A memoir on the h-principle

Benoît SENARD

Supervisor: Yann ROLLIN

Université de Nantes

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Abstract

This memoir aims to review the h-principle and convex integration, as tools to demonstrate the Hirsch theorem (and eventually Smale's sphere eversion theorem). The most known application of these concepts is Smale's sphere eversion. The use of compact-open C^r-topology is signalled by the abbreviation Com-Op.

To fulfill this objective, we will first define jets of order r (as vectors, and then as equivalence classes) and relations of order r. Secondly, we will deal with ample, open relations by using the theory of convex integration, particularly the immersion relation $\mathcal{I}$ defined from the subsection 1.3.

These tools will be used to demonstrate the Hirsch theorem taking the immersion curves $\gamma : [0,1] \to \mathbb{R}^2$ as an easily understandable example, and then we will address Smale's sphere eversion theorem which states that a homotopy of immersions allows to "turn" the unit sphere inside out without tearing nor cutting it, and without any crease.

Contents

1 Jets and relations

Let us denote $\mathcal{H}_r(n,q)$ the real vector space of homogeneous polynomials of total degree r, from $\mathbb{R}^n$ to $\mathbb{R}^q$. Such polynomials are written in the form

$$p(x) = \sum_{|\alpha|=r} c_\alpha \, x_1^{\alpha_1} \cdots x_n^{\alpha_n} \ , \quad c_\alpha \in \mathbb{R}^q$$

Let $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^q$ be open and $f \in C^r(U,W)$. If $v = (v_1, \cdots, v_n)$ is a basis of $\mathbb{R}^n$ with any point $x \in \mathbb{R}^n$ being written as the coordinates $(u_1, \cdots, u_n)$ in the basis v, then we can denote $\partial_j = \partial_{u_j}$ and $\partial_v^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Furthermore, for all $(x,h) \in U \times \mathbb{R}^n$ (with $h = (h_1, \cdots, h_n)$ in the basis v):

$$\begin{aligned} D^r f(x)(h) = D^r f(x)(h, \cdots, h) &= (h_1\partial_1 + \cdots + h_n\partial_n)^r f(x) \\ &= \sum_{|\alpha|=r} \frac{r!}{\alpha!} h^\alpha \, \partial_v^\alpha f(x) \end{aligned}$$

thus $D^r f \in C^0(U, \mathcal{H}_r(n,q))$.

From now, we adopt the following convention : any polynomial $p \in \mathcal{H}_r(n,q)$, written as $p(h) = \sum_{|\alpha|=r} c_\alpha \, h^\alpha$, is endowed with coordinates $(\alpha! \, c_\alpha)_{|\alpha|=r} \in \mathbb{R}^{q\,d_r}$ where $d_r =: \dim \mathcal{H}_r(n,q) = \binom{n+r-1}{r}$. In such weighed coordinates, we will have

$$\frac{1}{r!} D^r f(x) = (\partial_v^\alpha f(x))_{|\alpha|=r} = \left(\frac{\partial^r f}{\partial u_1^{\alpha_1} \cdots \partial u_n^{\alpha_n}}(x) \right)_{\alpha \text{ ordered}}$$

where the subscript «α ordered» means that the n-tuples α of length r are ordered lexicographically.
We define the space $J^r(U,W) = U \times W \times \prod_{s=1}^r \mathcal{H}_s(n,q)$, $r \geq 1$ (with $J^0(U,W) = U \times W$), whose expression as a Cartesian product allows for a natural projection $p_s^r : J^r(U,W) \to J^s(U,W)$ for any $0 \leq s \leq r-1$.

We can prove that $J^r(U,W)$ is identified to the space of r-jets (of germs) of maps within $C^r(U,W)$.

Remark. Two functions f and g are said to define the same germ at a point $x \in U$ if there is a neighborhood N around x such that $f_{|N} = g_{|N}$ (such equality can be extended to derivatives up to order r).

We introduce the continuous operator $j^r : C^r(U,W) \to C^0(U, J^r(U,W))$ which maps f to its r-jet, defined by

$$j^r f(x) = \left(x, \ f(x), \ \frac{1}{2} D^2 f(x), \ \cdots, \ \frac{1}{r!} D^r f(x) \right) \tag{1}$$

There are also projection maps s, τ from $J^r(U,W)$ defined by $\begin{cases} s \circ j^r f(x) = x \\ \tau \circ j^r f(x) = f(x) \end{cases}$
The source map s and the target map τ are natural projections just like

$$\begin{aligned} p_s^r : J^r(U,W) &\to J^s(U,W) \\ j^r f &\mapsto j^s f \end{aligned}$$

$(0 \leq s \leq r-1)$ with $s = p_0^r$ and $j^1 = (s \circ j^r, \tau \circ j^r)$.

Let $\sim_r$ be the following equivalence relation on $U \times C^r(U,W)$: $(x,f) \sim_r (y,g) \Leftrightarrow$ $\begin{cases} x = y \\ j^r f(x) = j^r g(x) \end{cases}$

If we denote $[f]_r(x)$ the equivalence classes, then the map $[f]_r(x) \mapsto j^r f(x)$ is well-defined and injective, and thus defines a bijection between the classes and $J^r(U,W)$.

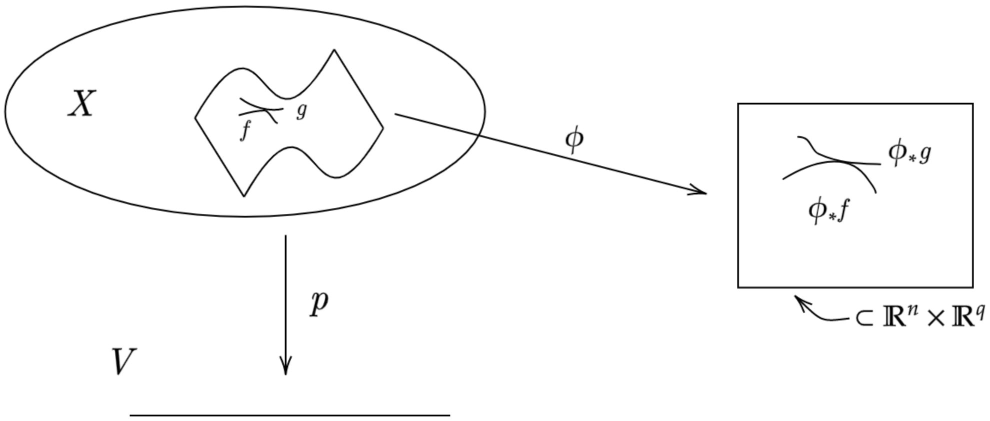

Figure 1: The bundle $p : X \to V$, with a local trivialization ϕ on a neighborhood of $x \in V$.

1.1 The manifold $X^{(r)}$

Now, to generalize the definition of $J^r(U,W)$ to any smooth fiber bundle $r : X \to V$, the above equivalence relation can be seen as a tangency relation between function up to order r, i.e., all derivatives of f and g up to order r coincide in an open neighborhood of x (denoted $\mathfrak{Op}\ x$) ; we also denote $X_{\mathfrak{Op}\ x} \subset X$ the subset defined by $X_{\mathfrak{Op}\ x} = p^{-1}(\mathfrak{Op}\ x)$.

