I_\alpha f(x) = \infty$ for all $x \in \mathbf{R}^n$, and, so :

$$\int_Q V(x)^q \, \mathrm{d}x \le \int_{\mathbf{R}^n} V(x)^q \, \mathrm{d}x \le C \frac{1}{\lambda^q} \|g\|_p^q$$

This for all $\lambda > 0$, so $\int V(x)^q \, \mathrm{d}x = 0$, and (2.2.2) is satisfied.

Now if $0 < A < \infty$, let $f = V^{-p'} \mathbb{1}_Q$. Then we have, for all $x \in Q$, $|x - y| \le \sqrt{n} m(Q)^{\frac{1}{n}}$. Then there is a $c > 0$ not depending on f such that :

$$I_\alpha f(x) = \int_Q \frac{f(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}y \ge c A m(Q)^{-1+\frac{\alpha}{n}}$$

Taking this as λ, we get :

$$\left(\int_Q V(x)^q \, \mathrm{d}x \right)^{\frac{1}{q}} \le C \frac{1}{cA} m(Q)^{1-\frac{\alpha}{n}} \left(\int_Q V(x)^{-p'} \, \mathrm{d}x \right)^{\frac{1}{p}}$$

So, by the definition of A :

$$\left(\fint_Q V(x)^q \, \mathrm{d}x \right)^{\frac{1}{q}} \left(\fint_Q V(x)^{-p'} \, \mathrm{d}x \right)^{-\frac{1}{p}} \le \frac{C}{c} m(Q)^{1-\frac{\alpha}{n}-\frac{1}{q}+\frac{1}{p}} \left(\int_Q V(x)^{-p'} \, \mathrm{d}x \right)^{-1}$$

Which reduces to (2.2.2), with C independant of Q.

If $p = 1$, let Q be a cube in $\mathbf{R}^n$, $A = \operatorname{ess\,inf}_{y \in Q} V(y)$. If $A = \infty$ then (2.2.26) is true. Otherwise, for all $\varepsilon > 0$, there exist a subset $E \subset Q$ with positive measure such that $V(x) < A + \varepsilon$ for all $x \in E$. Let $f = \mathbb{1}_E$, then for $x \in Q$:

$$I_\alpha f(x) \geq cm(E)m(Q)^{-1+\frac{\alpha}{n}}$$

And with this as λ :

$$\left(\int_Q V(x)^q \text{ d}x\right)^{\frac{1}{q}} \leq \frac{C}{c} m(E)^{-1} m(Q)^{1-\frac{\alpha}{n}} \int_E V(x) \text{ d}x$$

But $\int_E V(x) \text{ d}x \leq m(E)(A+\varepsilon)$, and so, for all $\varepsilon > 0$:

$$\left(\int_Q V(x)^q \text{ d}x\right)^{\frac{1}{q}} \leq Cm(Q)^{\frac{1}{q}}(A+\varepsilon)$$

And thus (2.2.26) holds. $\square$

3. Spectrum of the Schrödinger operator

The following is based on the article of Martin Schechter[7].

We are interested in the operator of Schrödinger of the form $H = -\Delta - V$ with the potential V a non-negative, locally integrable function. Our objective will be to establish estimates on $-\mu^2$, the lower bound of the spectrum of H, and to give some conditions for the operator to be positive.

We let $C_\lambda(V)$ be the smallest constant satisfying :

$$\langle V\psi,\, \psi\rangle \le C_\lambda(V)\left(\|\nabla\psi\|^2 + \lambda^2\|\psi\|^2\right), \quad \psi \in \mathcal{C}_c^\infty \tag{3.0.1}$$

This is equivalent to $\langle(-\Delta - C_\lambda(V)^{-1}V)\psi,\, \psi\rangle \ge -\lambda^2\|\psi\|^2$. Thus if $\langle H\psi,\, \psi\rangle \ge -\lambda^2\|\psi\|^2$ then $C_\lambda(V) \le 1$.

3.1 Estimating $C_\lambda(V)$

The goal of this section will be to gives estimates on $C_\lambda(V)$.

For a locally finite Borel measure μ, we define :

$$G_{s,\lambda} = (\lambda^2 - \Delta)^{\frac{s}{2}}, \quad G_{s\,\lambda}\mathrm{d}\mu(x) = \int_{\mathbf{R}^n} G_{s,\lambda}(x-y)\ \mathrm{d}\mu(y) \tag{3.1.1}$$

Where we write $G_{s,\lambda}(x)$ for the kernel of the operator $G_{s,\lambda}$.

$$I_{s,\delta}\mathrm{d}\mu(x) = \int_{B(x,\delta)} |x-y|^{s-n}\ \mathrm{d}\mu(y), \quad 0 < s \le n \tag{3.1.2}$$

$$M_{s,\delta}\mathrm{d}\mu(x) = \sup_{r<\delta}\left(r^{s-n}\int_{B(x,r)}\mathrm{d}\mu(x)\right), \quad 0 \le s \le n, \qquad M_s\mathrm{d}\mu = M_{s,\infty}\mathrm{d}\mu \tag{3.1.3}$$

