NANTES' UNIVERSITY

MASTER THESIS

Gromov-Witten-Welschinger invariants of $\mathbb{C}P^3$ and of $\mathbb{C}P^1 \times \mathbb{C}P^1$

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A thesis of Master of Mathematics

in

Algebraic Geometry
Department of Mathematics

July 2, 2018

NANTES' UNIVERSITY

Abstract

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Gromov-Witten-Welschinger invariants of $\mathbb{C}P^3$ and of $\mathbb{C}P^1 \times \mathbb{C}P^1$

by NGUYEN THI NGOC ANH

In this rapport, we study about the invariants when counting (real) rational curves of degree d which pass through $2d$ points in the $3-$dimentional complex projective space $\mathbb{C}P^3$ (in other words, Gromov-Witten-Welschinger invariants of $\mathbb{C}P^3$) and (real) rational curves of bidegree (a, b) which pass through $2(a+b)-1$ points in the $2-$dimentional complex projective space $\mathbb{C}P^1 \times \mathbb{C}P^1$ (in other words, Gromov-Witten-Welschinger invariants of $\mathbb{C}P^1 \times \mathbb{C}P^1$) and their relationship.

Acknowledgements

First and foremost, I offer my sincerest gratitude to my advisor, professor *Erwan BRUGALLÉ*, who has supported me throughout my master's thesis with his patience and knowledge. I attribute the level of my Master degree to his encouragement and effort. Without him, this thesis would not have been written or completed. One simply could not wish for a better or friendlier advisor.

I would like to thanks all of my dear friends, who are always by my side, encouraging me to overcome difficulties.

Finally, my special thank and deep gratitude are towards my family for supporting me throughout my study in France, especially at Nantes' University for two years.

Nantes, June 20, 2018

NGUYEN Thi Ngoc Anh

Contents

Introduction

This rapport is based on the paper [BG16].

Enumerative geometry aims to count how many geometric figures satisfy given conditions. The most basis example is that: How many lines passing through 2 distinct points? A natural extension of this question is the problem: How many rational curves of degree d pass through $(3d-1)$ generic points in the complex projective plane? (The number $(3d-1)$ is exactly the dimension of the space of rational (genus 0) degree d curves in $\mathbb{C}P^2$). This can be done by recursion by using Kontsevich's formula.

This formula is quite surprising relevant to a notion in symplectic geometry (which is Gromov's pseudoholomorphic curves), hence the number answering for the problem turns into the number called Gromov-Witten invariant. Indeed, Gromov-Witten invariant is a rigorous mathematical definition required moduli space of stable maps. So we can say the problem of counting rational curves in a projective space as the problem of finding the Gromov-Witten invariant.

In the context of enumerative real algebraic geometry, some of the invariants were discovered by Welschinger. In particular, Welschinger invariants are real analogues of certain Gromov-Witten invariants.

In this rapport, we concern about the Gromov-Witten invariant and Welschinger invariant of $\mathbb{C}P^3$. The number of rational curves of degree d passing through $2d$ generic points in $\mathbb{C}P^3$ is the Gromov-Witten invariant of $\mathbb{C}P^3$, denoted by $GW_{\mathbb{C}P^3}(d)$. If we consider the real case, then the number of real rational curves of degree d passing through $2d$ real generic points in $\mathbb{C}P^3$ counted with sign is the Welschinger invariant of $\mathbb{C}P^3$, denoted by $W_{\mathbb{R}P^3}(d,l)$. In Chapter 1, we prepare some backgrounds that we will use to study these invariants.

Following the idea of Kollár: there exists $2d$ distinct points in a degree 4 elliptic curve such that the number of rational curves of degree d passing through them are indeed $GW_{\mathbb{C}P^3}(d)$. These curves are also contained in a non-singular quadric Q which is in the pencil of quadric induced by this elliptic curve. Now we just count how many non-singular quadrics do we have (via elliptic curves corresponded) and how many curves lie on each quadric (that is invariant and that is exactly the number of curves of bidegree $(a, d-a)$ lying on Q and passing through $(2d-1)$ points, denoted by $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a, d-a)$). At the end of Chapter 2, we construct the relation between two invariants $GW_{\mathbb{C}P^3}(d)$ and $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a, d-a)$.

Once again, by Kollár's idea, but this time we note that the invariant is defined as the number of curves of degree d counted with sign, then we need to define the sign for each curve. It is done by studying certain real normal bundles (which are not easy to visualize). In order to determine the Welschinger invariant $W_{\mathbb{R}P^3}(d,l)$, we find the answer for two questions: How many REAL non-singular quadrics do we have? What is the invariant in each quadric? (that is the number of curves counted with sign of bidegree $(a, d-a)$ lying on Q and passing through $(2d-1)$ distinct points, denoted by $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a, d-a), l)$). At the end of Chapter 3, we construct the relation between two invariants $W_{\mathbb{R}P^3}(d, l)$ and $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a, d-a), l)$.

Chapter 1

Preliminaries

In this chapter, we recall some properties came from elliptic curves, especially about the elliptic curve as base locus of a pencil of quadrics. We also give the statement and a proof of Kollár's theorem which is the spirit of the two main theorems in the following chapters. And we introduce some remarks about normal bundles - the important tool used for real enumerative problem.

1.1 Elliptic curves

1.1.1 Complex elliptic curves

We works over the algebraically closed field $k = \bar{k}$, for example $k = \mathbb{C}$.
We consider a complex elliptic curve C_0 equipped with a distinguished point p_0. The set of points on this elliptic curve over a field k under point addition, denoted by $C_0(k)$, form a commutative group with the point at infinity ($O = (0 : 1 : 0)$) is the identity.
Given m a positive integer. Considering a homomorphism of groups:

$$[m] : C_0(k) \longrightarrow C_0(k)$$

$$P \longmapsto mP$$

This homomorphism allows us define the torsion points on C_0.

Definition 1.1.1. *The $m-$* ***torsion point*** *of elliptic curve C_0 is the kernel of the homomorphism $[m]$.*
i.e. P is the $m-$torsion point of C_0 if $mP = O$.

Example 1. *Given an elliptic curve $C_0 : y^2 = f(x)$ with $deg(f) = 3, char(k) \neq 2$. Let $x_i, i \in \{1, 2, 3\}$ be the solutions of $f(x) = 0$, then*
{$2-$torsion points of C_0}
=$\{P \in C_0 : 2P = O\}$
=$\{O, P_1, P_2, P_3$ where $P_i = (x_i, 0)\}$

Property 1.1.2. *The $m-$torsion points of C_0 form a subgroup of $C_0(k)$ with cardinal m^2.*

Geometrically, an elliptic curve over the complex numbers is obtained as a quotient of the complex plane by a lattice, i.e. $C_0 = \mathbb{C}/\Lambda$, such that p_0 is the orbit of O.
Recall: A lattice Λ of the complex numbers $\mathbb{C}$ is an additive subgroup free of rank two that generates $\mathbb{C}$ as a real vector space. One can write $\Lambda = u\mathbb{Z} + v\mathbb{Z}; u, v \in \mathbb{C}$.

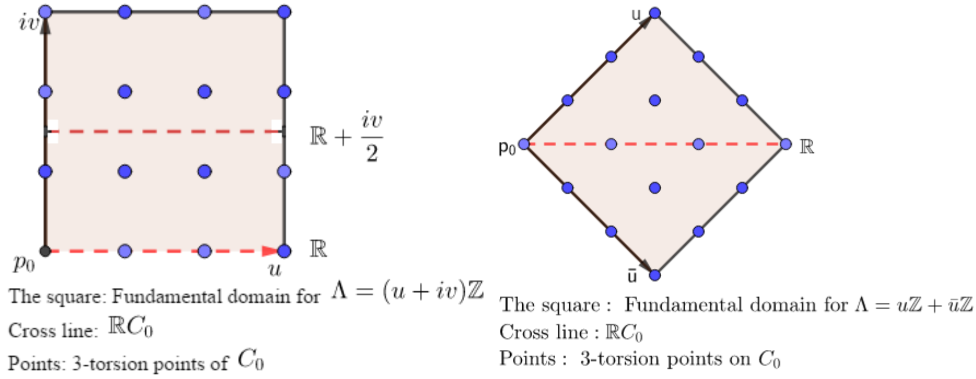

FIGURE 1.1: Two different fundamental domains for lattice of C_0 in case $m=3$

1.1.2 Real elliptic curves

Now we suppose that C_0 is real with its real part $\mathbb{R}C_0$ is nonempty containing p_0. Since we have $C_0=\mathbb{C}/\Lambda$ is real, that implies either $\Lambda=u\mathbb{Z}+iv\mathbb{Z}; u,v\in\mathbb{R}$ or $\Lambda=u\mathbb{Z}+\bar{u}\mathbb{Z}, u\in\mathbb{C}$. Thus, there are two cases for $\mathbb{R}C_0$ associated.

- Case 1: $\Lambda=u\mathbb{Z}+iv\mathbb{Z}; u,v\in\mathbb{R}$ then $\mathbb{R}C_0=\mathbb{R}/u\mathbb{Z}\sqcup(\mathbb{R}+\frac{iv}{2})/u\mathbb{Z}$. That means $\mathbb{R}C_0$ has two connected components, one contains p_0. In this case, if m is even, $\mathbb{R}C_0$ contains exactly $2m$ of real $m-$torsion points. If m is odd, $\mathbb{R}C_0$ contains exactly m of real $m-$torsion points, all lie on the connected component of $\mathbb{R}C_0$ containing p_0 (see Figure 1.1 left for the case $m=3$).
- Case 2: $\Lambda=u\mathbb{Z}+\bar{u}\mathbb{Z}, u\in\mathbb{C}$ then $\mathbb{R}C_0=\mathbb{R}/(u+\bar{u})\mathbb{Z}$. That means $\mathbb{R}C_0$ has only one connected components . In this case, for all m, $\mathbb{R}C_0$ contains exactly m of real $m-$torsion points (see Figure 1.1 right for the case $m=3$).

1.2 Pencils of quadrics

1.2.1 Complex pencils of quadric

Firstly, we need to give the definition of complete intersection which we will use frequently in the sequel.

Definition 1.2.1. *A projective variety $X\subset\mathbb{C}P^n$ of codimension m is a **complete intersection** if it is the intersection of m hypersurfaces that meet transversally at each point of intersection.*

For example, a degree 4 elliptic curve is the complete intersection of two irreducible quadric surfaces in $\mathbb{C}P^3$.

Property 1.2.2. *([Har97], Remark 6.4.1, p352)*
If Y is a non-singular curve in $\mathbb{C}P^3$, which is the complete intersection of non-singular surfaces of degree a,b for every $a,b\geq 1$ then $g_Y=\frac{1}{2}(ab(a+b-4)+1)$.

Property 1.2.3. *A curve C lying in quadric Q is of bidegree (d,d) iff it is the complete intersection of Q, i.e. it is the intersection of Q with a degree d surface in $\mathbb{C}P^3$.*

We note that the space of quadrics in $\mathbb{C}P^3$ is isomorphic to $\mathbb{C}P^9$. Now we imagine that we are in the 9−dimentional projective space (with its 'points' are quadrics), then there is a unique 'line' passing through two 'points' in $\mathbb{C}P^9$, this 'line' is called the **pencil of quadrics** in $\mathbb{C}P^3$, denoted by $\mathcal{Q}$. Two non-singular quadrics in $\mathcal{Q}$ intersect at a degree 4 elliptic curve. Indeed, let $C = Q_1 \cap Q_2$ then $deg(C) = 4$ and since C is the complete intersection of two quadrics surfaces so genus of C, g_C satisfies $g_C = \frac{1}{2}(2 \times 2 \times (2+2-4)) + 1 = 1$. Inversely, every non-degenerate degree 4 elliptic curve C_0 in $\mathbb{C}P^3$ can define a pencil of quadrics $\mathcal{Q}$ with base locus C_0. (that is a family of quadrics containing C_0)

Let $Pic_r(C_0)$ be the set of complex divisors of degree r in the Picard group of the complex elliptic curve C_0. Let $h \in Pic_4(C_0)$ be the hyperplane section class (the hyperplane class of the non-singular quadric surface restricts to C_0). Since a non-singular quadric in $\mathbb{C}P^3$ is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, at every point in this quadric, there are exactly two lines of $\mathbb{C}P^3$, which lie on the quadric, passing through. Let D_1, D_2 be two lines representing two families of lines in this quadric (we can say: $D_1 = \{P_1\} \times \mathbb{C}P^1, D_2 = \mathbb{C}P^1 \times \{P_2\}$ where P_1, P_2 are two fixed points in $\mathbb{C}P^1$).

In a non-singular quadric Q of $\mathcal{Q}$, C_0 is of bidegree $(2,2)$ (because of the property of complete intersection). On Q, we can define two elements E_i in $Pic_2(C_0)$ by taking $E_i = D_i \cap C_0$ and we also have $E_1 + E_2 = h$. Conversely, given $E \in Pic_2(C_0)$, we can construct a quadric Q_E (either singular if $2E = h$ or non-singular if $E \neq h - E$) in $\mathcal{Q}$. As a consequence, we get a ramified covering of degree 2 map:

$$\pi_Q : Pic_2(C_0) \longrightarrow \mathcal{Q}$$

$$E \longmapsto Q_E$$

In the complex pencil of quadrics $\mathcal{Q}$, we always get 4 ramifcation (or critical) values of π_Q (i.e. 4 singular quadrics in $\mathcal{Q}$). This is because there are four critical points of π_Q, these points are exactly the solutions of the equation $2E = h$, where these solutions have form $E + E_i$ with $2E_i = 0$ and since C_0 is the complex elliptic curve, we have $2^2 = 4$ of 2−torsion points E_i, then there are four solutions of the equation $2E = h$.