Definition 1.1. Let $p : X \to V$ be a smooth fiber bundle with fiber F, $\dim V = n$, $\dim F = q$. We call $\Gamma^r(X)$ the space of C^r-sections of p (in Com-Op). For any point $x \in V$, sections $f, g \in \Gamma^r(X_{\mathfrak{Op}\ x})$ have the same r-jet at x if $j^r f = j^r g$ on $\mathfrak{Op}\ x$, <u>in local coordinates</u>.

With respect to this equivalence relation, $X^{(r)}$ is defined as the set of all equivalence classes $[f]_r(x)$, $x \in V$. With the local trivialization ϕ below, we can write $X^{(r)} = \phi^{-1}(J^r(U,W))$.

Remark. Locally p is a product bundle $p_U : X_U \to U$ where $X_U = U \times F$, where $U \equiv \mathbb{R}^n$ is a chart on V and same for $W \equiv \mathbb{R}^q$; thus we can identify $\Gamma^r(X_U) = C^r(U, F)$. Then, for any $f \in \Gamma^r(X)$ such that $f(U) \subset W$, $f_{|U} \in \Gamma^r(X_U)$ and $[f]_r(x)$ is locally written as $j^r(f_{|U})(x)$ in the coordinates (1).

Definition 1.2. The above projections s, τ can be regarded as $s : X^{(r)} \to V$, $\tau : X^{(r)} \to X$ with $s \circ [f]_r(x) = x$, $\tau \circ [f]_r(x) = f(x)$.

We can also take, for $0 \leq s \leq r$, $p^r_s : X^{(r)} \to X^{(s)}$ defined by $p^r_s \circ [f]_r = [f]_s$.

The transition maps $g_{AB} : A \cap B \to \mathrm{Diff}(F)$ of the bundle p (with overlapping charts A, B within the base manifold V), verify, for $f_{|A} \in C^r(A, W_1)$, $f_{|B} \in C^r(B, W_2)$, $x \in A \cap B$:

$$f_{|B}(x) = g(x) \circ f_{|A}(x), \quad g(x) \in \mathrm{Diff}(W_1, W_2)$$

Using equation (1) on $j^r f_{|A}$, $j^r f_{|B}$ and identifying $g \equiv \mathrm{ev}\ g : (A \cap B) \times W_1 \to W_2$,

$$j^r f_{|B} = j^r g \circ (\mathrm{id}, f_{|A})(x) \tag{2}$$

This calculation induce a change of coordinates for overlapping charts $J^r(A, W_1)$, $J^r(B, W_2)$ in a fashion illustrated by the commutating diagram below (where $\phi^{-1}(U) \subset V$ and writing)

$$\begin{tikzcd}
\phi^{-1}(U) \arrow[r, "{[f]_r}"] \arrow[d, "\phi"'] & {X_U}^{(r)} \arrow[d, equal, "\text{local coordinates}"] \\
U \arrow[r, "j^r(\phi_* f)"'] & J^r(U, W)
\end{tikzcd}$$

Thus $X^{(r)}$ is the smooth manifold of r-jets of germs of C^r-sections of the bundle $p : X \to V$, topologized in this manner : the charts of $X^{(r)}$ are defined on $J^r(U, W)$ and, <u>in these chart coordinates</u>, $[f]_r(x) \in X^{(r)}$ is expressed as $j^r(\phi_* f)_{|U}(\phi(x))$.

Remark. Up to a chart of $X^{(r)}$, we can make the local interpretation $\phi \circ [f]_r(x) = j^r(\phi_* f)_{|U}(\phi(x))$.

Definition 1.3. $\Gamma(X^{(r)})$ is the space of continuous sections (in compact-open topology) of the bundle $s : X^{(r)} \to V$.

Let $U' = (U, k)$ be a chart on $\phi^{-1}(U)$ and set $W \supset f(U')$. If $\alpha \in \Gamma(X^{(r)})$ and $f = p^r_0 \circ \alpha \in \Gamma(X)$, then, interpreting (in local coordinates) the source map s as the projection $s : J^r(U,W) \to U$, for all $x \in U'$, there exist, for multi-indices β ($1 \le |\beta| \le r$), continuous functions $\psi_\beta \in C^0(U', \mathbb{R}^q)$ such that

$$\alpha(x) = \left(x, f(x), \left(\frac{1}{r!} \psi_\beta(x) \right)_{|\beta| \le r} \right)$$

For any $f \in \Gamma^r(X)$, the space $\Gamma(X^{(r)})$ has a continuous map $j^r f : x \mapsto [f]_r(x)$ such that, locally, $j^r f_{|U'}(x) \in \Gamma(X^{(r)}_{U'})$ has coordinates (1).

Remark. The projection $p^r_s : X^{(r)} \to X^{(s)}$ satisies $p^r_s \circ j^r f = j^s f \in \Gamma(X^{(s)})$.

Example 1.1. The most trivial example is with the trivial fibration $p : V \times Z \to V$ which is a projection onto the first factor. Generally, any smooth fiber bundle can be handled as a local product bundle, using a neighborhood of $x \in V$. But here, the projection p induces, not just a *local*, but a *global* product bundle, espectially if we have $\begin{cases} V = k(U) \\ Z = l(W) \end{cases}$ for k (respectively l) an invertible map on $\mathbb{R}^n$ (resp. $\mathbb{R}^q$). Thus, representing $[f]_r(x) \in X^{(r)}$ in local coordinates seems more obvious than with non-trivial bundles.

1.2 The manifold $X^\perp$ ("between" X^{r-1} and X^r)

The projection map $p^r_{r-1} : X^{(r)} \to X^{(r-1)}$ can be seen as an affine bundle, fiber $\mathcal{H}_r(n,q)$, for any $r \ge 1$. Let $h \in \Gamma^r(X)$, $x_0 \in U'$; employing local coordinates in a chart $J^r(U', \mathbb{R}^q)$ on $X^{(r)}$, the fiber $(p^r_{r-1})^{-1}(\{j^{r-1}h(x_0)\})$ contains exactly all the r-jet extensions $j^r(h + p(x - x_0))(x)$, $p \in \mathcal{H}_r(n,q)$. because, for all $|\beta| \le r-1$, $\frac{\partial^{|\beta|} p(\bullet - x_0)}{\partial^\beta_v}(x) = \sum_{|\alpha|=r} c_\alpha \frac{\alpha!}{(\alpha-\beta)!}(x-x_0)^{\alpha-\beta}$ vanishes at x_0. Applying equation (2) to $f = g_{AB} \circ (h + p(\bullet - x_0))$, since p is homogeneous,

$$j^{r-1} f(x_0) = j^{r-1}(g_{AB} \circ h)(x_0)$$

this change of coordinates induces an affine transformation on $\mathcal{H}_r(n,q)$ over the base point $j^{r-1}h(x_0)$.