3.1.1 Study of $I_{s,\delta}$

Theorem 3.1. *There is a constant $C_{s,q}$, depending only on s, n and q such that :*

$$\left\|I_{s,\delta}\mathrm{d}\mu\right\|_q \le C_{s,q}\left\|M_{s,\delta}\mathrm{d}\mu\right\|_q \tag{3.1.4}$$

Proof. Define, for $t > 0$,

$$S_t = \{x \in \mathbf{R}^n : I_{s,\delta}\mathrm{d}\mu(x) > t\} \tag{3.1.5}$$

If $S_t \neq \mathbf{R}^n$, then we can apply Theorem 1.2, to get

$$S_t = \bigcup_{j=1}^{\infty} Q_j \tag{3.1.6}$$

Where the Q_j are disjoints and each cube satisfy :

$$\frac{1}{2}\,\mathrm{diam} Q_j < \mathrm{d}(Q_j, S_t^c) \leq 3\,\mathrm{diam}(Q_j) \tag{3.1.7}$$

We additionally want for all cubes to satisfy :

$$\rho = 4\,\mathrm{diam}(Q_j) \leq \delta \tag{3.1.8}$$

Which we obtain by subdivising the cubes. We may lose (3.1.7), in which case we can ensure that $\delta \leq 2\rho_j$. That is to say, we get a decomposition of S_t into cubes Q_j, each satisfying (3.1.8), and each cube will satisfy either (3.1.7) or :

$$\delta \leq 2\rho_j \tag{3.1.9}$$

Now let $b, d > 0$, and define :

$$E_j = \left\{x \in Q_j;\ I_{s,\delta/2}\mathrm{d}\mu(x) > tb,\ M_{s,\delta}\mathrm{d}\mu(x) \leq td\right\} \tag{3.1.10}$$

Let Q be one of the Q_j, and E the associated E_j set. If Q satisfy both 3.1.8 and 3.1.9, then :

$$\begin{aligned} tb\, m(E) &\leq \int_Q I_{s,\delta/2}\mathrm{d}\mu(x)\ \mathrm{d}x \\ &\leq \int_Q \int_{B(x,\delta/2)} |x-y|^{s-n}\ \mathrm{d}\mu(y)\ \mathrm{d}x \\ &\leq \int \int_{\substack{|x-y|<\delta/2 \\ x\in Q}} |x-y|^{s-n}\ \mathrm{d}x\ \mathrm{d}\mu(y) \\ &\leq \frac{\omega}{s}\left(\frac{\delta}{2}\right)^s \mu\left(Q + \delta/2\right) \end{aligned} \tag{3.1.11}$$

Where ω refers to the surface of the unit sphere of $\mathbf{R}^n$, since $\int_{B(0,R)} |x|^{s-n}\ \mathrm{d}x = \frac{\omega}{s} R^s$, and $Q + \delta/2$ is the set of points $y \in R^n$ such that $\mathrm{d}(y, Q) \leq \delta/2$. This set is contained in the ball with center x_0 and radius $\mathrm{diam}(Q) + (\delta/2) \leq (\rho/4) + (\delta/2) \leq 3\delta/4$, by (3.1.8). We thus have, using (3.1.9), and since $x_0 \in E$:

$$\begin{aligned} tb\, m(E) &\leq \frac{\omega}{s}\left(\frac{\delta}{2}\right)^s \left(\frac{\rho}{4} + \frac{\delta}{2}\right)^{n-s} M_{s,\delta}\mathrm{d}\mu(x_0) \\ &\leq \frac{\omega}{s}\rho^s \left(\frac{5\rho}{4}\right)^{n-s} td \\ &\leq \frac{\omega}{s}\left(\frac{5}{4}\right)^{n-s} td\rho^n \\ &\leq \frac{\omega}{s} 4^s 5^{n-s} n^{\frac{n}{2}}\, td\, m(Q) \end{aligned} \tag{3.1.12}$$

And so we get :

$$m(E) \leq \frac{\omega}{s} 4^s 5^{n-s} n^{\frac{n}{2}} \frac{d}{b} m(Q) = c_{n,s} \frac{d}{b} m(Q) \tag{3.1.13}$$

And (3.1.13) is also true if E is empty.

Now, if $2\rho < \delta$, then Q satisfy (3.1.7) and (3.1.8). Let $x_1 \in S_t^c$, such that $\mathrm{d}(x_1, Q) < 4\,\mathrm{diam}(Q)$. If $x \in Q$ then $|x - x_1| < \rho$. Then for any point y such that $|y - x| > \rho$, we have :

$$|y - x_1| \leq |y - x| + |x - x_1| < 2|y - x| \tag{3.1.14}$$

Hence since $\rho < \delta/2$, we have :

$$\begin{aligned} I_{s,\delta/2}\mathrm{d}\mu(x) &= I_{s,\rho}\mathrm{d}\mu(x) + \int_{\rho \leq |y-x| < \delta/2} |y - x|^{s-n}\ \mathrm{d}\mu(y) \\ &\leq I_{s,\rho}\mathrm{d}\mu(x) + 2^{n-s} \int_{|y-x_1| < \delta} |y - x_1|^{s-n}\ \mathrm{d}\mu(y) \\ &\leq I_{s,\rho}\mathrm{d}\mu(x) + 2^{n-s} I_{s,\delta}\mathrm{d}\mu(x_1) \\ &\leq I_{s,\rho}\mathrm{d}\mu(x) + 2^{n-s} t \end{aligned} \tag{3.1.15}$$