1.2.2 Real pencils of quadrics

Now, we suppose that C_0 is a real elliptic curve with its real part is nonempty then the corresponding pencil $\mathcal{Q}$ is real with its real part $\mathbb{R}\mathcal{Q}$. If $Q \in \mathcal{Q}$ is a regular value then $\pi_Q^{-1}(\mathbb{R}Q)$ consists of two points (real or complex conjugate). If $Q \in \mathcal{Q}$ is a critical value then $\pi_Q^{-1}(\mathbb{R}Q)$ consists of only one point. Therefore, we have 3 possibilities for the map $\pi_{Q|\mathbb{R}C_0}$:

- When $\mathbb{R}C_0$ is not connected and the equation $2E = h$ have no real solution $E \in Pic_2(\mathbb{R}C_0)$ (in this case h does not lie on the real part of C_0 which contains p_0) then there is no real singular quadrics in $\mathbb{R}\mathcal{Q}$.

- When $\mathbb{R}C_0$ is not connected and the equation $2E = h$ have 4 real solutions $E \in Pic_2(\mathbb{R}C_0)$ (in this case h lies on the real part of C_0 which contains p_0) then there are 4 real singular quadrics in $\mathbb{R}\mathcal{Q}$.

- When $\mathbb{R}C_0$ is connected then the equation $2E = h$ have only 2 real solutions $E \in Pic_2(\mathbb{R}C_0)$ and there are 2 real singular quadrics in $\mathbb{R}\mathcal{Q}$.

1.3 Kollár's theorem

In all cases, an elliptic curve and a pencil of quadrics mean complex elliptic curve and complex pencil of quadrics.
There are some results that we use repeatedly in this text.

Theorem 1.3.1. *(Bézout's theorem) ([Har97], p47)*
Let Y, Z be varieties of dimensions r, s and of degree d, e in $\mathbb{C}P^n$. Assume that Y, Z are in a sufficiently general position so that all irreducible components of $Y \cap Z$ have dimension $r + s - n$ (assume that $r + s - n \geq 0$). For each irreducible component W of $Y \cap Z$, define the intersection multiplicity $i(Y, Z; W)$ of Y and Z along W. Then we have:

$$\Sigma i(Y, Z; W) \times degW = de$$

For example, in $\mathbb{C}P^3$, a quadric surface (a variety of dimension 2, degree 2) and a degree d irreducible curve (a variety of dimension 1, degree d) intersect at $2d$ points counted with multiplicity. Otherwise, this curve is contained in the quadric.

Theorem 1.3.2. *(Adjunction formula), ([Har97], Proposition 1.5, p361):*
If C is a non-singular curve of genus g_C on the non-singular surface Q and K is the canonical divisor on Q then

$$2g_C - 2 = C(C + K)$$

For example, for every $a, b \geq 1$, there are non-singular curves of bidegree (a, b) which lie on a non-singular quadric surface with degree $d = a + b$ and genus $g = (a - 1)(b - 1)$.

Theorem 1.3.3. *(Kollár's theorem)*
Let k be an algebraically closed field.
Let $C_0 \subset \mathbb{C}P^3$ be a non-degenerate degree 4 elliptic curve. Let $\mathcal{X} \subset C_0$ be the configuration of $2d$ general points. Let $\mathcal{Q}$ be the pencil of quadrics induced by C_0. Let $C(\mathcal{X})$ be the set of connected rational curves of degree d in $\mathbb{C}P^3$ passing through $\mathcal{X}$ (so not containing C_0).
Then, every curves C in $C(\mathcal{X})$ is irreducible and contained in a non-singular quadric Q of $\mathcal{Q}$.
Furthermore, $Q = \pi_{\mathcal{Q}}(E)$ where $E \in Pic_2(C_0)$ and $\pi_{\mathcal{Q}} : Pic_2(C_0) \to \mathcal{Q}$ is a ramified covering of degree 2 map. E is a solution of the equation:

$$(d - 2a)E = (d - a)h - \mathcal{X} \quad (*)$$

with condition $0 \leq a < \frac{d}{2}$ and h is the hyperplane class of Q restricted to C_0.
And $C \sim aD_1 + (d - a)D_2$ (linear equivalence in Q) where D_1, D_2 are two lines in Q such that $D_1 \cap C_0 = E, D_2 \cap C_0 = h - E$.

Proof. The idea of the proof is based on ([Kol14], Proposition 3). Supposing that C is irreducible then we show that it is contained in some quadric Q of $\mathcal{Q}$ and show that $Q = \pi_{\mathcal{Q}}(E)$ with E satisfies the equation $(*)$ and C is of bidegree $(a, d - a)$. To conclude, we need to exclude the case C is reducible.

Step 1: If C is an irreducible curve of degree d over k (i.e. C has only one irreducible component), then C is contained in some quadric Q (singular or non-singular). (That is because our curve is defined over an algebraically closed field then there are points in $C \backslash \mathcal{X}$ contained in C_0 then contained in some quadric, so there are more than $2d = deg(C) \times deg(Q)$ intersection points of C and Q. By Bézout's theorem, $C \subset Q$).

Let H be the hyperplane class of Q and $h = H|_{C_0}$ (i.e. $h = H \cap C_0$).
If the quadric Q is singular then $2C \sim dH$. Since $C \neq C_0$ then $2\mathcal{X} = 2C \cap C_0 \sim dh$. That is impossible as with generic configuration, the former varies by varying one point of the points of $\mathcal{X}$ but the latter is constant.
Thus the quadric Q is non-singular, then C is of bidegree $(a,b), a \neq b, a+b = d$. Otherwise, if $a = b$, i.e. bideg $C = (\frac{d}{2}, \frac{d}{2})$, then $C \sim \frac{d}{2}H$. Since $C \neq C_0$ then $\mathcal{X} = C \cap C_0 \sim \frac{d}{2}h$. It's impossible as above argument.
Moreover, we can choose on Q such that C is of bidegree $(a,b), a < b, a+b = d$ (i.e. C is of bidegree $(a, d-a), 0 \leq a < \frac{d}{2}$).

Note that C_0 is of bidegree $(2,2)$ in Q, and C is of bidegree $(a, d-a)$ (C does not contain C_0), so applying the formula of intersection points of curves in a quadric surface: $\sharp(C \cap C_0) = (2,2) \times (a, d-a) = 2d$. We can write $\mathcal{X} = C \cap C_0$. We choose two lines D_1, D_2 representing two families of lines in Q, such that $C_0 \sim 2D_1 + 2D_2$ and $(D_1 \cap C_0, D_2 \cap C_0) = (E, h-E)$. So $Q = \pi_Q(E)$ where π_Q is the map defined in the last section. Note that both C and C_0 lie on Q, we obtain:

$$\mathcal{X} = C_0 \cap C \sim (d-a)E + a(h-E) = ah + (d-2a)E$$

Or

$$(d-2a)E = (d-a)h - \mathcal{X}$$

Therefore, we get E is the solution of the equation:

$$(d-2a)E = (d-a)h - \mathcal{X}; 0 \leq a < \frac{d}{2}$$

Step 2: Suppose that C is reducible, i.e $C = \sum_i C_i$ where C_i is of degree d_i such that $0 < d_i < d$ and C_i does not contain C_0.

- Claim 1. Every C_i passes through exactly $2d_i$ points of $\mathcal{X}$.
  Otherwise, if one of the irreducible curves in C, let's call C_i, passes through more than $2d_i$ points in $\mathcal{X}$. Then $\sharp(C_i \cap Q) > 2d_i, \forall Q \in \mathcal{Q}$. By Bézout's theorem, $C_i \subset Q, \forall Q \in \mathcal{Q}$, that means $C_i \equiv C_0$, contradiction.
  As a consequence, different C_i passes through different points of $\mathcal{X}$.

- Claim 2. $C = \sum_i C_i$ is contained in only one quadric of $\mathcal{Q}$.
  Otherwise, suppose that there are two different irreducible curves lying in different quadrics, i.e. $C_i \subset Q_i, C_j \subset Q_j$. Then $C_i \cap C_j \subset Q_i \cap Q_j = C_0$, that implies two different curves $C_i \neq C_j$ pass through the same points in $\mathcal{X}$ (exclusively).

- Claim 3. There does not exist such a reducible curve $\mathrm{C} = \sum_i C_i$ satisfied.
  If C_i is of bidegree $(a_i, d_i - a_i)$ then by Claim 1, C_i passes through the set of $2d_i$ points of $\mathcal{X}$, denoted by $\mathcal{X}_i$. By **Step 1**, we get:

  $$\mathcal{X}_i \sim a_i h + (d_i - 2a_i)E$$

  But we also have:

  $$\mathcal{X} \sim ah + (d-2a)E$$

  Then:

  $$(d-2a)\mathcal{X}_i \sim a_i(d-2a)h + (d_i - 2a_i)(d-2a)E$$

$$\Rightarrow (d-2a)\mathcal{X}_i \sim (d_i-2a_i)\mathcal{X} + (a_i(d-2a) - a(d_i-2a_i))h$$
$$\Rightarrow (d-2a)\mathcal{X}_i - (d_i-2a_i)\mathcal{X} \sim (a_id - ad_i)h$$

It's impossible because the former varies whenever $\mathcal{X}_i \neq \mathcal{X}$ while the latter is constant.

In conclusion for **Step 2**: C is irreducible.
In conclusion for both step, for a generic configuration of $2d$ points $\mathcal{X} \subset C_0$, every connected rational curves of degree d in $\mathbb{C}P^3$ passing through $\mathcal{X}$ is irreducible and contained in a non-singular quadric Q of $\mathcal{Q}$. □

Remark:

Firstly, this theorem builds the relation between irreducible rational curves in $\mathbb{C}P^3$ and in $\mathbb{C}P^1 \times \mathbb{C}P^1$: counting connected (irreducible) rational curves C of degree d passing through $2d$ distinct points on an elliptic curve $C_0 \subset \mathbb{C}P^3$ is equivalent to counting quadrics in the pencils of quadrics induced by C_0 then counting the irreducible rational curves of bidegree $(a, d-a)$ passing through $2d$ distinct points on each quadric (we will prove in Chapter 2 that in fact every such curve only need to pass through $(2d-1)$ distinct points on C_0).

Secondly, it turns the enumerative problem of quadrics into of elliptic curves: to count such quadrics, we can count solutions of the equation $(*)$ which are divisors of degree 2 of $Pic(C_0)$.

Thirdly, this method works over the real case as well, that is counting real rational curves of degree d passing through $2d$ generic points in $\mathbb{C}P^3$ and their relationship with real rational curves in $\mathbb{C}P^1 \times \mathbb{C}P^1$. To apply this method to the real case, we note about choosing the real configuration containing at least one real point.

1.4 Normal bundles

We are familiar with the definition of normal bundle in term of differential geometry which is based on the notion of orthogonal complement of a vector subspace. However, it is no longer applicable in the algebraic situation. In algebraic geometry, we observe that the orthogonal complement is in fact isomorphic to the quotient of two vector bundles. That is the point we exploit, see [Sha13].

1.4.1 Complex normal bundles

Definition 1.4.1. *Let $j : Y \to X$ be an algebraic immersion. The normal bundle of Y in X, denoted by $\mathcal{N}_{Y/X}$, is the quotient of the pull-back of the tangent bundle of X to the tangent bundle of Y.*
i.e. Let TX, TY be the tangent bundles of X, Y respectively, then

$$\mathcal{N}_{Y/X} = j^*TX/TY$$

A normal bundle is in fact a vector bundle of rank $(n-k)$, where n and k are the rank of the vector bundle TX, TY respectively.

Example 2. *If $Y \subset X$ is a non-singular hypersurface, then the normal bundle $\mathcal{N}_{Y/X}$ is a line bundle.*

Let Q be a non-singular quadric surface in $\mathbb{C}P^3$ and f be an algebraic immersion $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ such that $f(\mathbb{C}P^1) \subset Q$, then we can define the following normal bundles:

$$f^*(\mathcal{N}_{f(\mathbb{C}P^1)/Q}) = f^*TQ/T\mathbb{C}P^1 := \mathcal{N}',$$

$$f^*(\mathcal{N}_{f(\mathbb{C}P^1)/\mathbb{C}P^3}) = f^*T\mathbb{C}P^3/T\mathbb{C}P^1 := \mathcal{N},$$

$$\mathcal{N}_{Q/\mathbb{C}P^3} = T\mathbb{C}P^3|Q/TQ := \mathcal{N}_Q.$$

We have a short exact sequence of normal bundles over $\mathbb{C}P^1$:

$$0 \to \mathcal{N}' \to \mathcal{N} \to f^*\mathcal{N}_Q \to 0 \quad (**)$$

Remark: This is the exact sequence of holomorphic vector bundles over $\mathbb{C}P^1$ so it does not split in general.

Property 1.4.2. *The exact sequence $(**)$ splits iff $f(\mathbb{C}P^1) \subset Q$ is a complete intersection.*

Recall:
A short exact sequence $0 \to B \to A \xrightarrow{\pi} C \to 0$ splits if there exists a section $C \xrightarrow{\sigma} A$ such that $\pi \circ \sigma = 1_C$, or equivalently, $A = B \oplus C$.