We keep the smooth fiber bundle $p : X \to V$, fiber F. Let $\tau \subset TV$ be a hyperplane field on V ($\dim \tau = n-1$), to which is associated a manifold $X^\perp$ and an affine $\mathbb{R}^q$-bundle $p^r_\perp : X^{(r)} \to X^\perp$. This bundle is defined to turn p^r_{r-1} into a product, via the diagram

$$\begin{tikzcd} & X^\perp \arrow[dr, "p^\perp_{r-1}"] & \\ X^{(r)} \arrow[ur, "p^r_\perp"] \arrow[rr, "p^r_{r-1}"] & & X^{(r-1)} \end{tikzcd}$$

where $p^r_{r-1} = p^\perp_{r-1} \circ p^r_\perp$. For any $f \in \Gamma^r(X)$, the $(r-1)$-jet extension $j^{r-1}f \in \Gamma(X^{(r-1)})$ is here given by the C^1 map $j^{r-1}f$.

Definition 1.4 ($\perp$-jet relation)**.** The manifold $X^\perp$ is constructed with the following equivalence relation on sections $f, g \in \Gamma^r(X)$ at $x \in V$. For an affine hyperplane $\tau_x \subset T_x\, V$, f, g have the same $\perp$-jet at x if :

(i) $j^{r-1}f(x) = j^{r-1}g(x)$

(ii) $D(j^{r-1}f) = D(j^{r-1}g) \;:\; \tau_{j^{r-1}g(x)} \to T_y\, X^{(r-1)}$

The equivalence class of f that relation is $[f]_\perp(x)$, and the projections above are verify

$$p^r_\perp \;:\; [f]_r(x) \mapsto [f]_\perp(x) \;;\; p^\perp_{r-1} \;:\; [f]_\perp(x) \mapsto [f]_{r-1}(x)$$

Lemma 1.1. *With respect to $v = (v_1, \cdots, v_n)$ any basis for $\mathbb{R}^n$, there is a splitting*

$$\mathcal{H}_r(n,q) = \mathcal{H}^\perp_r(n,q) \;\times\; L^q$$

Proof. A polynomial of the form $p(h) = \sum_{|\alpha|=r} h^\alpha\, c_\alpha$ can be written as $p = p^\perp + p_n$ with $p_n(h) = p(0, \cdots, 0, h_n) = h^r_n\; c_{(0,\cdots,0,r)}$. Then, it is enough conclusive to define L^q as the space of such polynomials p_n, with the local identification $L^q \equiv \mathbb{R}^q$. □

Corollary 1.1.1. Using weighted coordinates on $\mathcal{H}_r(n,q)$, $p_n(h) \equiv r!\, c \in \mathbb{R}^q$. For any $f \in C^r(U, W)$ where $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^q$ are open, for $(x, h) \in U \times \mathbb{R}^n$,

$$\frac{1}{r!} D^r f(x)(h) = \left(D^\perp f(x)(h),\; \frac{h^r_n}{r!} \partial^r_n f(x) \right)$$

Explicitly, $D^\perp f(x)(h)$ has weighted coordinates $(\partial^\alpha_v f(x))_{\partial^\alpha_v \neq \partial^r_n}$ (the pure derivative ∂^r_n is ignored).

Definition 1.5. The $\perp$-jet of $f \in C^r(U, W)$ at x is

$$j^\perp f(x) = (j^{r-1}f(x),\; D^\perp f(x))$$

and the space of all $\perp$-jets is $J^\perp(U, W) := J^{r-1}(U, W) \times \mathcal{H}^\perp_r(n,q)$.

Definition 1.6. We denote $\sim_\perp$ the $\perp$-jet equivalence relation on $U \times C^r(U, W)$. f, g have the same $\perp$-jet extension at x, i.e. $(x, f) \sim_\perp (y, g)$ if and only if $\begin{cases} x = y \\ j^\perp f(x) = j^\perp g(x) \end{cases}$

Remark. The map $[f]_\perp(x) \mapsto j^\perp f(x)$ induces locally a well-defined bijection. In fact, we could also be able to write $X^\perp = \phi^{-1}(J^\perp(U,W))$.

Returning to $p : X \to V$, we set $x \in V$ and $v = (v_1, \cdots, v_n)$ a basis of $T_x\ V$ adapted to τ, with all $v_i \in \tau_x$. For any $f \in \Gamma^r(X)$, in local coordinates in $V \times X$ at $(x, f(x))$, we assume $\phi_* f \in C^r(U,W)$. We will use directional derivatives (operators) $\partial_{v_i} = \partial/\partial_{u_i}$ in local coordinates $(u_1, \cdots, u_n)$ in the basis v for $\mathbb{R}^n$, since in general τ is not well covered by global smooth coordinates on some neighborhoods $\mathfrak{Op}\ x$. With such basis v, around $x \in \phi^{-1}(U)$, we can find again

$$j^r(\phi_* f)(\phi(x)) = \left(j^{r-1}(\phi_* f)(\phi(x)),\ \frac{1}{r!} D^r(\phi_* f)(\phi(x)) \right)$$

and

$$\begin{aligned} j^\perp(\phi_* f)(\phi(x)) &= (j^{r-1}(\phi_* f)(\phi(x)),\ D^\perp(\phi_* f)(\phi(x))) \\ &= (j^{r-1}(\phi_* f)(\phi(x)),\ \partial_v^\alpha(\phi_* f)(\phi(x)))_{\partial_v^\alpha \neq \partial_n^r} \end{aligned}$$

If we extend the relation $\sim_\perp$ to functions mapping to $X^\perp$, we can say that the equivalence $[f]_\perp(x) = [g]_\perp(y) \Leftrightarrow (\phi(x), \phi_* f) \sim_\perp (\phi(y), \phi_* g)$ holds in local coordinates of $\phi^{-1}(U)$.

Lemma 1.2. $X^\perp$ is the smooth manifold of $\perp$-jets of germs of C^r-sections of the bundle $p : X \to V$, such that, with respect to local coordinates, the below diagram commutes

$$\begin{array}{ccc} \phi^{-1}(U) & \xrightarrow{[f]_\perp} & {X_U}^\perp \\ {\scriptstyle \phi}\downarrow & & \Big\| {\scriptstyle \text{local coordinates}} \\ U & \xrightarrow[j^\perp(\phi_* f)]{} & J^\perp(U,W) \end{array}$$

In particular, the change of local coordinates (2) can be extended to $\perp$-jets.

Considering the hyperplane τ_x, in local coordinates, identifying $L^q \equiv \mathbb{R}^q$ from Lemma 1.1, we have $j^r f(x) = (j^\perp f(x), \partial_{u_n}^r f(x)) \in X^\perp \times \mathbb{R}^q$, therefore $X^{(r)}$ is a split manifold on r-jet spaces : $X^{(r)} = X^\perp \times \mathbb{R}^q$.