Now take $b = 2^{n+1-s}$. If $x \in E$, then

$$tb < I_{s,\rho}\mathrm{d}\mu(x) + \frac{tb}{2} \tag{3.1.16}$$

And so :

$$\frac{tb}{2} < I_{s,\rho}\mathrm{d}\mu(x) \tag{3.1.17}$$

Thus :

$$E \subseteq \left\{ x \in Q;\ I_{s,\rho}\mathrm{d}\mu(x) > \frac{tb}{2},\ M_{s,\delta}\mathrm{d}\mu(x) \leq td \right\} \tag{3.1.18}$$

Hence :

$$\begin{aligned} \frac{tb}{2} m(E) &\leq \int_Q I_{s,\rho}\mathrm{d}\mu(x)\ \mathrm{d}x \\ &\leq \int \int_{\substack{|x-y| < \rho \\ x \in Q}} |x - y|^{s-n}\ \mathrm{d}x\ \mathrm{d}\mu(y) \\ &\leq \left(\frac{\omega}{s}\right) \rho^s \mu(Q + \rho) \end{aligned} \tag{3.1.19}$$

Since $2\rho < \delta$, and $Q + \rho$ is contained in a ball of radius $\mathrm{diam}(Q) + \rho = 5\rho/4 < \delta$ about any point of Q, we get, if $x_0 \in E$:

$$\begin{aligned} \frac{tb}{2} m(E) &\leq \left(\frac{\omega}{s}\right) \rho^s \left(\frac{5\rho}{4}\right)^{n-s} M_{s,\delta}\mathrm{d}\mu(x_0) \\ &\leq \left(\frac{\omega}{s}\right) \left(\frac{5}{4}\right)^{n-s} (4\,\mathrm{diam}(Q))^n\, td \end{aligned} \tag{3.1.20}$$

And so we get :

$$m(E) \leq \left(\frac{\omega}{s}\right) 2^{2s+1} 5^{n-s} n^{\frac{n}{2}} \left(\frac{d}{b}\right) m(Q) \tag{3.1.21}$$

And (3.1.21) is also valid if E is empty. Notice that the constant in this last equation is greater than the one in (3.1.13), so (3.1.21) holds for all cubes Q_j. Now, summing over all cubes, we get :

$$m\big\{I_{s,\delta/2}\mathrm{d}\mu(x) \geq tb,\ M_{s,\delta}\mathrm{d}\mu(x) \leq td\big\} \leq C_{n,s} d\, m(S_t), \quad b \geq 2^{n+1-s} \tag{3.1.22}$$

With $C_{n,s} = \omega 5^{n-s} n^{n/2} 2^{3s-n}/s$.
Now, we get :

$$m\big\{I_{s,\delta/2}\mathrm{d}\mu(x) \geq tb\big\} \leq C_{n,s} d\, m(S_t) + m\{M_{s,\delta} > td\} \tag{3.1.23}$$

Integrating against qt^{q-1} $\mathrm{d}t$ from 0 to N, we get :

$$\int_0^N m\big\{I_{s,\delta/2}\mathrm{d}\mu > tb\big\} qt^{q-1}\ \mathrm{d}t \leq C_{n,s} d \int_0^N m(S_t) qt^{q-1}\ \mathrm{d}t + \int_0^N m\{M_{s,\delta}\mathrm{d}\mu > td\} qt^{q-1}\ \mathrm{d}t$$

Changes of variables give :

$$b^{-q} \int_0^{Nb} m\big\{I_{s,\delta/2}\mathrm{d}\mu > \tau\big\} q\tau^{q-1}\ \mathrm{d}\tau \leq C_{n,s} d \int_0^N m(S_t) qt^{q-1}\ \mathrm{d}t + d^{-q} \int_0^{Nd} m\{M_{s,\delta}\mathrm{d}\mu > \tau\} q\tau^{q-1}\ \mathrm{d}\tau$$

And letting $N \to \infty$, we have :

$$\left\|I_{s,\delta/2}\mathrm{d}\mu\right\|_q^q \leq C_{n,s} d b^q \left\|I_{s,\delta}\mathrm{d}\mu\right\|_q^q + \left(\frac{b}{d}\right)^q \left\|M_{s,\delta}\mathrm{d}\mu\right\|_q^q \tag{3.1.24}$$

And so :

$$\left\|I_{s,\delta/2}\mathrm{d}\mu\right\|_q \leq C_{n,s}^{1/q} d^{1/q} b \left\|I_{s,\delta}\mathrm{d}\mu\right\|_q + \frac{b}{d} \left\|M_{s,\delta}\mathrm{d}\mu\right\|_q \tag{3.1.25}$$