There is another definition which plays an important role in the sequel.

Definition 1.4.3. *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ be an algebraic immersion and d be the degree of $f(\mathbb{C}P^1)$. Then f is **balanced** if $\mathcal{N}$ is isomorphic to the direct sum of two holomorphic line subbundles of degree $(2d-1)$, i.e. $\mathcal{N} = \mathcal{O}(2d-1) \oplus \mathcal{O}(2d-1)$.*

Property 1.4.4. *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ be an algebraic immersion such that $f(\mathbb{C}P^1) \subset Q$ and bidegree of $f(\mathbb{C}P^1)$ is (a, b) with $a \neq b$. Then f is balanced.*

1.4.2 Real normal bundles

If f and Q are real such that its real part $\mathbb{R}Q$ is homeomorphic to $\mathbb{R}P^1 \times \mathbb{R}P^1$, then the restriction of f to $\mathbb{R}P^1$ is $f_{\mathbb{R}P^1} : \mathbb{R}P^1 \to \mathbb{R}P^3$ and $f(\mathbb{R}P^1) \subset \mathbb{R}Q$. We also have the corresponding real normal bundles $\mathbb{R}\mathcal{N}', \mathbb{R}\mathcal{N}, \mathbb{R}\mathcal{N}_Q$ and a short exact sequence of real normal bundles over $\mathbb{R}P^1$:

$$0 \to \mathbb{R}\mathcal{N}' \to \mathbb{R}\mathcal{N} \to f^*_{\mathbb{R}P^1}\mathbb{R}\mathcal{N}_Q \to 0$$

Remark: This is the exact sequence of smooth vector bundles over $\mathbb{R}P^1$ (so it always splits).

Chapter 2

Gromov-Witten invariants of $\mathbb{C}P^3$ and of $\mathbb{C}P^1 \times \mathbb{C}P^1$

In this chapter, we study rational curves of degree d in complex projective space $\mathbb{C}P^3$ and the idea of counting these curves is the same as in the case of counting curves in complex projective plane $\mathbb{C}P^2$ (using moduli space of stable maps), but it needs more additional arguments and it has an interesting relation with counting curves in $\mathbb{C}P^1 \times \mathbb{C}P^1$. In order to do the counting curve problem, we parametrize our curve in $\mathbb{C}P^3$ (resp. in $\mathbb{C}P^1 \times \mathbb{C}P^1$) by a balanced immersion (resp. immersion). And a balanced immersion (resp. immersion) can be considered as a regular point of an evaluation map. Then we deal with the counting map problem.

2.1 Definitions of Gromov-Witten invariants

Given d be a positive integer and a, b be non-negative integers.

2.1.1 Gromov-Witten invariants of $\mathbb{C}P^3$: $GW_{\mathbb{C}P^3}(d)$

Definition 2.1.1. *The Gromov-Witten invariant of $\mathbb{C}P^3$, denoted by $GW_{\mathbb{C}P^3}(d)$, is the number of rational curves of degree d passing through a generic configuration of $2d$ points in $\mathbb{C}P^3$.*

One can write:
$GW_{\mathbb{C}P^3}(d) = \sharp\{$C: rational curves of degree d pass through $2d$ generic points in $\mathbb{C}P^3\}$.

Why is $2d$ points?
The space of rational curves of degree d in $\mathbb{C}P^3$ has dimension $4d$. Indeed, consider the holomorphic map:

$$\begin{aligned} \phi : \quad \mathbb{C}P^1 &\longrightarrow \mathbb{C}P^3 \\ [x:y] &\longmapsto [g_1(x,y) : g_2(x,y) : g_3(x,y) : g_4(x,y)] \end{aligned}$$

where $g_i(x,y)$ are homogeneous polynomials of degree d for all $i \in \{1,2,3,4\}$ with no common factor. Since each $g_i(x,y)$ has $(d+1)$ coefficients, then for all $g_i(x,y)$ with $i \in \{1,2,3,4\}$ we have $4(d+1)$ coefficients. A rational curve of degree d in $\mathbb{C}P^3$ can be identified with a class of holomorphic map ϕ as follows:
$[g_1(x,y) : g_2(x,y) : g_3(x,y) : g_4(x,y)]$ and $\lambda \times [g_1(x,y) : g_2(x,y) : g_3(x,y) : g_4(x,y)]$ define the same curve (so we subtract one coefficient) and if $u : \mathbb{C}P^1 \to \mathbb{C}P^1$ is an automorphism of $\mathbb{C}P^1$ then ϕ and $\phi \circ u$ define the same curve (so we subtract 3 more coefficients). Thus we get the number of coefficients presenting rational curves of degree d in $\mathbb{C}P^3$ or the dimension of the space of rational curves of degree d in $\mathbb{C}P^3$

is $4(d+1)-1-3=4d$.

Let V be the subspace of all rational curves of degree d in $\mathbb{C}P^3$ passing through $2d$ generic points in $\mathbb{C}P^3$. We first observe that, the passage of a point make the dimension decrease by 2 since the subspace of all rational curves of degree d in $\mathbb{C}P^3$ passing through a point in $\mathbb{C}P^3$ has codimension 2. The same holds for passage of other points, if they are in general position with the previous ones. It follows that the dimension of V is $4d - 2\times 2d = 0$, i.e. V contains certain number of points which we want to count.

Proposition 2.1.2. *For a configuration of* $2d$ *points* $\mathcal{X}$ *in* $\mathbb{C}P^3$ *(not necessarily generic), if a rational curve of degree d in* $\mathbb{C}P^3$ *passing through* $\mathcal{X}$ *is parametrized by a balanced immersion* $f : \mathbb{C}P^1 \to \mathbb{C}P^3$*, then the number of these balanced immersions is exactly* $GW_{\mathbb{C}P^3}(d)$*. In particular, for a generic configuration of* $2d$ *points in* $\mathbb{C}P^3$*, all rational curves of degree d passing through them are parametrized by balanced immersions.*

Then, we can write:
$GW_{\mathbb{C}P^3}(d) = \sharp\{f : \mathbb{C}P^1 \to \mathbb{C}P^3$ balanced immersions: $degf(\mathbb{C}P^1) = d, \mathcal{X} \subset f(\mathbb{C}P^1)\}$

2.1.2 Gromov-Witten invariants of $\mathbb{C}P^1 \times \mathbb{C}P^1$: $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,b)$

Definition 2.1.3. *The Gromov-Witten invariant of* $\mathbb{C}P^1 \times \mathbb{C}P^1$*, denoted by* $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,b)$*, is the number of rational curves of bidegree* (a,b) *passing through a generic configuration of* $2(a+b)-1$ *points in* $\mathbb{C}P^1 \times \mathbb{C}P^1$ *.*

One can write:
$GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,b) = \sharp\{$C: rational curves of bidegree (a,b) pass through $2(a+b)-1$ generic points in $\mathbb{C}P^1 \times \mathbb{C}P^1\}$.

Why is $2(a+b)-1$ points?
The space of rational curves of bidegree (a,b) in $\mathbb{C}P^1 \times \mathbb{C}P^1$ has dimension $2(a+b)-1$. Indeed, consider the holomorphic map:

$$\begin{aligned} \phi : \quad \mathbb{C}P^1 &\longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1 \\ [x:y] &\longmapsto ([g_1(x,y) : g_2(x,y)], [g_3(x,y) : g_4(x,y)]) \end{aligned}$$

where $g_1(x,y), g_2(x,y)$ are homogeneous polynomials of degree a with no common factor; $g_3(x,y), g_4(x,y)$ are homogeneous polynomials of degree b with no common factor. We write g_i instead of $g_i(x,y)$ for short. Since g_1, g_2 has $(a+1)$ coefficients and $g_3, g_4($ has $(b+1)$ coefficients then, for all $g_i, i \in \{1,2,3,4\}$, we have $2(a+1)+2(b+1) = 2(a+b)+4$ coefficients.
A rational curve of bidegree (a,b) in $\mathbb{C}P^1 \times \mathbb{C}P^1$ can be identified with a class of holomorphic map ϕ as follows:
$([g_1 : g_2],[g_3 : g_4])$ and $(\lambda_1 \times [g_1 : g_2], \lambda_2 \times [g_3 : g_4])$ define the same curve (so we subtract 2 coefficients) and if $u : \mathbb{C}P^1 \to \mathbb{C}P^1$ is an automorphism of $\mathbb{C}P^1$ then ϕ and $\phi \circ u$ define the same curve (so we subtract 3 more coefficients). Thus we get the number of coefficients presenting rational curves bidegree (a,b) in $\mathbb{C}P^1 \times \mathbb{C}P^1$ or the dimension of the space of rational curves of bidegree (a,b) in $\mathbb{C}P^1 \times \mathbb{C}P^1$ is $2(a+b)+4-2-3 = 2(a+b)-1$.

Let U be the subspace of all rational curves of bidegree (a,b) passing through $2(a+b)-1$ generic points in $\mathbb{C}P^1 \times \mathbb{C}P^1$. We observe that, the passage of a point make the dimension decrease by 1 (the same as in the case of rational curves of degree d passing through $(3d-1)$ generic points in $\mathbb{C}P^2$). The same holds for passage of other points since they are in generic position. It follows that the dimension of U is 0, i.e. U contains certain number of points which we want to count.

Proposition 2.1.4. *For a configuration of $(2d-1)$ points $\mathcal{Y}$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$ (not necessarily generic), if a rational curve of bidegree $(a, d-a)$ passing through $\mathcal{Y}$ is parametrized by an immersion $f : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$, then the number of these immersions is exactly $GW_{\mathbb{C}P^1 \times \mathbb{C}P^1}(a, d-a)$.*
In particular, for a generic configuration of $(2d-1)$ points in $\mathbb{C}P^1 \times \mathbb{C}P^1$, all rational curves of bidegree $(a, d-a)$ passing through them are parametrized by balanced immersions.

Then, we can write:
$GW_{\mathbb{C}P^1 \times \mathbb{C}P^1}(a, d-a) = \sharp\{f : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ immersions: bidegree of $f(\mathbb{C}P^1)$ is $(a, d-a)$ and $\mathcal{Y} \subset f(\mathbb{C}P^1)\}$.

By Kollár's idea, there exists a particular configuration of $2d$ distinct points in $\mathbb{C}P^3$ (resp. a particular configuration of $(2d-1)$ distinct points in $\mathbb{C}P^1 \times \mathbb{C}P^1$) which are in fact contained in a degree 4 elliptic curve such that the number of rational curves of degree d in $\mathbb{C}P^3$ (resp. the number of rational curves of bidegree $(a, d-a)$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$) passing through them is the Gromov-Witten invariant of $\mathbb{C}P^3$ (resp. the Gromov-Witten invariant of $\mathbb{C}P^1 \times \mathbb{C}P^1$). We will see how it works in the next section.

2.2 A (balanced) immersion as a regular point of an evaluation map

In all cases, a quadric Q means a non-singular quadric.

We consider the evaluation map on the moduli space of stable maps in two following cases:

- **Case 1:** Let $\mathcal{M}^*(\mathbb{C}P^3, d)$ be the moduli space of stable maps (up to reparametrization) f from $(\mathbb{C}P^1; x_1, \ldots, x_{2d})$ with $2d$ marked points to $\mathbb{C}P^3$, whose image has degree d, i.e.

$$\mathcal{M}^*(\mathbb{C}P^3, d) = \{f : (\mathbb{C}P^1; x_1, \ldots, x_{2d}) \longrightarrow \mathbb{C}P^3 : deg(f(\mathbb{C}P^1)) = d\}/\sim$$

  where $f([x:y]) = [g_1(x,y) : g_2(x,y) : g_3(x,y) : g_4(x,y)] \in \mathbb{C}P^3$, $g_i(x,y)$ are homogeneous polynomials of degree d with no common factor, we write g_i instead of $g_i(x,y)$ for short, then $[g_1 : \ldots : g_4] \sim \lambda[g_1 : \ldots : g_4]$, and $f \sim f \circ u$ with $u \in Aut(\mathbb{C}P^1)$. Then this moduli space has dimension $4d + 2d = 6d$.

  Let ev_1 be an evaluation map defined as:

$$ev_1 : \mathcal{M}^*(\mathbb{C}P^3, d) \longrightarrow (\mathbb{C}P^3)^{2d}$$
$$f \longmapsto (f(x_1), \ldots, f(x_{2d}))$$

- **Case 2:** Let $\mathcal{M}^*(Q, (a, d-a))$ be the moduli space of stable maps (up to reparametrization) f from $(\mathbb{C}P^1; x_1, \ldots, x_{2d-1})$ with $(2d-1)$ marked points to a non-singular quadric $Q \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$, whose image has bidegree $(a, d-a)$, i.e.

$$\mathcal{M}^*(Q,(a,d-a)) = \{f : (\mathbb{C}P^1; x_1, \ldots, x_{2d-1}) \longrightarrow Q : bideg(f(\mathbb{C}P^1)) = (a,d-a)\}/\sim$$

where $f([x:y]) = ([g_1(x,y) : g_2(x,y)], [g_3(x,y) : g_4(x,y)]) \in \mathbb{C}P^1 \times \mathbb{C}P^1$, $g_i(x,y)$ are homogeneous polynomials of degree a, $i \in \{1,2\}$ with no common factor, $g_j(x,y)$ are homogeneous polynomials of degree $(d-a)$, $j \in \{3,4\}$ with no common factor. We write g_i instead of $g_i(x,y)$ for short, then we have $([g_1 : g_2],[g_3 : g_4]) \sim (\lambda_1[g_1 : g_2], \lambda_2[g_3 : g_4])$, and $f \sim f \circ u$, $u \in Aut(\mathbb{C}P^1)$. This implies the moduli space has dimension $(2d-1)+(2d-1) = 4d-2$.