1.3 On relations

When any subset $\mathcal{R} \subset C^r(U, W)$ is characterized by the condition imposed on the partial derivatives of any function $f(x) \in \mathcal{R}$, we say $\mathcal{R}$ be a *partial differential relation* on $C^r(U, W)$; a solution of $\mathcal{R}$ is any function which satisfies the condition/relation with respect to its partial derivatives up to order r.

For example, using directional derivative $\partial/\partial\theta$ with $\theta = (\cos x, \sin x) \in S^1$, we can define the immersion relation $\mathcal{I} \subset C^1(S^1, \mathbb{R}^2)$ as the set of all functions $f : S^1 \to \mathbb{R}^2$ such that $\frac{\partial f}{\partial \theta} \neq 0$.

More generally, for open $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^q$, $\mathcal{I} \subset C^r(U, W)$ is the space of functions $f : U \to W$ such that Df is an injective linear application, i.e. that the partial derivatives $\frac{\partial f}{\partial u_1}(x), \cdots, \frac{\partial f}{\partial u_n}(x)$ are linearly independent for all $x \in U$. This is equivalent to saying that $Df(x)$ has maximal rank n for every x.

But, to be able to use the h-principle properly, we will have to extend the concept of *relation* to smooth bundles, typically $p : X \to V$ and setting $\mathcal{R} \subset X^{(r)}$.

Definition 1.7. A relation $\mathcal{R}$ over $X^{(r)}$ is a continuous map $\rho : \mathcal{R} \to X^{(r)}$, where $X^{(r)}$ is defined in Section 1.1.
$\Gamma(\mathcal{R})$ is the space of continuous sections of the map $s \circ \rho : \mathcal{R} \to V$ in Com-Op, implying $\alpha \in \Gamma(\mathcal{R}) \Rightarrow \rho \circ \alpha \in \Gamma(X^{(r)})$.

Definition 1.8. A section $\alpha \in \Gamma(\mathcal{R})$ is said to be *holonomic* if $\exists f \in \Gamma(X^{(r)}),\ \rho \circ \alpha = j^r f$. This section $f = p_0^r \circ \rho \circ \alpha$ is unique within $\Gamma(X)$.

Example 1.2. The easiest example is the inclusion $i : \mathcal{R} \to X^{(r)}$, which extends classic *partial differential relations* into smooth bundles. Here, a section $\alpha \in \Gamma(\mathcal{R})$ is holonomic if there is a unique C^r-section f such that $\alpha = j^r f$.

Definition 1.9. A *formal solution* of $\mathcal{R}$ is any section $f : V \to \mathcal{R}$. A *genuine solution* of $\mathcal{R}$ is a section $f : V \to \mathcal{R}$ such that $[f]_r(V) \subset \mathcal{R}$.

Example 1.3. We start from the embedding $p : \mathbb{R}^2 \to S^1$. Let $i : \mathcal{I} \to X^{(1)}$ be the inclusion of the immersion relation, and $f : S^1 \to \mathbb{R}^2$ be written as a section of $X^{(1)}$. Thus, f is a genuine solution of $\mathcal{R}$ if and only if, for any $x = (\cos\theta,\ \sin\theta) \in S^1$, in local coordinates, $[f]_1(x) \in \{x\} \times f(S^1) \times (\mathbb{R}^2 \setminus \{0\})$.

Example 1.4. Let $\Psi : X^{(r)} \to \mathbb{R}$. Then, defining the relation $\mathcal{R} = \Psi^{-1}(\{0\})$, any section f being a genuine solution of $\mathcal{R}$ satisfy, in local coordinates on $X^{(r)}$, the partial differential equation

$$\Psi\left(x, f(x), \cdots, \frac{1}{r!}D^r f(x)\right) = 0$$

Definition 1.10. We denote $\Gamma_{\mathcal{R}}(X)$ the subspace of $\Gamma^r(X)$ such that $f \in \Gamma_{\mathcal{R}}(X) \Leftrightarrow j^r f \in \Gamma(\mathcal{R})$.

We introduce for the next section the following notation :
for, a relation $\mathcal{R}$ on $X^{(r)}$ and a chart U within a smooth fiber bundle $p : X \to V$, we set $\mathcal{R}_U = \mathcal{R} \cap X_U^{(r)}$. We analogously define $\mathcal{R}_z$ the fiber of $\mathcal{R}$ over a point $z \in X_U^{\perp}$ with respect to the projection $p_{\perp}^r$:

$$\begin{tikzcd} \mathcal{R} \arrow[r, "i"] & X^{(r)} \arrow[d, "p_{\perp}^r"] \\ & X^{\perp} \end{tikzcd}$$

where i is the inclusion $\mathcal{R} \to X^{(r)}$.

2 Ample and/or open relations and the h-principle

In this section we will deal with the h-principle applied to ample and/or open relations. We already saw that any relation $\mathcal{R}$ over $X^{(r)}$ is written from a **subset** of $X^{(r)}$, which leads to the following definition :

Definition 2.1. A relation $\mathcal{R}$ on $X^{(r)}$ is *open* or *closed* when $\mathcal{R}$ is an open or closed subset of $X^{(r)}$ in Com-Op.

In any two topological spaces E, F, two distinct functions f and g from E to F are homotopic to each other if they are linked together by a continuous *homotopy* $H : [0,1] \times E \to E$ such that $H(0) = f$ and $H(1) = g$.

Furthermore, for a compact $K \subset X$, we say that f and g are homotopic *relatively to* K when H satifsies, for all $t \in [0,1]$, $H(t)_{|K} = f_{|K} = g_{|K}$ (H is said to be a homotopy rel K).

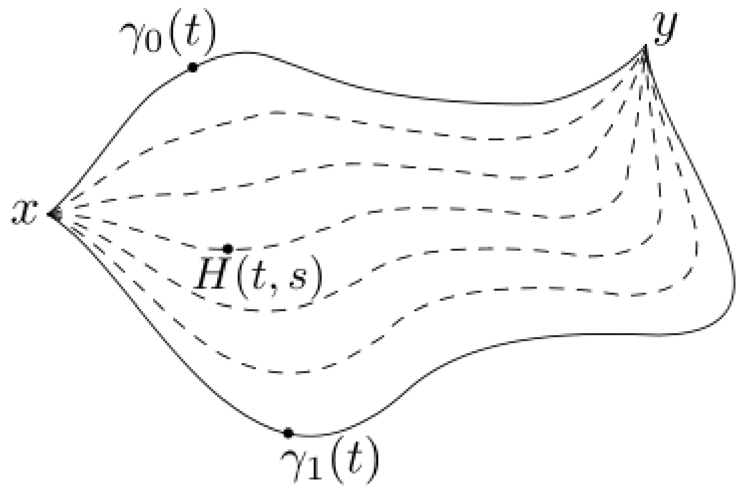

Figure 2: Classic examples of homotopy (courtesy of Wikipedia).

Homotopy principle (classic). A relation $\mathcal{R}$ over $X^{(r)}$ is said to satisfy it when any section $\alpha_0 \in \Gamma(\mathcal{R})$ is linked to a holonomic section α_1 through a homotopy of sections $\alpha_t \in \Gamma(\mathcal{R})$ with $0 \leq t \leq 1$.