But we also have :

$$\begin{aligned} I_{s,\delta}\mathrm{d}\mu(x) &= I_{s,\delta/2}\mathrm{d}\mu(x) + \int_{\delta/2 \leq |y-x| < \delta} |x-y|^{s-n}\ \mathrm{d}\mu(y) \\ &\leq I_{s,\delta/2}\mathrm{d}\mu(x) + 2^{n-s} M_{s,\delta}\mathrm{d}\mu(x) \end{aligned} \tag{3.1.26}$$

Thus :

$$\left\|I_{s,\delta}\mathrm{d}\mu\right\|_q - 2^{n-s} \left\|M_{s,\delta}\mathrm{d}\mu\right\|_q \leq \left\|I_{s,\delta/2}\mathrm{d}\mu\right\|_q \tag{3.1.27}$$

And so :

$$\left\|I_{s,\delta}\mathrm{d}\mu\right\|_q \leq C_{n,s}^{1/q} d^{1/q} b \left\|I_{s,\delta}\mathrm{d}\mu\right\|_q + \left(\frac{b}{d} + 2^{n-s}\right) \left\|M_{s,\delta}\mathrm{d}\mu\right\|_q \tag{3.1.28}$$

Take $1/d = C_{n,s} 2^q b^q$, i.e. $d^{1/q} = 2^{-1} b^{-1} C_{n,s}^{-1/q}$. Then

$$\|I_{s,\delta}\|_q \leq \left(2bd^{-1} + 2^{n-s+1}\right) \|M_{s,\delta}\mathrm{d}\mu\|_q \tag{3.1.29}$$

With $b = 2^{n-s+1}$, we have :

$$\|I_{s,\delta}\|_q \leq b\left(2d^{-1} + 1\right) \|M_{s,\delta}\mathrm{d}\mu\|_q = C_{n,s,q} \|M_{s,\delta}\mathrm{d}\mu\|_q \tag{3.1.30}$$

□

3.1.2 Study of $G_{s,\lambda}$

Theorem 3.2. *There is a constant $C'_{s,n,q}$ depending only on those parameters, such that :*

$$\|G_{s,\lambda}\mathrm{d}\mu\|_q \leq C'_{s,n,q} \left\|M_{s,1/\lambda}\mathrm{d}\mu\right\|_q \tag{3.1.31}$$

Proof. We will use the following result by Aronszajn-Smith[1] : $G_{s,\lambda}(x)$ satisfies

$$G_{s,\lambda}(x) \leq \begin{cases} c_0|x|^{s-n}, & \lambda|x| \leq 1, \\ c_1\lambda^{n-s}|\lambda x|^{\gamma}\mathrm{e}^{-\lambda|x|}, & \lambda|x| > 1. \end{cases} \tag{3.1.32}$$

With $\gamma = (n-s-1)/2$, and the c_j do not depend on λ. We let :

$$\widetilde{G}_{s,\lambda}(x) = \begin{cases} 0, & \lambda|x| \leq 1, \\ G_{s,\lambda}(x), & \lambda|x| > 1. \end{cases} \tag{3.1.33}$$

We have :

$$\left\|\left(G_{s,\lambda} - \widetilde{G}_{s,\lambda}\right)\mathrm{d}\mu\right\|_q \leq c_0 \left\|I_{s,1/\lambda}\mathrm{d}\mu\right\|_q \tag{3.1.34}$$

And so, using Theorem 3.1, to prove Theorem 3.2, it will suffices to show that for some constant C depending only on n, s, q, we have :

$$\left\|\widetilde{G}_{s,\lambda}\mathrm{d}\mu\right\|_q \leq C \left\|M_{s,1/\lambda}\mathrm{d}\mu\right\|_q \tag{3.1.35}$$

Now, using (3.1.32) and the definition of $\widetilde{G}_{s,\lambda}$, we have :

$$\begin{aligned} \widetilde{G}_{s,\lambda}\mathrm{d}\mu(y) &\leq c_1 \int_{\lambda|x-y|>1} \lambda^{n-s} \left|\lambda(x-y)\right|^{\gamma} \mathrm{e}^{-\lambda|x-y|} \,\mathrm{d}\mu(x) \\ &\leq c_1\lambda^{n-s} \sum_{k=1}^{\infty} \int_{k<\lambda|x-y|<k+1} (k+1)^{\gamma}\mathrm{e}^{-k} \,\mathrm{d}\mu(x) \end{aligned} \tag{3.1.36}$$

The set $R_k = \{k < |x| < k+1\}$ can be covered by $N(k)$ balls of radius 1 and centers $z^{(1)}, \dots, z^{N(k)}$, with $N(k) \leq c_2 k^{n-1}$.

Indeed, we let $A \subset R_k$ be maximal such that for all $x, y \in A$, $x \neq y$, then $|x-y| > 1$. Then if $x \in R_k$, there is a $y \in A$ such that $|x-y| \leq 1$, otherwise A would not be maximal. Thus $R_k \subset \bigcup_{x\in A} B(x,1)$. Moreover the balls with center in A and with radius $1/2$ are disjoints, and we also have :

$$\bigcup_{x\in A} B\left(x, \frac{1}{2}\right) \subseteq B\left(0, k+\frac{3}{2}\right) \setminus B\left(0, k-\frac{1}{2}\right) \tag{3.1.37}$$

And so :

$$2^{-n}\#A \leq \left(k+\frac{3}{2}\right)^n - \left(k-\frac{1}{2}\right)^n \sim c\,k^{n-1} \tag{3.1.38}$$

And so we can indeed impose $N(k) \leq c_2 k^{n-1}$.