Let ev_2 be an evaluation map defined as:

$$ev_2 : \mathcal{M}^*(Q,(a,d-a)) \longrightarrow Q^{2d-1}$$
$$f \longmapsto (f(x_1), \ldots, f(x_{2d-1}))$$

By [Web05, lemma 1.2], we have two followings results:

Lemma 2.2.1. *A stable map $f \in \mathcal{M}^*(\mathbb{C}P^3, d)$ is a regular point of ev_1 iff f is a balanced immersion from $\mathbb{C}P^1$ to $\mathbb{C}P^3$.*

Lemma 2.2.2. *A stable map $f \in \mathcal{M}^*(Q,(a,d-a))$ is a regular point of ev_2 iff f is an immersion from $\mathbb{C}P^1$ to $\mathbb{C}P^1 \times \mathbb{C}P^1$.*

Let C_0 be a non-degenerate degree 4 elliptic curve in $\mathbb{C}P^3$.

We can choose a particular configuration $\mathcal{X}$ of $2d$ distinct points in $\mathbb{C}P^3$ such that every $f = ev_1^{-1}(\mathcal{X})$ is a balanced immersion, i.e $\mathcal{X}$ is a regular value of ev_1. Then the number of rational curves of degree d passing through such $\mathcal{X}$ is exactly the Gromov-Witten invariant of $\mathbb{C}P^3$.

If $\mathcal{X}$ is *a configuration of* $2d$ *distinct points lying on* C_0 then we can choose such $\mathcal{X}$ in C_0 satisfied. Indeed, let V_n be the set of configurations of n distinct points on C_0, $V_n \subset (C_0)^n \subset (\mathbb{C}P^3)^n$. We have $f(\{x_1, \ldots, x_{2d}\}) = \mathcal{X} \in V_{2d}$. Applying Sard's theorem to the holomorphism ev_1, there is a dense open subset $U \subset V_{2d}$ such that ev_1 is regular on U. Thus, we choose $\mathcal{X} \in U \subset V_{2d}$, we get $ev_1^{-1}(\mathcal{X})$ is a regular point of ev_1.

By Kollár's theorem, if $\mathcal{X}$ is a generic configuration of $2d$ points lying on C_0, then all connected rational curves of degree d passing through $\mathcal{X}$ are contained in a quadric Q which is in the pencil of quadrics induced by C_0. Moreover, if $\mathcal{Y} \subset \mathcal{X}$ as *a configuration of* $(2d-1)$ *distinct points lying on* C_0 then we can choose such $\mathcal{Y}$ that the number of rational curves of bidegree $(a, d-a)$ passing through them on each quadric is exactly the Gromov-Witten invariant of $\mathbb{C}P^1 \times \mathbb{C}P^1$. By the same argument, applying Sard's theorem to holomorphism ev_2, there exists $U' \subset V_{2d-1}$ a dense open subset such that $ev_2^{-1}(\mathcal{Y})$ is a regular point of ev_2 for all $\mathcal{Y} \in U' \subset V_{2d-1}$.

2.3 Relation between two GW-invariants: $GW_{\mathbb{C}P^3}(d)$ and $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,b)$

Theorem 2.3.1. *Let d be a positive integer then:*

$$GW_{\mathbb{C}P^3}(d) = \sum_{0\le a<\frac{d}{2}} (d-2a)^2 GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,d-a)$$

Proof. Let $C_0 \subset \mathbb{C}P^3$ be a non-degenerate degree 4 elliptic curve.
Let $\mathcal{X} \subset C_0$ be a configuration of $2d$ distinct points, i.e. $\mathcal{X} \in V_{2d}$; let $\mathcal{Y} \subset C_0$ be a configuration of $(2d-1)$ distinct points of $\mathcal{X}$, i.e. $\mathcal{Y} \subset \mathcal{X}, \mathcal{Y} \in V_{2d-1}$.
Let $C(\mathcal{X})$ be the set of connected rational curves of degree d in $\mathbb{C}P^3$ containing $\mathcal{X}$ (then every $C \in C(\mathcal{X})$ is irreducible).
Let $Q \in \mathcal{Q}$ be a non-singular quadric in the pencil of quadrics induced by C_0.
Let $C_{Q,a}(\mathcal{Y})$ be the set of irreducible rational curves of bidegree $(a, d-a)$ in Q containing $\mathcal{Y}$.

We have $C_{Q,a}(\mathcal{Y}) \subset C(\mathcal{X})$, i.e. every curve in $C_{Q,a}(\mathcal{Y})$ containing $\mathcal{Y} \subset \mathcal{X}$ then contains $\mathcal{X}$. Indeed, suppose that C_d, C'_d are two curves of bidegree $(a, d-a)$ such that $C_d \cap C_0 = p_1 + p_2 + \ldots + p_{2d}$ and $C'_d \cap C_0 = p'_1 + p_2 + \ldots + p_{2d}$. Since C_0 is of bidegree $(2,2)$ and D_1, D_2 are two families of lines in Q such that $D_1 \cap C_0 = E$, $D_2 \cap C_0 = h - E$ then we have linear equivalences:

$$C_d \cap C_0 \sim a(h-E) + (d-a)E \sim C'_d \cap C_0$$

Thus $p_1 \sim p'_1$, but p_1, p'_1 are in the elliptic curve C_0 so $p_1 = p'_1$. That means all curves passing through $(2d-1)$ points in the configuration of $2d$ points $\mathcal{X}$ in C_0 pass through the last point for free.
Now we consider:
$C(\mathcal{X}) = \{$ connected rational curves of degree d in $\mathbb{C}P^3$, contain $\mathcal{X}\}$

$$= \{f : \mathbb{C}P^1 \to \mathbb{C}P^3 \text{ balanced immersions: } degf(\mathbb{C}P^1) = d, \mathcal{X} \subset f(\mathbb{C}P^1)\}.$$

On the one hand, by Lemma 2.2.1, if we choose $\mathcal{X} \in U \subset V_{2d}$ is a regular value of ev_1, then $\sharp(C(\mathcal{X})) = GW_{\mathbb{C}P^3}(d)$.

On the other hand, by Kollár's theorem, every curve in $C(\mathcal{X})$ is contained in a quadric $Q \in \mathcal{Q}$ then has bidegree $(a, d-a)$. Moreover, this quadric is the image under π_Q of $E \in Pic_2(C_0)$, which can exist if $0 \le a < \frac{d}{2}$. We note that $C_{Q,a}(\mathcal{Y}) \subset C(\mathcal{X})$. Therefore,
$C(\mathcal{X}) = \bigcup_{0\le a<\frac{d}{2}} \bigcup_{Q\in\mathcal{Q}} \{f : \mathbb{C}P^1 \to Q$ immersions, bidegree of $f(\mathbb{C}P^1)$ is $(a, d-a)$, $\mathcal{Y} \subset f(\mathbb{C}P^1)\}$.

By the property of the torsion points in Chapter 1, we have exactly $(d-2a)^2$ solutions in $Pic_2(C_0)$ of the equation:

$$(d-2a)E = (d-a)h - \mathcal{X}; \quad 0 \le a < \frac{d}{2} \quad (*)$$

(Indeed, the solutions of the equation $(*)$ have form $E + E_i$ where $(d-2a)E_i = 0$ and the number of $(d-2a)-$torsion points in C_0 is $(d-2a)^2$, that means there are $(d-2a)^2$ quadrics associated in $\mathcal{Q}$). Thus,

$$\begin{aligned} C(\mathcal{X}) &= \bigcup_{0\le a<\frac{d}{2}} \bigcup_{\substack{Q=\pi_Q(E)\\(d-2a)E=(d-a)h-\mathcal{X}\\ \mathcal{Y}\subset\mathcal{X}}} \{f:\mathbb{C}P^1\to Q \text{ immersions}, \mathrm{bideg} f(\mathbb{C}P^1)=(a,d-a), \mathcal{Y}\subset f(\mathbb{C}P^1)\} \\ &= \bigcup_{0\le a<\frac{d}{2}} \bigcup_{\substack{Q=\pi_Q(E)\\(d-2a)E=(d-a)h-\mathcal{X}\\ \mathcal{Y}\subset\mathcal{X}}} C_{Q,a}(\mathcal{Y}) \end{aligned}$$

By Lemma 2.2.2, if we choose $\mathcal{Y} \in U' \subset V_{2d-1}$ is a regular value of ev_2, then $\sharp(C_{Q,a}(\mathcal{Y})) = GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,d-a)$, that is the number of elements of $C(\mathcal{X})$ in each quadric Q of $\mathcal{Q}$.

Thus,

$$\sharp(C(\mathcal{X})) = \sum_{0\le a<\frac{d}{2}} \sum_{\substack{Q=\pi_Q(E)\\(d-2a)E=(d-a)h-\mathcal{X}\\ \mathcal{Y}\subset\mathcal{X}}} \sharp(C_{Q,a}(\mathcal{Y}))$$

Remark:

- $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,b) = GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(b,a)$.
  Indeed, $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,b)$ is the number of rational curves passing through a configuration of $(2d-1)$ general points in $\mathbb{C}P^1\times\mathbb{C}P^1$ and intersecting D_1, D_2 at a and b points respectively. We fix a configuration in $\mathbb{C}P^1\times\mathbb{C}P^1$, we change the role of (D_1, D_2) into (D_2, D_1) then the number of curves doesn't change but they now have bidegree (b,a).

- $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(1,0) = 1$ and $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,0) = 0, \forall a > 1$.
  Indeed, $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(1,0)$ is the number of lines in the family D_1 which pass through 1 point in $\mathbb{C}P^1\times\mathbb{C}P^1$. But every point in $\mathbb{C}P^1\times\mathbb{C}P^1$ is the intersection of exactly two lines, one in the family D_1, the other in the family D_2. So $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(1,0) = GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(0,1) = 1$.
  Otherwise, $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,0)$ is the number of rational curves which intersect D_1 at a points but don't intersect D_2 (up to isotopy class, these curves are collection of a lines in the same family D_2) and pass through $2a-1 \ge 3$ general points in $\mathbb{C}P^1\times\mathbb{C}P^1$. Since $2a-1$ points are general, they can not lie in the same line, so there doesn't exist any such curve. Thus, $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,0) = 0$ for every $a > 1$.

For every $0 \le a < \frac{d}{2}$ (more precisely, if d is odd, then $a \in \{1,2,\ldots,\frac{d-1}{2}\}$; if d is even, then $a \in \{1,2,\ldots,\frac{d-2}{2}\}$), there are exactly $(d-2a)^2$ non-singular quadrics Q: $Q = \pi_Q(E)$. On each quadric, there are $\sharp(C_a(\mathcal{Y})) = GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,d-a)$ rational curves of bidegree $(a,d-a)$ containing $\mathcal{Y}$, i.e. $(d-2a)^2 \times GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,d-a)$ curves in $C(\mathcal{X})$.

Hence, for all $d \ge 1$,

$$\sharp(C(\mathcal{X})) = \sum_{0\le a<\frac{d}{2}} (d-2a)^2 GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,d-a).$$

□

Example 3. *Compute the Gromov-Witten invariants in the case $d = 4$.*

Let $\mathcal{X}$ be a configuration of 8 distinct points on the elliptic curve C_0. Then

$$GW_{\mathbb{C}P^3}(4) = (4-2)^2 GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(1,3) + 4^2 GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(0,4)$$

We have $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(0,4) = 0$, so we only need to compute $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(1,3)$.
In each quadric, there is a unique rational curve C of bidegree $(1,3)$ passing through $\mathcal{X}$. Indeed, C can be viewed as the graph of a degree 3 map:

$$\begin{aligned} \mathbb{C}P^1 &\longrightarrow \mathbb{C}P^1 \\ [x:y] &\longmapsto [g_1(x,y) : g_2(x,y)] \end{aligned}$$

where $g_i(x,y) = a_i x^3 + b_i y^3 + c_i x^2 y + d_i x y^2; i \in \{1,2\}$. Passing through 7 distinct points gives 7 linear equations on the 8 coefficients then gives a unique pair $(g_1(x,y), g_2(x,y))$ up to scalar. Thus, $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(1,3) = 1$. Therefore, $GW_{\mathbb{C}P^3}(4) = 4 \times 1 = 4$.

Chapter 3

Welschinger invariants of $\mathbb{C}P^3$ and of $\mathbb{C}P^1 \times \mathbb{C}P^1$

In this chapter, we consider real rational curves so we don't simply count curves but curves with sign. We need to define the sign for each real curve in $\mathbb{C}P^3$ and in $\mathbb{C}P^1 \times \mathbb{C}P^1$ such that we can make the comparison between their Welschinger invariants. That can be done thanks to their link with the real normal bundles $\mathbb{R}\mathcal{N}'$.