Definition 2.2. For an integer $k \geq 1$, the k-th homotopy group $\pi_k(E, p)$ is the set of all homotopy classes between functions $f : S^k \to E$ through a basepoint $p \in E$, S^k being the unit sphere within $\mathbb{R}^{k+1}$.

Remark. For $k = 1$, the set of path components of E based at $p \in E$ may be seen as the set of homotopy classes of loops in E based at p, this is called the *fundamental group* of E.

Weak homotopy equivalence. A continuous function $f : E \to F$ is a weak homotopy equivalence when it meets both following conditions :

(i) f induces a bijection f_* between the sets of path components of E and of F.

(ii) for all $x \in E,\ k \in \mathbb{N}^*$, the morphism $f_* : \pi_k(E, x) \to \pi_k(F, f(x))$ is bijective.

2.1 Convex integration and the Integral Representation Theorem

The Integral Representation Theorem represents a continuous function $f : E \to \mathbb{R}^q$ with values in the convex hull of a connected open set $X \subset \mathbb{R}^q$, as the Riemann integral of a continuous function h from $[0, 1] \times E$ to X ; for all $x \in E$,

$$f(x) = \int_0^1 h(t, x)\, dt$$

We define below the tools used to formulate the Integral Representation Theorem.

Definition 2.3. Let $\rho : X \to \mathbb{R}^q$ be continuous, where we suppose X is path connected. We define the convex hull of $\rho(X)$,

$$\mathrm{Conv}_\rho(X) = \left\{ \sum_{i=1}^{N} p_i\ \rho(x_i)\ ;\ x_i \in X,\ p_1 + \cdots + p_N = 1,\ 0 \le p_i \le 1 \right\}$$

and $\mathrm{IntConv}_\rho(X)$ the interior of $\mathrm{Conv}_\rho(X)$.
If X is not path connected, we denote $\mathrm{Conv}_\rho(X, y) = \mathrm{Conv}_\rho(Y)$ where Y is the path component in X containing y ; and $\mathrm{IntConv}_\rho(X, y) = \mathrm{IntConv}_\rho(Y)$.

Example 2.1. The most obvious example of convex hull is for the inclusion $i : X \to \mathbb{R}^q$. In this case, we can drop the letter ρ from the (interior) convex hull notation : $\mathrm{Conv}(X) = \mathrm{Conv}_i(X)$.

Definition 2.4. A subset $X \subset \mathbb{R}^q$ is said to be *ample* (for ρ) when, for every $x \in X$, we have $\mathrm{Conv}_\rho(X, x) = \mathbb{R}^q$.
A relation $\rho : \mathcal{R} \to X^{(r)}$ is ample if $\mathrm{Conv}_\rho(\mathcal{R}, f) = X^{(r)}$ for any $f \in \mathcal{R}$.

Alongside with these definitions, quoting the Integral Representation Theorem will involve a concept of contraction on loops, which is central to convex integration.

Definition 2.5. Fix a basepoint $x \in X$, and let $g : [0,1] \to X$ be a continuous loop such that $g(0) = g(1) = x$ (we say that g is based at x). A point $z \in \mathrm{Conv}_\rho(X, x)$ is given the adjective *surrounded* with the convention

$$\begin{cases} z \text{ surrounded by } g & \text{if } z \in \mathrm{Conv}_\rho(g([0,1])) \\ z \text{ strictly surrounded by } g & \text{if } z \in \mathrm{IntConv}_\rho(g([0,1])) \end{cases}$$

Definition 2.6. A loop $g : [0,1] \to X$ based at x is *contractible* if it is homotopic to the constant path $[0,1] \to \{x\}$. The homotopy between g and the constant path is a *contraction* of g.

Definition 2.7. We suppose $z \in \mathrm{Conv}_\rho(X, x)$. X_x^z is the space of pairs (g, G) satisfying the following properties :

(i) $g : [0,1] \to X$ is a contractible loop based at x and surrounds z.

(ii) $G : [0,1]^2 \to X$ is a basepoint-preserving contraction of g to $[0,1] \to \{x\}$

that is, for $(t, s) \in [0,1]^2$, $\begin{cases} G(t,0) = x \\ G(t,1) = g(t) \end{cases}$ and $G(0,s) = G(1,s) = x$.
int X_x^z is the space of (almost) such pairs (g, G) but where g strictly surrounds $z \in \mathrm{IntConv}_\rho(X, x)$.

Remark. int X_x^z and X_x^z are subspaces of $C^0([0,1], X) \times C^0([0,1]^2, X)$ in Com-Op. In the next paragraph, *second-countable* (for E) means there exists a countable family $U = (U_i)_{i \in \mathbb{N}}$ of open subsets of E, such that any open $O \subset B$ can be written as the union of some opens from the family U.

The spaces int X_x^z can be generalized for relations $\rho : \mathcal{R} \to Z$. We set $p : Z \to E$ an affine $\mathbb{R}^q$-bundle, with E supposed to be a second-countable paracompact base space ; such affine bundle is the restriction over a submanifold $E \subset X^\perp$ of the affine $p_\perp^r : X^{(r)} \to X^\perp$ (cf. Subsection 1.2). Just as $\Gamma(Z)$ is the space of continuous sections of Z in Com-Op, we can define spaces of sections for such a relation below.

Definition 2.8. A *relation* over Z is a continuous map $\rho : \mathcal{R} \to Z$, where $\mathcal{R} \subset Z$. $\Gamma(\mathcal{R})$ is the space of continuous section of $p \circ \rho : \mathcal{R} \to E$ in Com-Op. $\Gamma_K(Z)$, $\Gamma_K(\mathcal{R})$ are the analogous space of sections over some subspace $K \subset E$.

Definition 2.9. Let $b \in E$ be a basepoint, then $Z_b = p^{-1}(\{b\})$ is the $\mathbb{R}^q$-fiber over b within the bundle Z, and $\mathcal{R}_b = \mathcal{R} \cap \rho^{-1}(Z_b)$. When $a \in \mathcal{R}_b$, then $\mathrm{Conv}_\rho(\mathcal{R}_b, a)$ is the convex hull (within Z_b) of $\rho(S)$, S being the path component of $\mathcal{R}_b$ containing a.

Remark. With the above setting, we can define ampleness in the following sense : $\mathcal{R}$ (precisely $\rho : \mathcal{R} \to Z$) is ample when $\mathrm{Conv}_\rho(\mathcal{R}_b, a) = Z_b$ for every $b \in E,\ a \in \mathcal{R}_b$.

Definition 2.10. For a relation $\rho : \mathcal{R} \to Z$ over Z, let $f \in \Gamma(Z),\ \beta \in \Gamma(\mathcal{R})$ such that, for all $b \in E,\ f(b) \in \mathrm{IntConv}_\rho(\mathcal{R}_b, \beta(b))$.
For some subset $K \subset E$, a C-*structure* over K with respect to f, β is a pair (g, G) satisfying the following properties :

(i) $g : [0,1] \to \Gamma_K(\mathcal{R})$ is a contractible loop based at $\beta_{|K}$ and fiberwise strictly surrounds $f_{|K}$, i.e., for all $b \in K$, defining $g_b : \begin{array}{l} [0,1] \to \mathcal{R}_b \\ t \mapsto g(t)(b) \end{array}$ then the path g_b strictly surrounds $f(b)$.