Then the set $k < \lambda|x| < k+1$ can be covered by $N(k)$ balls with centers $z^{(1)}/\lambda, \ldots, z^{N(k)}/\lambda$ with radius $1/\lambda$. Then :

$$\begin{aligned}\widetilde{G}_{s,\lambda}\mathrm{d}\mu(y) &\leq c_1\lambda^{n-s}\sum_{k=1}^{\infty}(k+1)^{\gamma}\mathrm{e}^{-k}\sum_{j=1}^{N(k)}\int_{|x-y-z^{(j)}/\lambda|<1/\lambda}\mathrm{d}\mu(x)\\ &\leq c_1\sum_{k=1}^{\infty}(k+1)^{\gamma}\mathrm{e}^{-k}\sum_{j=1}^{N(k)}M_{s,1/\lambda}\mathrm{d}\mu\left(y+\frac{z^{(j)}}{\lambda}\right)\end{aligned} \tag{3.1.39}$$

And finally :

$$\left\|\widetilde{G}_{s,\lambda}\mathrm{d}\mu\right\|_q \leq c_1\sum_{k}^{\infty}N(k)(k+1)^{\gamma}\mathrm{e}^{-k}\left\|M_{s,1/\lambda}\mathrm{d}\mu\right\|_q \tag{3.1.40}$$

And (3.1.35) holds. □

3.1.3 Estimate on $C_\lambda(V)$

Theorem 3.3. *For each $p>1$, there is a constant C_p, depending only on n and p such that :*

$$C_\lambda(V) \leq C_p \sup_x \left(M_{2p,1/\lambda}V(x)^p\right)^{1/p}, \quad \lambda \geq 0. \tag{3.1.41}$$

Moreover, there is a constant C_1 depending only on n such that :

$$C_\lambda(V) \geq C_1 M_{2,1/\lambda}V \tag{3.1.42}$$

Proof. Let $\delta = 1/\lambda$, and define :

$$K_p = \sup_x \left(M_{2p,\delta}V^p\right)^{\frac{1}{p}} \tag{3.1.43}$$

For $q = 2p > 2$, then by Hölder's inequality we have :

$$M_{1,\delta}\left(V^{\frac{1}{2}}\psi\right) \leq M_{q,\delta}\left(V^{\frac{q}{2}}\right)^{\frac{1}{q}} M_{0,\delta}\left(|\psi|^{q'}\right)^{\frac{1}{q'}} = K_p^{\frac{1}{2}} M\left(|\psi|^{q'}\right)^{\frac{1}{q'}} \tag{3.1.44}$$

And so :

$$\left\|M_{1,\delta}\left(V^{\frac{1}{2}}\psi\right)\right\|_2 \leq K_p^{\frac{1}{2}}\left\|M_{0,\delta}\left(|\psi|^{q'}\right)^{\frac{1}{q'}}\right\|_2 = K_p^{\frac{1}{2}}\left\|M_{0,\delta}|\psi|\right\|_{\frac{2}{q'}}^{\frac{1}{q'}} \tag{3.1.45}$$

Then since $q' < 2$ we have, since $M_{0,\delta}$ is bounded on L^r for all $r > 1$:

$$\left\|M_{1,\delta}\left(V^{\frac{1}{2}}\psi\right)\right\|_2 \leq CK_p^{\frac{1}{2}}\|\psi\|_2 \tag{3.1.46}$$

Then by Theorem 3.2, we have :

$$\left\| G_{1,\lambda} \left(V^{\frac{1}{2}} \psi \right) \right\|_2 \leq C C'_{s,n,2} K_p^{\frac{1}{2}} \left\| \psi \right\|_2 \tag{3.1.47}$$

The adjoint of $G_{1,\lambda} V^{1/2}$ is $V^{1/2} G_{1,\lambda}$, since both $V^{1,2}$ and $G_{1,\lambda} = (\lambda^2 - \Delta)^{-1/2}$ are self-adjoint, and so we have :

$$\left\| V^{\frac{1}{2}} G_{1,\lambda} \phi \right\|_2 \leq C C'_{s,n,2} K_p^{\frac{1}{2}} \left\| \phi \right\|_2 \tag{3.1.48}$$

If we let $\widehat{\phi}(\xi) = \left(\lambda^2 + |\xi|^2\right)^{\frac{1}{2}} \widehat{\psi}(\xi)$, then :

$$\|\phi\|_2^2 = \lambda^2 \|\psi\|_2^2 + \|\nabla\psi\|_2^2 \tag{3.1.49}$$

And :

$$\langle V\psi,\, \psi \rangle = \left\| V^{\frac{1}{2}} G_{1,\lambda} \phi \right\|_2^2 \leq C^2 (C'_{s,n,2})^2 K_p \|\phi\|_2^2 \tag{3.1.50}$$

Finally :

$$\langle V\psi,\, \psi \rangle \leq C^2 (C'_{s,n,2})^2 K_p \left(\lambda^2 \|\psi\|_2^2 + \|\nabla\psi\|_2^2 \right) \tag{3.1.51}$$

Which gives (3.1.41) by the definition of K_p.