3.1 Definitions of Welschinger invariants:

3.1.1 Welschinger invariants of $\mathbb{C}P^1 \times \mathbb{C}P^1$: $W_{\mathbb{R}P^1 \times \mathbb{R}P^1}((a,b),l)$

Let a, b be two nature numbers, $a + b = d$.
Let $\mathcal{Y}$ be a real generic configuration of $(2d-1)$ points (including l pairs of complex conjugated points) in $\mathbb{C}P^1 \times \mathbb{C}P^1$. Let $\mathbb{R}C(\mathcal{Y})$ be the set of all real rational curves of bidegree (a,b) in $\mathbb{C}P^1 \times \mathbb{C}P^1$ passing through $\mathcal{Y}$. For each curve C in $\mathbb{R}C(\mathcal{Y})$, we define its sign, denoted by $s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(C)$, so that there exists an invariant only depending on a, b and l in $\mathcal{Y}$. This invariant is called **the Welschinger invariant of** $\mathbb{C}P^1 \times \mathbb{C}P^1$, denoted by $W_{\mathbb{R}P^1 \times \mathbb{R}P^1}((a,b),l)$.

Definition 3.1.1.

$$W_{\mathbb{R}P^1 \times \mathbb{R}P^1}((a,b),l) := \sum_{C \in \mathbb{R}C(\mathcal{Y})} (-1)^{s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(C)}$$

On the one hand, we define $s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(C)$ as the number of elliptic real nodes (they are the intersection points of two complex conjugated branches) of C.

On the other hand, we are looking for the parity of this sign, i.e. $s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(C)$ mod 2, so we can describe it by the following. We know that, by a generic configuration $\mathcal{Y}$, every curve $C \in \mathbb{R}C(\mathcal{Y})$ is parametrized by a real algebraic immersion $f : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ with bideg$f(\mathbb{C}P^1) = (a,b)$ and $\mathcal{Y} \subset f(\mathbb{C}P^1)$. We also have $f_{\mathbb{R}P^1} : \mathbb{R}P^1 \to \mathbb{R}P^1 \times \mathbb{R}P^1$ is an immersion.

A trivialization of the tangent bundle over $\mathbb{R}P^1$ deduces a trivialization ϕ_0 over the tangent bundle of its product:

$$\phi_0 : T(\mathbb{R}P^1 \times \mathbb{R}P^1) \longrightarrow \mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}^2$$

By the canonical orientation and scalar product on $\mathbb{R}^2$, we can deduce an orientation and a Riemannian metric on $T(\mathbb{R}P^1 \times \mathbb{R}P^1)$. Taking the pull-back of this tangent bundle by the immersion $f_{\mathbb{R}P^1}$, we deduce a trivialization and a Riemannian metric on $f^*_{\mathbb{R}P^1}T(\mathbb{R}P^1 \times \mathbb{R}P^1)$. In $f^*_{\mathbb{R}P^1}T(\mathbb{R}P^1 \times \mathbb{R}P^1)$, we have a natural $\mathbb{R}-$subbundle $T\mathbb{R}P^1$ by the universal propriety of the pull-back, ($T\mathbb{R}P^1$ is a rank 1 real vector bundle over $\mathbb{R}P^1$) and we call E its orthogonal $\mathbb{R}-$subbundle (E is also a rank 1 real vector bundle over $\mathbb{R}P^1$).

Note that, we have an isomorphism $\mathbb{R}P^1 \simeq \mathbb{S}^1$, so we can choose a non-vanishing smooth section $\sigma_T : \mathbb{R}P^1 \to T\mathbb{R}P^1$ and then choose a section $\sigma_E : \mathbb{R}P^1 \to E$ such that (σ_T, σ_E) is a positive basis of $f^*_{\mathbb{R}P^1}T(\mathbb{R}P^1 \times \mathbb{R}P^1)$.

$$
\begin{array}{ccccc}
f^*_{\mathbb{R}P^1}T(\mathbb{R}P^1 \times \mathbb{R}P^1) = T\mathbb{R}P^1 \oplus E & & & & T(\mathbb{R}P^1 \times \mathbb{R}P^1) \xrightarrow{\phi_0} \mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}^2 \\
\downarrow \qquad \uparrow \sigma_T \uparrow \sigma_E & & & & \downarrow \\
\mathbb{R}P^1 & & \xrightarrow{f_{\mathbb{R}P^1}} & & \mathbb{R}P^1 \times \mathbb{R}P^1
\end{array}
$$

Suppose we have a non-vanishing map $g : \mathbb{S}^1 \to \mathbb{R}^2$. Dividing by the norm, we obtain a map $g : \mathbb{S}^1 \to \mathbb{S}^1, z \mapsto g(z)$, and we can count how many times $g(z)$ goes around $\mathbb{S}^1$ when z goes around $\mathbb{S}^1$. The map $g : \mathbb{S}^1 \to \mathbb{S}^1$ is the Gauss map, and the number of times $g(z)$ rotates is the Gauss index of g. For examples, the Gauss map $z \mapsto c$ with c is a constant has Gauss index 0; the Gauss map $z \mapsto z(resp. z \mapsto \bar{z} = \frac{1}{z})$ has Gauss index 1 (resp. -1). Let N be the parity of the degree of the Gauss map of $f(\mathbb{R}P^1)$, that is the Gauss index of the Gauss map from $f(\mathbb{R}P^1)$ to $\mathbb{R}^2$. We have the following lemma.

Lemma 3.1.2. *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ be a real algebraic immersion then:*

$$s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)) = N \mod 2$$

Proof. We fix an orientation for $f(\mathbb{R}P^1)$.

If $f(\mathbb{C}P^1)$ has bidegree (a, b), then $f(\mathbb{C}P^1)$ has exactly $(a-1)(b-1)$ nodes. Note that a node of $f(\mathbb{R}P^1)$ is exactly a hyperbolic node of $f(\mathbb{C}P^1)$. By smoothing each node of $f(\mathbb{R}P^1)$ according to the orientation of $f(\mathbb{R}P^1)$, we obtain a collection γ of n disjoint oriented circles embedded in $\mathbb{R}P^1 \times \mathbb{R}P^1$. Hence, the Gauss index of the Gauss map of $f(\mathbb{R}P^1)$ is the sum of the Gauss index of the Gauss map of all $\gamma_i \in \gamma$ and the Gauss index of the Gauss map of γ_i is either 0, 1 or -1. Note that whenever we smooth a node, the number of embedded circles is changed by 1. After smoothing, we get $n = 1 + k \mod 2$.

Moreover, the oriented circles embedded in $\mathbb{R}P^1 \times \mathbb{R}P^1$ are either of the $(p, q)-$class with $pgcd(p, q) = 1$ or of the $(0, 0)-$class in $H_1(\mathbb{R}P^1 \times \mathbb{R}P^1, \mathbb{Z})$. The ones are of the $(p, q)-$class have Gauss index 0 (see the presenting of these circles on Figure 3.1, with an orientation, the Gauss map associated is constant); the ones are of the $(0, 0)-$class have Gauss index ± 1 (see the presenting of these circles on Figure 3.1, with an orientation, the Gauss map associated is orientation preserving or not , this implies the Gauss index is 1 or -1 respectively).
Indeed, if γ_i is not a trivial class, up to isotopy, it is parametrized by an embedding $\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1, t \mapsto (pt, qt)$ where $p, q \in \mathbb{Z}^*$, see ([Hat03], Torus knots, p47). If $pgcd(p, q) = d$ then this embedding is a $(d : 1)$ map, since this map is injective so $d = 1$. Moreover, if γ_i are not of $(0, 0)-$class and they are all disjoint, then such γ_i are of the same class (p, q). Otherwise, suppose that γ_i is of class (p, q), γ_j is of class $(p', q') \neq (p, q)$ then $\sharp(\gamma_i \cap \gamma_j) = pq' - qp' \neq 0$, i.e γ_i intersects γ_j, contradiction.

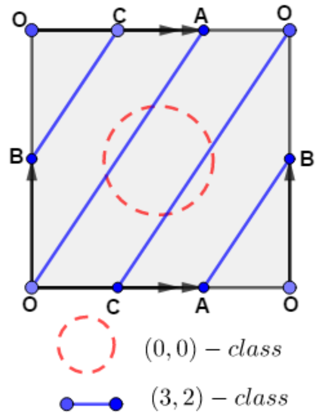

FIGURE 3.1: Examples of homology classes on the torus

Thus, we can suppose that (up to orientation) there are m oriented circles of γ being of the same class $(p,q) \in H_1(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{Z})$, $(pgcd(p,q) = 1)$. Therefore, there are $(n-m)$ oriented circles in γ are of the class $(0,0) \in H_1(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{Z})$ with Gauss index ± 1. We can write $f(\mathbb{R}P^1)$ be of the class $m(p,q)$.

Remark: If C is of bidegree (a,b) then $\mathbb{R}C$ is of bidegree $(a,b) \mod 2$.

So $(a,b) = m(p,q) \mod 2$. Since $pgcd(p,q) = 1$ then: if m is even then $a \vee b$ is even, so $(a-1)(b-1)$ is odd; if m is odd then $a \vee b$ and $a \wedge b$ are odd, so $(a-1)(b-1)$ is even. Thus, we have $(a-1)(b-1) = m-1 \mod 2$.

In conclusion, we have:

$$
\begin{aligned}
s_{\mathbb{R}P^1 \times \mathbb{R}P^1}(f(\mathbb{C}P^1)) &= \sharp \text{ (elliptic nodes of } f(\mathbb{C}P^1)) \\
&= \sharp\text{(nodes of } f(\mathbb{C}P^1)) \text{ - } \sharp\text{(hyperbolic nodes of } f(\mathbb{C}P^1)) \mod 2 \\
&= (a-1)(b-1) - k \mod 2 \\
&= (m-1-k) \text{ mod } 2 \\
&= ((1+k) - m) \text{ mod } 2 \\
&= (n-m) \text{ mod } 2 \\
&= \sharp \text{ (oriented cirles in } \mathbb{R}P^1 \times \mathbb{R}P^1) \text{ -}\sharp\text{(oriented circles in } \mathbb{R}P^1 \times \mathbb{R}P^1 \\
&\qquad \text{of Gauss index 0) mod 2} \\
&= \sharp \text{ (oriented cirles in } \mathbb{R}P^1 \times \mathbb{R}P^1 \text{ of Gauss index } \pm 1) \text{ mod 2} \\
&= \text{(degree of Gauss map of } f(\mathbb{R}P^1)) \text{ mod 2} \\
&= N \text{ mod } 2.
\end{aligned}
$$

That completed the proof of the lemma.

□

3.1.2 Welschinger invariants of $\mathbb{C}P^3$: $W_{\mathbb{R}P^3}(d,l)$

Let d be a nature number.
Let $\mathcal{X}$ be the real generic configuration of $2d$ points (including l pairs of complex conjugated points) in $\mathbb{C}P^3$. Let $\mathbb{R}\mathcal{C}(\mathcal{X})$ be the set of all real rational curves of degree d in $\mathbb{C}P^3$ passing through $\mathcal{X}$. For each curve C in $\mathbb{R}\mathcal{C}(\mathcal{X})$, we define its sign, denoted by $s_{\mathbb{R}P^3}(C)$, so that there exist an invariant only depending on d and l in $\mathcal{X}$. This invariant is called **Welschinger invariant of** $\mathbb{C}P^3$, denoted by $W_{\mathbb{R}P^3}(d,l)$.

Definition 3.1.3.
$$W_{\mathbb{R}P^3}(d,l) := \sum_{C \in \mathbb{R}C(\mathcal{X})} (-1)^{s_{\mathbb{R}P^3}(C)}$$

Now we need to define $s_{\mathbb{R}P^3}(C)$ for every curve $C \in \mathbb{R}C(\mathcal{X})$.

For a real generic configuration $\mathcal{X}$, every curve $C \in \mathbb{R}C(\mathcal{X})$ is parametrized by a real *balanced* algebraic immersion $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ with $\deg f(\mathbb{C}P^1) = d$ and $\mathcal{X} \subset f(\mathbb{C}P^1)$. We also have $f_{\mathbb{R}P^1} : \mathbb{R}P^1 \to \mathbb{R}P^3$ is an immersion.

We fix an orientation on $\mathbb{R}P^3$. We can choose a compatible trivialization ϕ_0 of the tangent bundle over $\mathbb{R}P^3$:

$$\phi_0 : T\mathbb{R}P^3 \longrightarrow \mathbb{R}P^3 \times \mathbb{R}^3$$

The canonical Euclidean scalar product on $\mathbb{R}^3$ deduces a Riemannian metric on $\mathbb{R}P^3$. Taking the pull-back of the tangent bundle $T\mathbb{R}P^3$ by the immersion $f_{\mathbb{R}P^1}$, we deduce a trivialization and a Riemannian metric on $f^*_{\mathbb{R}P^1}T\mathbb{R}P^3$. In $f^*_{\mathbb{R}P^1}T\mathbb{R}P^3$, we have a natural $\mathbb{R}-$subbundle $T\mathbb{R}P^1$ by the universal propriety of the pull-back ($T\mathbb{R}P^1$ is a rank 1 real vector bundle over $\mathbb{R}P^1$) and we call $\mathcal{N}_\mathbb{R}$ its orthogonal $\mathbb{R}-$subbundle ($\mathcal{N}_\mathbb{R}$ is a rank 2 real vector bundle over $\mathbb{R}P^1$) .