(ii) $G : [0,1]^2 \to \Gamma_K(\mathcal{R})$ is a fiberwise basepoint-preserving contraction of g to $[0,1] \to \{\beta_K\}$

that is, for $(t,s) \in [0,1]^2$, $\begin{cases} G(t,0) = \beta_K \\ G(t,1) = g(t) \end{cases}$ and $G(0,s) = G(1,s) = \beta_K$

C-structures defined in int X_x^z allow us to represent z as the Riemann integral of a function with values in X, which leads to the Integral Representation Theorem.

Integral Representation Theorem. For the affine bundle $p : Z \to E$ defined as above, we suppose $\mathcal{R} \subset Z$ is an open subset, and $f \in \Gamma(Z),\ \beta \in \Gamma(\mathcal{R})$ such that, for all $b \in E,\ f(b) \in \mathrm{Conv}_\rho(\mathcal{R}_b, \beta(b))$.
Then, each C-structure (g, G) over E with respect to $f,\ b$ can be reparametrized into a C-structure (h, H), with $h : [0,1] \to \Gamma(\mathcal{R})$ such that, for every $b \in E,\ f(b) = \int_0^1 h(t,b)\, dt$.

2.2 One particular example: the Hirsch theorem.

Setting $p : X \to V$ a smooth fiber bundle over a smooth manifold V ($\dim V = n$, fiber dimension q), we work with some relation $\mathcal{R}$ in the space of 1-jets $X^{(1)}$. Locally at any point in X, $p : X \to V$ is a product bundle $X_U = U \times \mathbb{R}^q \to U$, with $U \subset V$

working as a chart. With respect to local coordinates $(u_1, \cdots, u_n)$ in U, a section $f \in \Gamma^1(X_U) = C^1(U, \mathbb{R}^q)$ induces (for $\partial_i = \partial/\partial u_i$) the 1-jet extension, within $X_U^{(1)}$,

$$j^1 f(x) = (\underbrace{x, f(x)}_{j^0 f(x)}, \partial_1 f(x), \cdots, \partial_n f(x))$$

we have also $X_U^{(1)} = X_U^{\perp} \times \mathbb{R}^q$ with the equation $j^1 f(x) = (j^{\perp} f(x), \partial_n f(x))$, thus, at a local level, the projection $p_{\perp}^1$ is a product $\mathbb{R}^q$-bundle. We denote $X_z^{(1)} = \{z\} \times \mathbb{R}^q$ the fiber of this bundle over a point $z \in X_U^{\perp}$.

For this case, we will say that $\mathcal{R} \subset X^{(1)}$ is ample if, with respect to any local coordinates, for all $z \in X_U^{\perp},\ w \in \mathcal{R}_z,\ \mathrm{Conv}(\mathcal{R}_z, w) = X_z^{(1)} \equiv \mathbb{R}^q$ (convex hull using the inclusion $i : \mathcal{R} \to X^{(1)}$).

We suppose $V,\ W$ to be smooth manifolds, with dim $V = n$, dim $W = q \geq n$. A map $f \in C^1(V, W)$ is an immersion if the tangent bundle map $df : TV \to TW$ has maximal rank n. The inequality $n \leq q$ can be split in two cases :

1. the extra dimensional case $n < q$,
2. the equidimensional case $n = q$.

Now, for $p : V \times W \to V$ the product bundle, fiber W, the immersion relation $\mathcal{I}$ is the subspace of germs of functions $f \in \Gamma^1(X)$ such that all partial derivatives $\partial_i f(x) \in \mathbb{R}^q$ are linearly independent for any $x \in U$, with respect to local coordinates of $j^1 f$ on U. Likewise, we can say that $\mathcal{I}_U$ is the subspace of vectors $(x, y, v_1, \cdots, v_n)$ such that the vectors v_i are linearly independent within $\mathbb{R}^q$. It is obvious that $\mathcal{I}_U$ is open in $X_U^{(1)}$ and therefore $\mathcal{I} \subset X^{(1)}$ is open too.

When is $\mathcal{I}$ ample ?
Let us write $w = (x, y, v_1, \cdots, v_n) \in \mathcal{I}_U$ and $L = \mathrm{Vect}(v_1, \cdots, v_{n-1}) \subset \mathbb{R}^q$. For any $z = p_{\perp}^1(w) = (x, y, v_1, \cdots, v_{n-1})$, the fiber $\mathcal{I}_z$ is formed by points (z, v) such that $v \notin L$, thus we have $\mathcal{I}_z = \mathbb{R}^q \setminus L$, identifying $\{z\} \times \mathbb{R}^q \equiv \mathbb{R}^q$, all in local coordinates, whence $\dim \mathcal{I}_z = q - n + 1$. In the equidimensional case, $\mathcal{I}_z = q - n + 1$ is just a line with two path connected 1/2-spaces I_+ and I_- ; we have for $a \in I_{\pm}$, $\mathrm{Conv}(\mathcal{I}_z, a) = I_{\pm} \neq \mathbb{R}^q$, thus $\mathcal{I}$ is not ample. Only in the extra dimensional case $(q \geq n + 1)$ $\mathcal{I}_z$ is path connected in $\mathbb{R}^q$ since $\dim \mathcal{I}_z = q - n + 1 \geq 2$, letting $\mathcal{I}$ be ample.

Theorem 2.1. (Hirsch) We suppose that $q \geq n + 1$. Then $\begin{array}{rl} J: & \Gamma_{\mathcal{I}}(X) \to \Gamma(\mathcal{I}) \\ & h \mapsto j^1 h \end{array}$ induces a weak homotopy equivalence.

C^0-dense h-principle. For the above bundle $p : X \to V$, let $\mathcal{R} \subset X^{(1)}$ be open and ample, $\phi \in \Gamma(\mathcal{R})$ and a section $h = p_0^1 \circ \phi \in \Gamma(X)$. We suppose $K_0 \subset V$ is a closed subset such that h is C^1 on $\mathfrak{Op}\ K_0$ and $j^1 h = \phi$ on K_0 ; and N_h a neighborhood of $h(V)$ in X.
Then, there exists a section $f \in \Gamma^1(X)$ and a homotopy $F : [0, 1] \to \Gamma(\mathcal{R})$ between ϕ and $j^1 f$, such that

(i) $j^1 f \in \Gamma(\mathcal{R})$;

(ii) $f(V) \subset N_h$;

(iii) $F(0) = \phi,\ F(1) = j^1 f$;

(iv) for any $0 \le t \le 1,\ p_0^1 \circ F(t)(V) \subset N_h$

(v) (relative theorem) for any $0 \le t \le 1,\ F(t)_{|K} = \phi_{|K}$.

Proof. The general proof for the above principle is hard to understand, so we will illustrate it with the immersion relation $\mathcal{I}$.