Now, to prove (3.1.42), let ϕ be a test function equal to 1 on $|x| < 1$ and to 0 on $|x| > 2$. Let $z \in \mathbf{R}^n$ and define :

$$\phi_\lambda(x) = \phi\left(\lambda(x - z)\right) \tag{3.1.52}$$

Then :

$$\begin{aligned} \langle V\phi_\lambda,\, \phi_\lambda \rangle &\leq C_\lambda(V) \left(\lambda^2 \|\phi_\lambda\|_2^2 + \|\nabla\phi_\lambda\|_2^2 \right) \\ &\leq C_\lambda(V) \lambda^{2-n} \left(\|\phi\|_2^2 + \|\nabla\phi\|_2^2 \right) \\ &\leq C \lambda^{2-n} C_\lambda(V) \end{aligned} \tag{3.1.53}$$

Hence :

$$\lambda^{n-2} \int_{\lambda|x-z|<1} V(x)\, \mathrm{d}x \leq C C_\lambda(V) \tag{3.1.54}$$

Since $\lambda \mapsto C_\lambda(V)$ is decreasing, then for all positives $r \leq 1/\lambda$:

$$r^{2-n} \int_{|x-z|<r} V(x)\, \mathrm{d}x \leq C C_{1/r}(V) \leq C C_\lambda(V) \tag{3.1.55}$$

And so $M_{2,1/\lambda} V(z) \leq C C_\lambda(V)$, for all $z \in \mathbf{R}^n$. □

The following corollary will finally gives u the desired estimates on μ^2, but we will first need to establish some facts on $C_\lambda(V)$ before proving it. This result was initially established by C. Fefferman and D-H Phong, see [3] for their proof.

Corollary 3.1. *If $-\mu^2$ is the lowest point of the spectrum of $-\Delta - V$, then :*

$$\begin{aligned}\mu^2 &\leq \sup_{\delta>0}\left(2C_p\delta^{-2}\sup_x \left(M_{2p,\delta}V^p\right)^{1/p} - \delta^{-2}\right)\\ &\leq \sup_{x,\delta}\left(2C_p\left(\delta^{-n}\int_{B(x,\delta)} V(y)^p\,\mathrm{d}y\right)^{1/p} - \delta^{-2}\right)\end{aligned}\tag{3.1.56}$$

And :

$$\begin{aligned}\mu^2 &\geq \sup_{\delta>0}\left(C_1\delta^{-2}\sup_x M_{2,\delta}V - \delta^{-2}\right)\\ &\geq \sup_{x,\delta}\left(C_1\delta^{-n}\int_{B(x,\delta)} V(y)\,\mathrm{d}y - \delta^{-2}\right)\end{aligned}\tag{3.1.57}$$

Corollary 3.2. *If $C_p^pM_{2p}V^p \leq 1$ then $\mu = 0$*

3.2 Properties of $C_\lambda(V)$

Theorem 3.4. *$C_\lambda(V)$ is continuous in λ in $[0,\infty)$.*

Proof. Let $A \geq 0$, suppose that for all $\nu > \lambda$, we have $C_\nu(V) \leq A$. Then $C_\lambda(V) \leq A$. Indeed, we have :

$$\langle V\psi,\,\psi\rangle \leq A\left(\|\nabla\psi\|^2 + \nu^2\|\psi\|^2\right),\quad \psi \in \mathcal{C}_c^\infty \tag{3.2.1}$$

And so taking $\nu \to \lambda$,

$$\langle V\psi,\,\psi\rangle \leq A\left(\|\nabla\psi\|^2 + \lambda^2\|\psi\|^2\right),\quad \psi \in \mathcal{C}_c^\infty \tag{3.2.2}$$

And $C_\lambda(V) \leq A$.

Next, suppose $\lambda > 0$ and, for all $\nu < \lambda$, $C_\nu(V) \geq A$, then $C_\lambda(V) \geq A$. Indeed, if $C_\lambda(V) \leq A - \varepsilon$, with $\varepsilon > 0$, we can find for each ν a function $\psi_\nu \in \mathcal{C}_c^\infty$ such that :

$$\|\nabla\psi_\nu\|^2 + \nu^2\|\psi_\nu\|^2 = 1 \tag{3.2.3}$$

And :

$$C_\nu(V) - \frac{\varepsilon}{2} \leq \langle V\psi_\nu,\,\psi_\nu\rangle \leq C_\lambda(V)\left(\|\nabla\psi_\nu\|^2 + \lambda^2\|\psi_\nu\|^2\right) \tag{3.2.4}$$

Then by (3.2.3) we have :

$$A - \frac{\varepsilon}{2} \leq C_\lambda(V)\left(1 + \left(\lambda^2 - \nu^2\right)\|\psi_\nu\|^2\right) \leq C_\lambda(V)\frac{\lambda^2}{\nu^2} \tag{3.2.5}$$

Indeed,

$$\|\psi_\nu\|^2 = \frac{1 - \|\nabla\psi_\nu\|^2}{\nu^2} \leq \frac{1}{\nu^2} \tag{3.2.6}$$

And so $1 + (\lambda^2 - \nu^2)\|\psi_\nu\|^2 \leq \lambda^2/\nu^2$. Now if we let $\nu \to \lambda$; we get :

$$A - \frac{\varepsilon}{2} \leq C_\lambda(V) \leq A - \varepsilon \tag{3.2.7}$$

Which is a contradiction. Thus $C_\lambda(V) \geq A$.