Fixing an orientation on $\mathbb{R}P^1$, we can choose a positive orthonormal section $\sigma_T : \mathbb{R}P^1 \to T\mathbb{R}P^1$. We can also choose a line $\mathbb{R}-$subbundle E of $\mathcal{N}_\mathbb{R}$ together with its non-vanishing section σ_E such that (σ_T, σ_E) is an orthonormal section of $T\mathbb{R}P^1 \oplus E$. Then there is a unique way to choose the second section σ_N of $\mathcal{N}_\mathbb{R}$ to make $(\sigma_T, \sigma_E, \sigma_N)$ form a positive orthonormal section of $f^*_{\mathbb{R}P^1}T\mathbb{R}P^3$.

$$\begin{array}{ccccc}
f^*_{\mathbb{R}P^1}T(\mathbb{R}P^3) = T\mathbb{R}P^1 \oplus \mathcal{N}_\mathbb{R} & & & & T(\mathbb{R}P^3) \xrightarrow{\phi_0} \mathbb{R}P^3 \times \mathbb{R}^3 \\
\downarrow \quad \uparrow \sigma_T \uparrow \sigma_E \uparrow \sigma_N & & & & \downarrow \\
\mathbb{R}P^1 & & \xrightarrow{f_{\mathbb{R}P^1}} & & \mathbb{R}P^3
\end{array}$$

Topologically, we have a homeomorphism $\mathbb{R}P^1 \simeq \mathbb{S}^1/\{\text{antipodal points}\} = \mathbb{S}^1$. Note that $SO_3(\mathbb{R}) = \{$ positively orthonormal basis of $\mathbb{R}^3\}$ and $\pi_1(SO_3(\mathbb{R})) = \mathbb{Z}_2$. Thus, the section $(\sigma_T, \sigma_E, \sigma_N)$ defines a loop in $SO_3(\mathbb{R})$ (that is the continuous map $\mathbb{S}^1 \to SO_3(\mathbb{R}); u \mapsto (\sigma_T(u), \sigma_E(u), \sigma_N(u))$). In fact, this loop is characterized by the section of the line $\mathbb{R}-$subbundle E of $\mathcal{N}_\mathbb{R}$, σ_E. From now, we can associate a number for the line $\mathbb{R}-$ subbundle E of $\mathcal{N}_\mathbb{R}$, denoted by $s(E)$, be either 0 or 1, depending whether the loop characterized in $\pi_1(SO_3(\mathbb{R}))$ is trivial or non-trivial respectively.

Remark: $s(E)$ depends on the isotopy class of E as a line $\mathbb{R}-$ subbundle of $\mathcal{N}_\mathbb{R}$ and on the homotopy class of the restriction of the trivialization ϕ_0 to $T\mathbb{R}P^3|_{f(\mathbb{R}P^1)}$. At the end of this section, we fix the trivialization ϕ_0, so $s(E)$ only depends on the isotopy class of E. We will emphasize on the line $\mathbb{R}-$subbundles of $\mathcal{N}_\mathbb{R}$ which realize two different isotopy classes then define two different loops in $\pi_1(SO_3(\mathbb{R}))$. That is the case of line $\mathbb{R}-$ subbundles of $\mathcal{N}_\mathbb{R}$ of degree $(2d-2)$ which we are interested in.

We distinguish the holomorphic line $\mathbb{R}-$subbundle of $\mathcal{N}_\mathbb{R}$ of degree $(2d-1)$ or $(2d-2)$ in the consequence of the following lemma.

Lemma 3.1.4. *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^3$ be a real balanced immersion with $\deg f(\mathbb{C}P^1) = d$ (so $\mathcal{N} \simeq \mathcal{O}(2d-1) \oplus \mathcal{O}(2d-1) := H \oplus K$). A holomorphic line subbundle of $\mathcal{N}$ is*

in $1-1$ correspondence with a rational function $F:\mathbb{C}P^1\to\mathbb{C}P^1$: its fiber over u has equation $w=F(u)z$ where u,z,w are complex numbers and $(u,(z,w)),(u,z),(u,w)$ are local coordinates of $\mathcal{N},H,K$ respectively; its degree is $2d-1-degF$.
In particular, a holomorphic line $\mathbb{R}-$subbundle of $\mathcal{N}$ is in $1-1$ correspondence with a real rational function $F_{\mathbb{R}P^1}:\mathbb{R}P^1\to\mathbb{R}P^1$.

Proof. Let M be a holomorphic line subbundle of $\mathcal{N}$. Let u,z,w be complex numbers as in the statement. The slope of a fiber of M over u varies depending on the position of $u\in\mathbb{C}P^1$, so it corresponds to the rational function $F:\mathbb{C}P^1\to\mathbb{C}P^1; u\mapsto F(u)$. Then its fiber over u is $M_u: w=F(u)z$.

To determine the degree of M, we can count the number of zeros and poles of its section σ_M. In fact, when $F(u)=0$ then $w=0$, so the zeros of σ_M is equal to the zeros of the section of $H=\mathcal{O}(2d-1)$, so the number of zeros of σ_M is $(2d-1)$. The poles of σ_M is the points u where $F(u)=+\infty$ (that is when $z=0, w\neq 0$, in other words, when the fiber $M_u\equiv K_u$), so the number of poles of σ_M is $degF$. Therefore, $M\simeq\mathcal{O}(2d-1-degF)$. □

Remark: The Riemannian metric on $\mathbb{R}P^3$ allows us to identify $\mathcal{N}_{\mathbb{R}}$ with $\mathbb{R}\mathcal{N}$.

As the consequence, depending on degree of the real rational map $F_{\mathbb{R}P^1}$ we can determine the holomorphic line $\mathbb{R}-$subbundle of $\mathcal{N}$ associated:

- When $degF_{\mathbb{R}P^1}=0$, i.e. $F_{\mathbb{R}P^1}(u)=constant,\forall u$, then up to real isotopy, there is a unique holomorphic line $\mathbb{R}-$subbundle of $\mathcal{N}$ of degree $(2d-1)$.
- When $degF_{\mathbb{R}P^1}=1$, i.e. $F_{\mathbb{R}P^1}(u)=\frac{Au+B}{Cu+D}, AD-BC\neq 0$, then depending whether the value of $AD-BC$ is positive or negative, i.e. $F_{\mathbb{R}P^1}$ is orientation preserving or not, and up to real isotopy, there are two holomorphic line $\mathbb{R}-$subbundles of $\mathcal{N}$ of degree $(2d-2)$, let's call L and L' respectively. In other words, we distinguish two (real isotopy classes of) holomorphic line $\mathbb{R}-$subbundles of $\mathcal{N}$ of degree $(2d-2)$ depending on whether their real fibers rotate positively or negatively in local holomorphic coordinate of $\mathcal{N}$. We always choose L *belonged to the former case* and L' belonged to the latter case. Since the difference between L and L' is exactly one full rotation , so $s(L)\neq s(L')$.

In conclusion, for a generic configuration $\mathcal{X}$, the sign of a curve $C\in\mathbb{R}C(\mathcal{X})$ is defined to be equal to the number $s(L)$, i.e. $s_{\mathbb{R}P^3}(C):=s(L)\in\{0,1\}$.

Remark: We fix a trivialization ϕ_0 such that for a line $D\subset\mathbb{C}P^3: s_{\mathbb{R}P^3}(D)=0$.

3.2 Relation between two W-invariants $W_{\mathbb{R}P^3}(d,l)$ and $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a,b),l)$

In this section, we always suppose that Q is a real quadric in $\mathbb{C}P^3$ whose real part is homeomorphic to the torus, i.e. $\mathbb{R}Q\simeq\mathbb{R}P^1\times\mathbb{R}P^1$.
We also suppose that D_1,D_2 are real and $f:\mathbb{C}P^1\to\mathbb{C}P^3$ is a real algebraic immersion.

Remark: If $s(\mathbb{R}\mathcal{N}'_1)=s(\mathbb{R}\mathcal{N}_{D_1/Q})=0$, we say (D_1,D_2) form a positive basis.

In the Proposition 3.2.1 and Proposition 3.2.2), we suppose that $f(\mathbb{C}P^1)\subset Q$ with $bideg(f(\mathbb{C}P^1))=(a,b)$ in the positive basis with $a\neq b$ and $a+b=d$. In here, our convention is $s(\mathbb{R}\mathcal{N}'_1) = 0$ and $s(\mathbb{R}\mathcal{N}'_2) = 1$. That means the line $\mathbb{R}-$bundle $\mathbb{R}\mathcal{N}'_1$ defines a trivial loop in $\pi_1(SO_3(\mathbb{R}))$ while $\mathbb{R}\mathcal{N}'_2$ defines a non-trivial one.

Recall: If E is a line $\mathbb{R}-$subbundle of $\mathcal{N}$ then one of its sections σ_E defines a loop in $\pi_1(SO_3(\mathbb{R}))$. We have $s(E) \in \{0,1\}$. Precisely, if the loop in $\pi_1(SO_3(\mathbb{R}))$ defined by E is non-trivial then $s(E) = 1$ and $s(E) = 0$ otherwise.

Thus, the line $\mathbb{R}-$bundle $\mathbb{R}\mathcal{N}' = f^*T\mathbb{R}Q/T\mathbb{R}P^1$ also defines a loop in $\pi_1(SO_3(\mathbb{R}))$, and the line $\mathbb{R}-$bundle $T\mathbb{R}Q|_{\gamma_i}/T\gamma_i$ will define a loop in $\pi_1(SO_3(\mathbb{R}))$ for each $\gamma_i \in \gamma$ defined as in Lemma 3.1.2.

Now, let's see how is the relation between $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1))$, $s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1))$ and $s(\mathbb{R}\mathcal{N}')$ in the two following propositions.

Proposition 3.2.1.

$$s(\mathbb{R}\mathcal{N}') = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + b \mod 2.$$

Proof. Firstly, we fix an orientation on $\mathbb{R}P^1$ and smooth each node of $f(\mathbb{R}P^1)$ similarly as in the proof of Lemma 2.2.1, we obtain a collection γ of n disjoint oriented circles γ_i embedded in $\mathbb{R}P^1 \times \mathbb{R}P^1$. Moreover, γ_i is either of the trivial class or of $(p,q)-$class with $pgcd(p,q) = 1$ in the homology group $H_1(\mathbb{R}P^1 \times \mathbb{R}P^1, \mathbb{Z})$.

Secondly, we consider the loops in $\pi_1(SO_3(\mathbb{R}))$ defined in two ways: one way by the line $\mathbb{R}-$bundle $\mathbb{R}\mathcal{N}'$, we denote the loop associated $\tilde{\gamma}$; the other way by the line $\mathbb{R}-$ bundles $T\mathbb{R}Q|_{\gamma_i}/T\gamma_i$, we denote the loops associated $\tilde{\gamma}_i$. We have a free homotopy (homotopy of free base points) of loops in $\pi_1(SO_3(\mathbb{R}))$:

$$\tilde{\gamma} \sim \prod_{\gamma_i\in\gamma} \tilde{\gamma}_i$$

Then

$$s(\mathbb{R}\mathcal{N}') = \sum_{\gamma_i\in\gamma} s(T\mathbb{R}Q|_{\gamma_i}/T\gamma_i) = \sum_{\gamma_i\in\gamma} s(\mathbb{R}\mathcal{N}_{\gamma_i/\mathbb{R}Q})$$

$$\Rightarrow s(\mathbb{R}\mathcal{N}') = \underbrace{\sum_{\gamma_i\in(0,0)-class} s(\mathbb{R}\mathcal{N}_{\gamma_i/\mathbb{R}Q})}_{(I)} + \underbrace{\sum_{\gamma_j\in(p,q)-class} s(\mathbb{R}\mathcal{N}_{\gamma_j/\mathbb{R}Q})}_{(II)}$$

Consider (I):
We know that for each γ_i of class $(0,0) \in H_1(\mathbb{R}P^1 \times \mathbb{R}P^1; \mathbb{Z})$, (i.e. γ_i and $[\mathbb{R}D_1]$, $[\mathbb{R}D_2]$ have no intersection point counted with sign), $\mathbb{R}\mathcal{N}_{\gamma_i/\mathbb{R}Q}$ defines a non-trivial loop in $\pi_1(SO_3(\mathbb{R}))$, then $s(\mathbb{R}\mathcal{N}_{\gamma_i/\mathbb{R}Q}) = 1, \forall\gamma_i \in (0,0)-$ class. By Lemma 2.2.1, we have proven $\sharp\{$ circles in γ of class $(0,0)\} \equiv s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1))$ mod 2.

So $(I) = \Sigma_{\gamma_i\in(0,0)-class} s(\mathbb{R}\mathcal{N}_{\gamma_i/\mathbb{R}Q}) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1))$ mod 2.

Consider (II):
We have γ_j is of $(p,q)-$ class, i.e. $\gamma_j \sim p[\mathbb{R}D_1] + q[\mathbb{R}D_2]$. So $\mathbb{R}\mathcal{N}_{\gamma_j/\mathbb{R}Q}$ defines p times loop defined by $\mathbb{R}\mathcal{N}'_1$ and q times loop defined by $\mathbb{R}\mathcal{N}'_2$ in $\pi_1(SO_3(\mathbb{R}))$. In other words, for each circle γ_j of class $(p,q) \in H_1(\mathbb{R}P^1 \times \mathbb{R}P^1; \mathbb{Z}), pgcd(p,q) = 1$, $\mathbb{R}\mathcal{N}_{\gamma_j/\mathbb{R}Q}$ defines p times trivial loop and q times non-trivial loop in $\pi_1(SO_3(\mathbb{R}))$.