We set $X = [0,1] \times \mathbb{R}^2,\ V = [0,1], W = \mathbb{R}^2$. We have an immersion $\gamma : [0,1] \to \mathbb{R}^2$, defining $h(x) = (x, \gamma(x))$ and $\phi(x) = (x, \gamma(x), \gamma'(x))$, all in local coordinates. We will have $K_0 = [a,b] \subset [0,1]$. When we suppose the local representation $h \equiv j^0\gamma$ on a subinterval $[\tilde{a}, \tilde{b}] \subset [0, a[\cup]b, 1]$, let $(h_i)_{i \ge 1}$ be a sequence of sections such that each h_i improves h_{i-1} by solving $\mathcal{I}$ on some increasing neighborhoods of $[\tilde{a}, \tilde{b}]$, according to an induction, the main step of which is illustrated below with some couple (g, ρ). We can take $f = \lim_{i\to\infty} h_i$ for the conclusions of the theorem.

For $[c,d] \subset [0,1]$, we have $\rho \in \Gamma^\infty(\mathcal{I}_U)$, with U a chart on $[0,1]$ whose interior contain a closed set ω. We suppose the following conditions hold :

(a_0) $\rho = j^1\gamma$ on $U \cap \mathfrak{Op}[c,d]$

(b_0) $\rho(x) = (x, \gamma(x), \psi(x))$ where $\psi \in C^\infty([0,1], \mathbb{R}^2)$, for all $x \in \omega$

see One-Dimensional Theorem as a tool to get the conditions $(a_1), (b_1)$. For M a neighbourhood of the image $\gamma(U)$ in X_U, and a small δ, there is also a map $q \in C^\infty(U, \mathbb{R}^2)$ and a smooth homotopy $Q : [0,1] \to \Gamma(\mathcal{I}_U)$ such as ($\|\cdot\|$ being the sup-norm on $C^0([0,1], \mathbb{R}^2)$)

- $\|q - g\| < \delta$, and $\forall t \in [0,1],\ (p_0^1 \circ Q(t))(U) \subset M$
- $Q(0) = \rho,\ p_0^1 \circ Q(1) = f$, and for all $x \in \omega$,

$$Q(1)(x) = (x, \gamma(x), \gamma'(x))$$

- $q = g$ on $U \cap \mathfrak{Op}\ [c,d]$.

□

Remark. There is a homotopy F from a true immersion $f : [0,1] \to \mathbb{R}^2 \equiv \mathbb{C}$ to $t \mapsto (t, 0, e^{2\pi ikt})$:

$$F_\epsilon(t) = (t,\ \theta(\epsilon)f(t),\ \theta(\epsilon)f'(t) + \theta(1-\epsilon)e^{2\pi ikt})$$

with $\theta : [0,1] \to [0,1]$ a continuous function satisfying $\theta(0) = 0$ and $\theta(1) = 1$.

3 Analytic theory and the proof of Hirsch

In this section, we set $E = B \times \mathbb{R}^q$, with a B being a compact Hausdorff space (separate means that two distinct points are always separated by respective neighborhoods not intersecting each other). In case $B = [0,1]^n \equiv [0,1]^{n-1} \times [0,1]$ when we denote $(u_1, \cdots, u_{n-1}, t)$ the coordinates on B, for any $\alpha = (\alpha_1, \cdots, \alpha_{n-1})$ we write

$$\partial_U^\alpha = \partial_{u_1}^{\alpha_1} \circ \cdots \circ \partial_{u_{n-1}}^{\alpha_{n-1}}$$

Moreover, all used norms $\|\cdot\|$ without subscript are C^0 function sup-norms.

One-Dimensional Theorem. (general enunciation) Let $\pi : E \to B$ be the product $\mathbb{R}^q$-bundle over some space $B = C \times [0,1]$ $\mathcal{R} \subset E$ an open relation, and suppose sections $\beta \in \Gamma(\mathcal{R}), f_0 \in \Gamma(E)$ such that f_0 is C^1in t, and for all $b \in B,\ \partial_t f_0(b) \in \mathrm{Conv}(\mathcal{R}_b, \beta(b))$.

Then there is a C^1 (in t)-section $f_\epsilon \in \Gamma(\mathcal{R})$ and a continuous homotopy $F : [0,1] \to \Gamma(\mathcal{R})$ such that

(i) $\lim_{\epsilon\to 0} \|f_\epsilon - f_0\| = 0$

(ii) $\partial_t f \in \Gamma(\mathcal{R}); F_0 = \beta, F_1 = \partial_t f$, and the image of F is contained in the image of a C-structure with respect to $\partial_t f,\ \beta$.

... where, for $s \geq r \geq 0$,

$$\|f\|^{s,r} = \sup\{\|\partial_t^k \circ \partial_U^\alpha f\| \ :\ k + |\alpha| \leq s;\ k \leq r\}$$

and $\|f\|^s := \|f\|^{s,0}$.

Proposition 3.1. Given $\pi : B \times \mathbb{R}^q$ the product $\mathbb{R}^q$-bundle over B, let $g_0 \in \Gamma(E)$ be C^1 in t and a continuous map $\psi : [0,1] \to \Gamma(E)$ be an integral representation of $\partial_t g_0$, that is, for all $b \in B$,

$$\partial_t g_0(b) = \int_0^1 \psi(s)(b)\ ds$$

Then, for any $\epsilon > 0$, there is a continuous function $\theta_\epsilon : [0,1] \to [0,1]$ such that the section $g_\epsilon \in \Gamma(E)$ defined for $(c,t) \in C \times [0,1]$ by

$$g_\epsilon(c,t) = g_0(c,0) + \int_0^t \psi(\theta_\epsilon(s))(c,s)\ ds$$

is C^1 with respect to the variable t, and $\lim_{\epsilon\to 0} \|\gamma_\epsilon - \gamma_0\| = 0$.

Proof. To prove this proposition in a simplified manner, we study the case of $\gamma : [0,1] \to \mathbb{R}^2$ being an immersion over $[0,1]$; with the change of notation from the above statement to hereafter.

Statement	Proof
g_ϵ	γ_ϵ
g_0	γ_0
$\psi(f(s))(c,s)$	$g \circ f(s)$, with $f : [0,1] \to [0,1]$

We suppose, along with γ_0 being an immersion, that $B = \{0\} \times [0,1] \equiv [0,1]$, so we can drop the variable c.

For some $0 < \epsilon < 1$, we define by $I_j = [t_j, t_j + \epsilon]$, $1 \le j \le m$ a sequence of disjoint subintervals in $[0,1]$ suth that the total sum of their lengths is $|I_1| + \cdots + |I_m| \ge 1 - \epsilon$. While specifying below the construction of a suitable continuous function $\theta = \theta_\epsilon : [0,1] \to [0,1]$, we set auxiliary step functions $l, k : [0,1] \to \mathbb{R}^2$

$$l(t) = \gamma_0(0) + \epsilon \sum_{j,\ I_j \subset [0,1]} \gamma_0'(t_j)$$
$$k(t) = \gamma_0(0) + \sum_{j,\ I_j \subset [0,1]} \int_{I_j} g(\theta_\epsilon(s))\ ds$$

where we have the estimate $\max\{j,\ I_j \subset [0,1]\} \le 1/\epsilon$. We also set the section γ_ϵ,

$$\gamma_\epsilon(t) = \gamma_0(0) + \int_0^t g(\theta_\epsilon(s))\ ds$$

then from the hypothesis on ψ in the statement, it is obvious that γ_ϵ is C^1 with derivative $\gamma_\epsilon'(t) = g(\theta_\epsilon(t))$.