Moreover $C_\lambda(V)$ is a decreasing function of λ. Combined with the above properties, if $\varepsilon > 0$, then there is a $\delta > 0$ such that, for all $\nu \in (\lambda - \delta, \lambda)$, $C_\lambda(V) \leq C_\nu(V) \leq C_\lambda(V) + \varepsilon$. And so $C_\lambda(V) = \inf\{C_\nu(V),\, \nu < \lambda\}$. Similarly, $C_\lambda(V) = \sup\{C_\nu(V),\, \nu > \lambda\}$.

Thus, $\lambda \mapsto C_\lambda(V)$ is continuous. □

Theorem 3.5. *Let $-\mu^2$ be the lowest point of the spectrum of $H = -\Delta - V$, then :*

$$\begin{aligned} \mu^2 &= \inf_{C_\lambda(V) \leq 1} \lambda^2 = \sup_{C_\lambda(V) > 1} \lambda^2 \\ &= \inf_{C_\lambda(V) \leq 1} \lambda^2 C_\lambda(V) = \sup_{C_\lambda(V) > 1} \lambda^2 C_\lambda(V) \end{aligned} \tag{3.2.8}$$

In particular :

- *If the set $\{C_\lambda(V) \leq 1\}$ is empty, then $\mu = \infty$.*
- *If the set $\{C_\lambda(V) > 1\}$ is empty, then $\mu = 0$.*

Proof. If $C_\lambda(V) \leq 1$, then

$$\langle V\psi,\, \psi\rangle \leq C_\lambda(V) \left(\|\nabla\psi\|^2 + \lambda^2 \|\psi\|^2\right) \tag{3.2.9}$$

implies :

$$-C_\lambda(V)\lambda^2\|\psi\|^2 \leq \|\nabla\psi\|_2 - \langle V\psi,\, \psi\rangle = \langle H\psi,\, \psi\rangle \tag{3.2.10}$$

Then taking the infimum for $\|\psi\| = 1$, we get :

$$-C_\lambda(V)\lambda^2 \leq -\mu^2 \tag{3.2.11}$$

And so :

$$\mu^2 \leq \lambda^2 C_\lambda(V) \leq \lambda^2 \tag{3.2.12}$$

If $C_\lambda(V) > 1$, then for any $\varepsilon > 0$, there is a $\psi \in \mathcal{C}_c^\infty$, $\|\psi\| = 1$, such that :

$$\langle V\psi,\, \psi\rangle \geq (C_\lambda(V) - \varepsilon) \left(\|\nabla\psi\|^2 + \lambda^2\|\psi\|^2\right) \tag{3.2.13}$$

Thus :

$$(1 + \varepsilon - C_\lambda(V))\|\nabla\psi\|^2 \geq \langle H\psi,\, \psi\rangle + \lambda^2 \left(C_\lambda(V) - \varepsilon\right) \|\psi\|^2 \tag{3.2.14}$$

For ε small enough, then this is non-positive. Then :

$$\langle H\psi,\, \psi\rangle \leq -\lambda^2(C_\lambda(V) - \varepsilon) \tag{3.2.15}$$

And so $\mu^2 \geq \lambda^2(C_\lambda(V) - \varepsilon)$. Taking $\varepsilon \to 0$, we get :

$$\mu^2 \geq \lambda^2 C_\lambda(V) \geq \lambda^2, \quad C_\lambda(V) > 1. \tag{3.2.16}$$

From this, if $\mu \neq 0$, we must have $C_\mu(V) \leq 1$. But this is also true if $\mu = 0$: then, since, for any λ with $C_\lambda(V) > 1$, we have $\mu^2 > \lambda$, then for any $\lambda > 0$, $C_\lambda(V) \leq 1$. Then by continuity, we also have $C_\mu(V) \leq 1$.