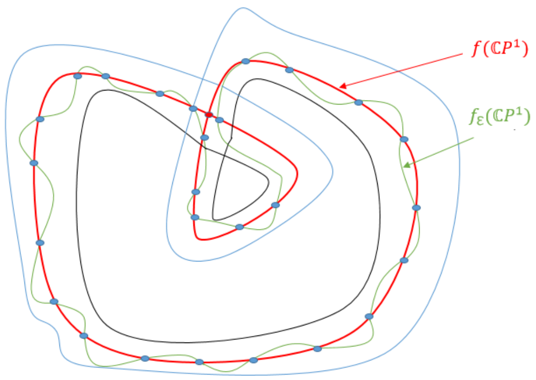

FIGURE 3.2: Intersection points of $f(\mathbb{C}P^1)$ with $f_\epsilon(\mathbb{C}P^1)$

Since $s(\mathbb{R}\mathcal{N'}_1) = 0$ and $s(\mathbb{R}\mathcal{N'}_2) = 1$, we have:

$$s(\mathbb{R}\mathcal{N}_{\gamma_j/\mathbb{R}Q}) = q \times s(\mathbb{R}\mathcal{N'}_2) = q$$

Suppose that there are m circles in γ being of the $(p,q)-$class.

$$\sum_{\gamma_j\in(p,q)-class} s(\mathbb{R}\mathcal{N}_{\gamma_j/\mathbb{R}Q}) = m \times s(\mathbb{R}\mathcal{N}_{\gamma_j/\mathbb{R}Q}) = mq = b \mod 2$$

Note that $(a,b) = (mp, mq) \mod 2$ implies $mq = b \mod 2$.

So $(II) = b$.

In conclusion,

$$s(\mathbb{R}\mathcal{N'}) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + b \mod 2$$

□

If a line $\mathbb{R}-$subbundle of $\mathbb{R}\mathcal{N}$ has degree $(2d-2)$ then it can realize the isotopy class either L or L'. The next proposition confirms that $\mathbb{R}\mathcal{N'}$ is the case and realize the isotopy class L when $a > b$ and L' otherwise.

Proposition 3.2.2. *$\mathbb{R}\mathcal{N'}$ realizes the isotopy class L if and only if $a > b$.*

Proof. We need to prove that $\mathcal{N'}$ is a line subbundle of $\mathcal{N}$ of degree $(2d-2)$ where $d = a+b$. Then we find a suitable way to determine the isotopy class of its real part. Lastly, we show that $\mathbb{R}\mathcal{N'}$ and L have the same isotopy class only when $a > b$.

Step 1: $\mathcal{N'}$ is the line subbundle of $\mathcal{N}$ of degree $2d-2$.
We have $\mathcal{N'} = f^*TQ/T\mathbb{C}P^1$, then it is a line bundle over $\mathbb{C}P^1$, let $\mathcal{N'} = \mathcal{O}(h)$. To determine h, we count the vanishing points of non-zero smooth section of $\mathcal{N'}$ and note that whenever we have a node of $f(\mathbb{C}P^1)$ then the two intersecting points of $f(\mathbb{C}P^1) \cap f_\epsilon(\mathbb{C}P^1)$ around this node are not counted. See Figure 3.2.

Moreover, on a non-singular quadric surface, two curves $f(\mathbb{C}P^1)$ and $f_\epsilon(\mathbb{C}P^1)$ are both of bidegree (a,b) so $\sharp(f(\mathbb{C}P^1) \cap f_\epsilon(\mathbb{C}P^1)) = (a,b)\times(a,b) = ab + ab = 2ab$ and by adjunction formula, $f(\mathbb{C}P^1)$ has exactly $(a-1)(b-1)$ nodes.
Therefore,

$$\begin{aligned} h &= \sharp(f(\mathbb{C}P^1) \cap f_\epsilon(\mathbb{C}P^1)) - 2 \times \sharp\{nodes \in f(\mathbb{C}P^1)\} \\ &= 2ab - 2 \times (a-1)(b-1) \\ &= 2(a+b) - 2 \\ &= 2d - 2. \end{aligned}$$

That means $\mathcal{N}' = \mathcal{O}(2d-2)$.

Step 2: One way to define the isotopy class of $\mathbb{R}\mathcal{N}'$.

Recall: When f is balanced, if $degF_{\mathbb{R}P^1} = 0$, there is a unique isotopy class of holomorphic line $\mathbb{R}-$subbundle of $\mathcal{N}$ of degree $(2d-1)$, let's call H. If $degF_{\mathbb{R}P^1} = 1$, there are two isotopy classes of holomorphic line $\mathbb{R}-$subbundles of $\mathcal{N}$ of degree $(2d-2)$ whose real fibers rotate positively or negatively in local holomorphic coordinate of $\mathcal{N}$, we call them L and L' respectively.

Suppose that we have H such that fibers over u_0: H_{u_0} and $\mathbb{R}\mathcal{N}'_{u_0}$ are coincide, then looking at $u > u_0$, we see that the fibers over u: $H_u = H_{u_0}$ but $\mathbb{R}\mathcal{N}'_u \neq \mathbb{R}\mathcal{N}'_{u_0}$. So $\mathbb{R}\mathcal{N}'$ might be in the isotopy class either L or L'. Therefore, we need to find such u_0 and H.

Let $C_0 \subset Q$ be a real elliptic curve of bidegree $(2,2)$ with $\mathbb{R}C_0 \neq \emptyset$ and C_0 intersects $f(\mathbb{C}P^1)$ transversely at $p_0 = f(u_0), u_0 \in \mathbb{R}P^1$. Let $\mathcal{Q}$ be the real pencil of quadrics induced by C_0. We have $f(\mathbb{C}P^1) \cap C_0 = \{p_0, p_1, \dots, p_{2d-1}\}$. Let f_ϵ be a first order real deformation of f in the pencil $\mathcal{Q}$ such that: for all ϵ, we have $f_\epsilon(\mathbb{C}P^1) \cap C_0 = \{p_\epsilon, p_1, \dots, p_{2d-1}\}$.

This deformation corresponds to a non-null real holomorphic section σ of $\mathcal{N}$, $\sigma : \mathbb{C}P^1 \to \mathcal{N}$ such that $\sigma(f^{-1}(C_0 \backslash \{p_0\})) = 0$ and $\sigma(f^{-1}(p_0)) = \sigma(u_0) \neq 0$ (equivalent to $\sigma(f^{-1}(C_0 \backslash \{p_0\})) \in T\mathbb{C}P^1$ and $\sigma(u_0) \notin T\mathbb{C}P^1$). Let H be the line holomorphic $\mathbb{R}-$subbundle of $\mathcal{N}$ of degree $(2d-1)$ such that $H_{u_0} =< \sigma(u_0) >$. Then we claim that:

- σ is also a section on H, i.e. $\sigma(u) \in H_u, \forall u \in \mathbb{C}P^1$.
  Indeed, σ induces a holomorphic section of the line bundle $\mathcal{N}/H$, $\sigma_{\mathcal{N}/H} : \mathbb{C}P^1 \to \mathcal{N}/H$. We have $deg(\mathcal{N}/H) = 2d-1$, i.e. $\mathcal{N}/H = \mathcal{O}(2d-1)$ but $\sigma_{\mathcal{N}/H}$ vanishes at $2d$ points of $f^{-1}(C_0)$ so it is a null-section.
- The fibers over u_0: $H_{u_0} \equiv \mathcal{N}'_{u_0}$.
  Indeed, $\sigma(u_0)$ corresponds to the pull-back of the deformation of p_0 to p_ϵ and $p_0, p_\epsilon \in C_0 \subset Q$ so $\overrightarrow{p_0p_\epsilon} \in TQ_{u_0}$. Since $p_0 \neq p_\epsilon$ so $\overrightarrow{p_0p_\epsilon} \in TQ/Tf(\mathbb{C}P^1)$, so $\sigma(u_0) \in f^*(TQ/Tf(\mathbb{C}P^1)) = \mathcal{N}'$.
- The direction of $\sigma(u_0)$ determines the isotopy class realized by $\mathbb{R}\mathcal{N}'$.

  Indeed, for $Q \in \mathcal{Q}$, let σ_Q be a holomorphic section of $T\mathbb{C}P^3/TQ$ and let $\mathbb{R}\sigma_Q$ be a fixed smooth non-vanishing section of $T\mathbb{R}P^3/T\mathbb{R}Q$ (we can fix direction of $\mathbb{R}\sigma_Q$ because of the orientation on $\mathbb{R}P^3$ and $\mathbb{R}Q$). Since $C_0 \subset Q$ then $\sigma_Q(C_0) = 0$, then $\mathbb{R}Q \backslash \mathbb{R}C_0$ is divided into two parts depending on the direction of σ_Q. Let $\mathbb{R}Q_+ \subset \mathbb{R}Q \backslash \mathbb{R}C_0$ be the one which σ_Q and $\mathbb{R}\sigma_Q$ have the same direction. This choice (of σ_Q and $\mathbb{R}\sigma_Q$) together with a choice of orientation on $\mathbb{R}P^1$ induce an orientation on $f(\mathbb{R}P^1)$ such that $f(\mathbb{R}P^1)$ points toward $\mathbb{R}Q_+$ at $f(u_0)$.

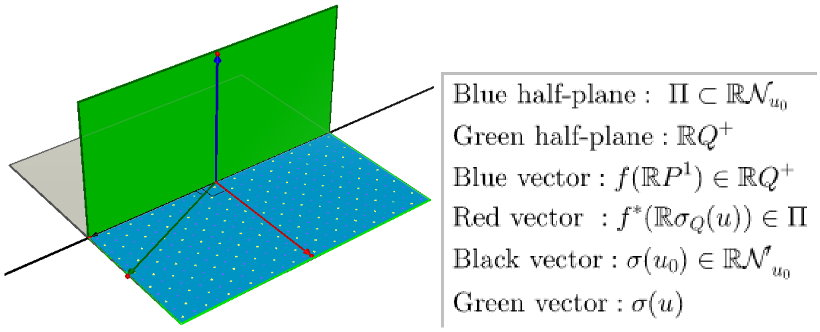

FIGURE 3.3: One way to define the isotopy class of $\mathbb{R}\mathcal{N}'$

Remark: We can identify $f^*(T\mathbb{R}P^3/T\mathbb{R}Q)$ with $\mathbb{R}\mathcal{N}'^{\perp} \subset \mathbb{R}\mathcal{N}$, this implies $\forall u \in \mathbb{R}P^1$, $f^*(\mathbb{R}\sigma_Q(u)) \in \mathbb{R}\mathcal{N}'^{\perp}$. We have a split short exact sequence of real normal bundles over $\mathbb{R}P^1$: $0 \to \mathbb{R}\mathcal{N}' \to \mathbb{R}\mathcal{N} \to f^*_{\mathbb{R}P^1}\mathbb{R}\mathcal{N}_Q \to 0$.

Therefore, for $u \in \mathbb{R}P^1$ which is close enough to u_0, we can decompose vector $\sigma(u) \in \mathbb{R}\mathcal{N}$ depending on vectors $\sigma(u_0) \in \mathbb{R}\mathcal{N}'$ and $f^*(\mathbb{R}\sigma_Q(u)) \in \mathcal{N}'^{\perp}$ as follows:

$$\sigma(u) = g_1(u) \times \sigma(u_0) + g_2(u) \times f^*(\mathbb{R}\sigma_Q(u))$$

Where g_1, g_2 are smooth functions of u such that $g_1(u_0) = 1, g_2(u_0) = 0$ and $g_2(u) > 0, \forall u > u_0$.
That means the choice of σ_Q and $\mathbb{R}\sigma_Q$ also induces an orientation of the fiber $\mathbb{R}\mathcal{N}_{u_0}$ together with a half-plane $\Pi \subset \mathbb{R}\mathcal{N}\backslash\mathbb{R}\mathcal{N}'$ which contains $\sigma(u), \forall u > u_0$. The orientation of this fiber (i.e. the direction of the vector $\sigma(u_0) \in \mathbb{R}\mathcal{N}'$) decides the direction of the rotation from $\mathbb{R}\mathcal{N}'$ to H.

Step 3: Compare the isotopy class of $\mathbb{R}\mathcal{N}'$ and L.

By the two steps above, we have $\mathbb{R}\mathcal{N}'$ is the holomorphic line $\mathbb{R}-$subbundle of $\mathcal{N}$ of degree $(2d-2)$ and its isotopy class is determined by $\sigma(u_0)$. So whether the direction of $\sigma(u_0)$ makes its fibers rotate positively in the half-plane Π, the isotopy class of $\mathbb{R}\mathcal{N}'$ and L are the same.

One the one hand, we have $p_\epsilon \in f_\epsilon(\mathbb{C}P^1) \cap C_0, p_\epsilon = p_0 + \epsilon\bar{p}_0 \neq p_0$, then $p'_\epsilon(0) = \bar{p}_0 \neq 0$ is the direction of deformation from p_0 to p_ϵ. The direction of the vector $\sigma(u_0)$ in fact corresponds to the direction of the vector $p'_\epsilon(0)$.