Now, by (3.2.12), if $\mu \neq 0$, we have :

$$C_\mu(V) = 1 \tag{3.2.17}$$

Moreover, (3.2.12) also implies :

$$\mu^2 \leq \inf_{C_\lambda(V) \leq 1} \lambda^2 C_\lambda(V) \leq \inf_{C_\lambda(V) \leq 1} \lambda^2 \tag{3.2.18}$$

And with (3.2.17), equality holds. Similarly,

$$\mu^2 \geq \sup_{C_\lambda(V) > 1} \lambda^2 C_\lambda(V) \geq \sup_{C_\lambda(V) > 1} \lambda^2 \tag{3.2.19}$$

And if $\mu^2 > \sup_{C_\lambda(V)>1} \lambda^2$, then there is a positive ν such that $\mu^2 > \nu^2 > \sup_{C_\lambda(V)>1} \lambda^2$. Thus $\nu < \mu$ and $C_\nu(V) \leq 1$. Which is a contradiction with (3.2.18). Thus there is equality, and the theorem holds. □

Corollary 3.3.

$$\mu^2 \leq \sup_\lambda \lambda^2 \left(2C_\lambda(V) - 1\right) \tag{3.2.20}$$

$$\mu^2 \geq \sup_\lambda \lambda^2 \left(C_\lambda(V) - 1\right) \tag{3.2.21}$$

Proof. If $C_\lambda(V) > 1$, then $\lambda^2 \leq \lambda^2(2C_\lambda(V) - 1)$. Then taking the supremum over the set $C_\lambda(V) > 1$, we get :

$$\mu^2 \leq \sup_{C_\lambda(V) > 1} \lambda^2 \left(2C_\lambda(V) - 1\right) \tag{3.2.22}$$

And the right hand side is clearly less than that of (3.2.20).

If $C_\lambda(V) > 1$, then $\lambda^2 C_\lambda(V) \geq \lambda^2(C_\lambda(V) - 1)$, and if $C_\lambda(V) \leq 1$, then the right hand side is non-positive. Then :

$$\mu^2 \geq \sup_{C_\lambda(V) > 1} \lambda^2 \left(C_\lambda(V) - 1\right) = \sup_\lambda \lambda^2 \left(C_\lambda(V) - 1\right) \tag{3.2.23}$$

□

Proof of Corollary 3.1. By (3.2.20) and Theorem 3.3, (3.1.41), we have :

$$\begin{aligned} \mu^2 &\leq \sup_{\lambda > 0} \lambda^2 \left(2C_p \sup_x \left(M_{2p,1/\lambda} V^p\right)^{\frac{1}{p}} - 1\right) \\ &\leq \sup_{\delta > 0} \left(2C_p \delta^{-2} \sup_x \left(M_{2p,\delta} V^p\right)^{\frac{1}{p}} - \delta^{-2}\right) \end{aligned} \tag{3.2.24}$$

And so the first inequality of (3.1.56) holds. The right hand side is equal to :

$$K = \sup_{x,\delta} \left(2C_p \delta^{-2} \left(M_{2p,\delta} V^p\right)^{\frac{1}{p}} - \delta^{-2}\right) \tag{3.2.25}$$

We will show it is actually equal to the second expression in (3.1.56), which we will write L. Recall :

$$L = \sup_{x,\delta} \left(2C_p \left(\delta^{-n} \int_{|y-x|<\delta} V(y)^p \, \mathrm{d}y \right)^{\frac{1}{p}} - \delta^{-2} \right)$$

We have

$$\left(\delta^{-n} \int_{|y-x|<\delta} V(y)^p \, \mathrm{d}y \right)^{\frac{1}{p}} \leq \frac{L+\delta^{-2}}{2C_p}, \quad \delta > 0 \tag{3.2.26}$$

And so :

$$(M_{2p,\delta} V^p)^{\frac{1}{p}} \leq \frac{\delta^2 L + 1}{2C_p} \tag{3.2.27}$$

And we finally get :

$$\mu^2 \leq K \leq \sup_{x,\delta} \left(\delta^{-2} \left(\delta^2 L + 1 \right) - \delta^{-2} \right) = L \tag{3.2.28}$$

And so, since $K \geq L$ is obvious, we have $K = L$.
Similarly, using (3.2.21) and Theorem 3.3, (3.1.42), we have :

$$\begin{aligned} \mu^2 &\geq \sup_{\lambda>0} \lambda^2 \left(C_1 M_{2,1/\lambda} V - 1 \right) \\ &\geq \sup_{\delta>0} \left(C_1 \delta^{-2} \sup_x M_{2,\delta} V - \delta^{-2} \right) \end{aligned} \tag{3.2.29}$$

Which is the first expression of (3.1.57). We obtain the second in the same way as above. □

Proof of Corollary 3.2. Taking $\lambda = 0$ in (3.1.41), we have :

$$C_0(V) \leq C_p \left(M_{2p} V^p \right)^{\frac{1}{p}} \tag{3.2.30}$$

Then, if $C_p^p M_{2p} V^p \leq 1$, using Theorem 3.5, $\mu = 0$. □

Corollary 3.4. *If $V(x)$ satisfy the A_∞ condition, then there is a $p > 1$ such that :*

$$C_\lambda(V) \leq N_p \left\| M_{2,1/\lambda} V \right\|_\infty \tag{3.2.31}$$

Proof. With $p > 1$ such that the reverse Hölder holds, there is a constant L_p such that :

$$(M_{2p,\delta} V^p)^{\frac{1}{p}} \leq L_p M_{2,\delta} V \tag{3.2.32}$$

Then using Theorem 3.3 :

$$C_\lambda(V) \leq C_p L_p \sup_x M_{2,1/\lambda} V \tag{3.2.33}$$

□

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