On the other hand, we have

$$f_\epsilon(\mathbb{C}P^1) \sim aD_{1,\epsilon} + bD_{2,\epsilon} = (a-b)D_{1,\epsilon} + bH$$

$$\Rightarrow f_\epsilon(\mathbb{C}P^1) \cap C_0 \sim (a-b)(D_{1,\epsilon} \cap C_0) + b(H \cap C_0) = (a-b)E_{1,\epsilon} + bh \in Pic_{2d}(C_0).$$

So $p_\epsilon \in f_\epsilon(\mathbb{C}P^1) \cap C_0 \sim (a-b)E_{1,\epsilon} + bh$. If $a > b$ then the direction of the vector $p'_\epsilon(0)$ is the same as the direction of the line D_1. If $a < b$ then the direction of the vector $p'_\epsilon(0)$ is opposite the direction of the line D_1.

Therefore, the direction of $\sigma(u_0)$ is the same as of the line D_1 iff $a > b$.

We only need to check for the case $d = 1$, i.e. given a real line D in $\mathbb{C}P^3$, if $f(\mathbb{C}P^1) = D$ then $\mathbb{R}\mathcal{N}'$ realizes the isotopy class L iff $(a,b) = (1,0)$ (or $D \sim D_1$).

Recall that we are working on the positive basis (D_1, D_2), i.e. $s(\mathbb{R}\mathcal{N}'_1) = 0$ and $s(\mathbb{R}\mathcal{N}'_2) = 1$.

Indeed, by definition, $s_{\mathbb{R}P^3}(D) = s(L(D))$ where $L(D)$ is the (real isotopy classes of) holomorphic line $\mathbb{R}-$subbundles of $\mathcal{N}$ of degree $(2d-2) = 0$ whose real fibers rotate positively in local holomorphic coordinate of $\mathcal{N}$. By convention in the last section, $s_{\mathbb{R}P^3}(D) = 0$ so $s(L(D)) = 0$.

- If $D \sim D_1$, i.e. D has bidegree $(a,b) = (1,0)$, then $s_{\mathbb{R}P^3}(D_1) = s_{\mathbb{R}P^3}(D) = 0$, this implies $s(L(D_1)) = 0$. Moreover, $s(\mathbb{R}\mathcal{N}'_1) = 0$ so $s(\mathbb{R}\mathcal{N}'_1) = s(L(D_1))$, i.e. $\mathbb{R}\mathcal{N}'_1$ realizes the isotopy class $L(D_1)$.
- If $D \sim D_2$, i.e. D has bidegree $(a,b) = (0,1)$, then $s_{\mathbb{R}P^3}(D_2) = s_{\mathbb{R}P^3}(D) = 0$, this implies $s(L(D_2)) = 0$. But $1 = s(\mathbb{R}\mathcal{N}'_2) \neq s(L(D_2)) = 0$, i.e. $\mathbb{R}\mathcal{N}'_2$ does not realize the isotopy class $L(D_2)$.

In conclusion, $\mathbb{R}\mathcal{N}'$ realizes the isopoty class L iff $a > b$. ☐

As the consequence of the proposition 3.2.2, if $a < b$ then $\mathbb{R}\mathcal{N}'$ realizes the isotopy class L', i.e. $s(\mathbb{R}\mathcal{N}') = s(L') = s(L) - 1 \mod 2$.

As the consequence of Proposition 3.2.1 and Proposition 3.2.2, we have found the relation between $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1))$ and $s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1))$:

- If $a > b$ then $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s(L) = s(\mathbb{R}\mathcal{N}') = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + b \mod 2$.
- If $a < b$ then $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s(L) = s(\mathbb{R}\mathcal{N}') + 1 = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + b + 1 \mod 2$.

According to Kollár's theorem, there exists $0 \leq a < \frac{d}{2}$ such that $f(\mathbb{C}P^1)$ has bidegree $(a, d-a)$ or $(d-a, a)$ in the positive basis (D_1, D_2) of Q. Moreover, We are under the condition $a + b = d$, d is odd, so a and $(d-a)$ have different parity. Therefore, we have:

- If a is even then b is odd, so $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) \mod 2$.
- If a is odd then b is even, so $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + 1 \mod 2$.

In other words, $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + a \mod 2$, for all $0 \leq a < \frac{d}{2}$.

Theorem 3.2.3. *Let d be an odd positive integer and $0 \leq l < d$, then:*

$$W_{\mathbb{R}P^3}(d,l) = \sum_{0\leq a<\frac{d}{2}} (-1)^a (d-2a) W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a, d-a), l)$$

Remark: G.Milkhalkin proved that when d is even and $0 \leq l < d$ then one has $W_{\mathbb{R}P^3}(d,l) = 0$. One has also calculated $W_{\mathbb{R}P^3}(d,d)$ in a non-trivial method.

Proof. Recall: We have proven in Theorem 2.3.1 that: $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(0,1) = 1$ and $GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(0,a) = 0, \forall a > 1$. That implies $W_{\mathbb{R}P^3}(1,0) = W_{\mathbb{R}P^1\times\mathbb{R}P^1}((0,1),0) = 1$ and $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((0,a),l) = 0, \forall a > 1, \forall 0 \leq l < a$.

We now consider the case $d>1$ odd.
First, using Kollár's theorem and the propriety of the torsion points in chapter 1, we have exactly $(d-2a)$ real solutions in $Pic_2(\mathbb{R}C_0)$ of the equation:

$$(d-2a)E=(d-a)h-\mathcal{X} \quad (*)$$

(Indeed, $(d-2a)$ is odd so there are only $(d-2a)$ of real $(d-2a)-$torsion points E_i in $\mathbb{R}C_0$ for both cases of $\mathbb{R}C_0$ and the solutions of the equation $(*)$ is of the form $E+E_i$, that means there are $(d-2a)$ real quadrics $\mathbb{R}Q$ associated in $\mathcal{Q}$.)

Remark: we need to choose $\mathcal{X}$ as a real configuration containing at least one real point. Otherwise, if $\mathcal{X}$ contains all complex conjugate point pairs then we can not choose the real configuration $\mathcal{Y}\subset\mathcal{X}$ of $(2d-1)$ points so that we can connect two Welschinger invariants.

By the same argument as in Chapter 2, we can choose $\mathcal{X}$ as a real configuration of $2d$ distinct points (with at least one real point) in the elliptic curve C_0 and choose $\mathcal{Y}$ as a real configuration of $(2d-1)$ distinct points in C_0, $\mathcal{Y}\subset\mathcal{X}$ so that the number of rational curves of degree d counted with sign passing through such $\mathcal{X}$ is exactly the Welschinger invariant of $\mathbb{C}P^3$ and the number of rational curves of bidegree $(a,d-a)$ counted with sign passing through such $\mathcal{Y}$ is exactly the Welschinger invariant of $\mathbb{C}P^1\times\mathbb{C}P^1$.

Recall that:
The Welschinger invariant of $\mathbb{C}P^1\times\mathbb{C}P^1$ is $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a,b),l)=\sum_{C\in\mathbb{R}C(\mathcal{Y})}(-1)^{s_{\mathbb{R}P^1\times\mathbb{R}P^1}(C)}$,
The Welschinger invariant of $\mathbb{C}P^3$ is $W_{\mathbb{R}P^3}(d,l)=\sum_{C\in\mathbb{R}C(\mathcal{X})}(-1)^{s_{\mathbb{R}P^3}(C)}$.
Thus, for all $d>1$ odd:

$$\begin{aligned}
W_{\mathbb{R}P^3}(d,l) &= \sum_{C\in\mathbb{R}C(\mathcal{X})}(-1)^{s_{\mathbb{R}P^3}(C)}\\
&= \sum_{\substack{f:\mathbb{C}P^1\to\mathbb{C}P^3 real,balanced,immersion\\ \mathcal{X}\subset f(\mathbb{C}P^1)}}(-1)^{s_{\mathbb{R}P^3}(f(\mathbb{C}P^1))}\\
&= \begin{cases}
(d-2)\times\sum_{\substack{f:\mathbb{C}P^1\to Q\ real,immersion\\ \mathcal{Y}\subset f(\mathbb{C}P^1)}}-(-1)^{s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1))} & if \quad a=1,\\
\ldots & \\
1\times\sum_{\substack{f:\mathbb{C}P^1\to Q\ real,immersion\\ \mathcal{Y}\subset f(\mathbb{C}P^1)}}(-1)^{\frac{d-1}{2}}(-1)^{s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1))} & if \quad a=\frac{d-1}{2}.
\end{cases}\\
&= \sum_{0<a<\frac{d}{2}}(-1)^a(d-2a)\sum_{\substack{f:\mathbb{C}P^1\to Q\ real,immersion\\ \mathcal{Y}\subset f(\mathbb{C}P^1)}}(-1)^{s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1))}\\
&= \sum_{0<a<\frac{d}{2}}(-1)^a(d-2a)W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a,d-a),l)
\end{aligned}$$

□

Remark: We can prove $W_{\mathbb{R}P^3}(d,l)=0$ for all d even and for all $0\le l<d$ by this method.

Indeed, suppose that $\mathbb{R}C_0$ has one connected component, then the equation $(*)$ still has exactly $(d-2a)$ real solutions.

According to Kollár's theorem, there exists $0 \leq a < \frac{d}{2}$ such that $f(\mathbb{C}P^1)$ has bidegree $(a, d-a)$ or $(d-a, a)$ in the positive basis (D_1, D_2) of Q. We have proven that: if $f(\mathbb{C}P^1)$ has bidegree (a, b) in the positive basis then, one has

$$s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = \begin{cases} s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + b + 1, & \text{if } a < b \\ s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + b, & \text{if } a > b \end{cases}$$

Since d is even then a and $(d-a)$ have the same parity. In the first case, if a is even then $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + 1 \mod 2$; otherwise $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) \mod 2$. In the second case, if a is even then $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) \mod 2$; otherwise $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + 1 \mod 2$.

In other words,

- If $\mathrm{bideg} f(\mathbb{C}P^1) = (a, d-a)$ then $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + a + 1 \mod 2$.
- If $\mathrm{bideg} f(\mathbb{C}P^1) = (d-a, a)$ then $s_{\mathbb{R}P^3}(f(\mathbb{C}P^1)) = s_{\mathbb{R}P^1\times\mathbb{R}P^1}(f(\mathbb{C}P^1)) + a \mod 2$.

Then, for all d even and $0 \leq l < d$:

$$W_{\mathbb{R}P^3}(d, l) = \sum_{0 \leq a < \frac{d}{2}} (d-2a)((-1)^{a+1} W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a, d-a), l) + (-1)^a W_{\mathbb{R}P^1\times\mathbb{R}P^1}((d-a, a), l))$$

Since $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a, d-a), l) = W_{\mathbb{R}P^1\times\mathbb{R}P^1}((d-a, a), l)$ then $W_{\mathbb{R}P^3}(d, l) = 0$ for all d even and $0 \leq l < d$.

Example 4. *Compute the Welschinger invariants in the following cases:*

- $d = 1$. *Since* $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((0,1), 0) = 1$, *then* $W_{\mathbb{R}P^3}(1, 0) = 1$.
- $d = 2$. *Since* $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((0,2), l) = 0$, *then* $W_{\mathbb{R}P^3}(2, l) = 0$.
- $d = 3$. *Since* $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((0,3), l) = 0$, *then* $W_{\mathbb{R}P^3}(3, l) = -W_{\mathbb{R}P^1\times\mathbb{R}P^1}((1,2), l)$ *for all* $l \in \{0, 1, 2\}$. *By the same method as shown in Example 3, we have:* $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((1,2), l) = 1$, *then* $W_{\mathbb{R}P^3}(3, l) = -1$.
- $d = 4$. *Since* $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((0,4), l) = 0$ *and* d *is even then:* $W_{\mathbb{R}P^3}(4, l) = 2 \times (-1)^2 \times W_{\mathbb{R}P^1\times\mathbb{R}P^1}((1,3), l) + 2 \times (-1)^3 \times W_{\mathbb{R}P^1\times\mathbb{R}P^1}((3,1), l)$ *for all* $l \in \{0, 1, 2, 3\}$.
  We have: $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((1,3), l) = W_{\mathbb{R}P^1\times\mathbb{R}P^1}((3,1), l) = 1$, *then* $W_{\mathbb{R}P^3}(3, l) = 0$.
- $d = 5$. *Similarly,* $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((0,5), l) = 0$ *and* $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((1,4), l) = 1$. *Thus,* $W_{\mathbb{R}P^3}(5, l) = -3 + W_{\mathbb{R}P^1\times\mathbb{R}P^1}((2,3), l)$ *for all* $l \in \{0, \ldots, 4\}$.
  In this case, computing $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((2,3), l)$ *needs more argument than in this rapport.*

Conclusion

In this rapport, we have constructed the relation between Gromov-Witten-Welschinger invariants of $\mathbb{C}P^3$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ by the particular method. As the consequence, we can turn the enumerative problem of counting rational curves (or real rational curves with sign) of degree d passing by certain number of points in the $3-$ dimensional projective space $\mathbb{C}P^3$ into an easier enumerative problem, it is counting rational curves (or real rational curves with sign) of bidegree (a,b) passing by certain number of points in the $2-$ dimensional projective space $\mathbb{C}P^1 \times \mathbb{C}P^1$. We give some examples about computation in some simple cases.

The questions might be asked as: Can we using these methods to solve other problems in enumerative geometry which are more complicated? Or are there any other methods to count more effectively? The relationship between Gromov-Witten invariant and Welschinger invariant might be interesting? For example of some other enumerative problems: counting curves with higher genus; counting rational curves in higher projective space; counting surfaces with some fixed conditions...

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