The function θ_ϵ is specified as a piecewise linear function which highly oscillates, such that, its restriction to each subinterval I_j is a linear homeomophism onto $[0,1]$. When ϵ is small enough, changing variables by θ_ϵ allows for the following approximation, for $1 \le j \le m$,

$$\int_{I_j} g(\theta_\epsilon(s)) \approx \epsilon \int_0^1 \psi(s)(t_j)\ ds = \epsilon\ \gamma_0'(t_j)$$

which proves that $\lim_{\epsilon \to 0} \|l - k\| = 0$.

As for θ_ϵ, we can assert that it is the piecewise C^1 function defined by

$$\forall s \in I_j,\ \theta_{\epsilon|I_j}(s) = \frac{s - t_j}{\epsilon},\ 1 \le j \le m$$

which allows for the change of variables $u = \theta_\epsilon(s)$ on the subinterval I_j, hence

$$\int_{I_j} g(\theta_\epsilon(s))\ ds = \int_{I_j} \psi(\theta_\epsilon(s))(s)\ ds = \epsilon \int_0^1 \psi(u)(t_j + \epsilon u)\ du$$

thus we see that $l,\ k,\ \gamma_\epsilon$ converge together towards γ_0 as ϵ vanishes. $\square$

Example 3.1. For the constant path $\gamma_0(t) = 0$ to be approximated by an immersion γ_ϵ, we will have $l(t) = k(t) = 0$ for all t, and we can set

$$\psi(t) = \begin{cases} e^{4\pi it} & \text{if } t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right] \\ e^{-4\pi it} & \text{if } \frac{1}{4} \leq t \leq \frac{3}{4} \end{cases}$$

Then using the change of variables $u = \theta\epsilon(s)$ in the above proof and supposing there is some subinterval I_j covering almost the interval $[0, 1]$, we will have

$$\begin{aligned} \gamma_\epsilon(t) &= \int_0^t \psi(\theta_\epsilon(s))\, ds \\ &= \int_{I_j} \psi(\theta_\epsilon(s))\, ds \\ &= \epsilon \int_0^1 \psi(u)\, du \end{aligned}$$

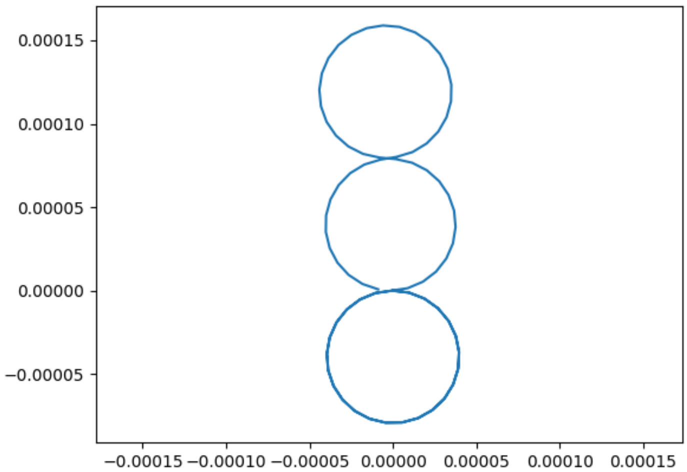

Figure 3: Graph of the integral of $s \mapsto e^{16\pi is}\ \psi(\theta_\epsilon(s))$ from 0 to t, as a path $[0, 1] \to \mathbb{R}^2$. Here, we have $\epsilon = 0.001 - 10^{-12}$ and for $0 \leq\leq 999,\ I_j = [0.001\, j\ ;\ 0.001\, j + \epsilon]$

The following theorem generalizes the C^0-dense h-principle to the case of an auxiliary compact space of parameters P. Its consequence is the Weak Homotopy Equivalence Theorem, which states that for any relation $\mathcal{R} \subset X^{(1)}$ that is open and ample,

$$\begin{aligned} J: \ \ \Gamma_{\mathcal{R}}(X) &\to \Gamma(\mathcal{R}) \\ h &\mapsto j^1 h \end{aligned}$$

is a weak homotopy equivalence, is obtained by proving that, for any $h \in \Gamma_{\mathcal{R}}(X),\ k \geq 1$ the induced map $J_* : \pi_k(\Gamma_{\mathcal{R}}(X), h) \to \pi_k(\Gamma(\mathcal{R}), j^1 h)$ is bijective. This is done by applying the Parametric h-principle to it, in the following manner :

(i) on the one hand, by setting $P = S^i \times [0,1]$, to prove the injectivity of J_*.

(ii) on the other hand, by setting $P = S^i$, to prove the surjectivity of J_*

Parametric h-principle For some open and ample relation $\mathcal{R} \subset X^{(1)}$, let $\phi \in C^0(P, \Gamma(R))$ and $h \in C^0(P, \Gamma(X))$ such that $h = p_0^1 \circ \phi$. Suppose that P_0 is closed with $\partial P \subset P_0 \subset P$, such that h is C^1 on $\mathfrak{Op}\ P_0$ and $j^1(h(p)) = \phi(p)$ for all $p \in P_0$.

Then, there exists a function $f \in C^1(P, \Gamma(X))$ and a homotopy rel P_0, $F : P \times [0,1] \to \Gamma(\mathcal{R})$ such that

(i) $H(0) = \phi$.

(ii) for all $p \in P,\ H(p)(1) = j^1(f(p))$.

Proof of Hirsch theorem

We still review the immersion relation $\mathcal{I}$ with respect to the (trivial) product bundle $X = [0,1] \times \mathbb{R}^2$, to give the proof of Hirsch theorem. From the subsection 2.2 we recall that $\mathcal{I}$ was open and ample, and is the subspace of vectors (x, y, v) with $v \neq 0$ in $\mathbb{R}^2$. Then we apply the Weak Homotopy Equivalence Theorem to $\mathcal{I}$. $\square$

4 Smale's sphere eversion

Let S^2 be the unit sphere in $\mathbb{R}^3$. We build the following functions on $\mathbb{R}^3$, using a positive real $\delta > 1$, and the following spherical coordinates :

$$P : (u, \theta, \phi) \mapsto u \begin{pmatrix} \cos\theta \; \cos\phi \\ \sin\theta \; \cos\phi \\ \sin\phi \end{pmatrix}$$

Denoting $A := \left]\frac{1}{\delta}, \delta\right[\cdot S^2$ the spherical "annulus" defined for the Euclidean norm in $\mathbb{R}^3$ by $\frac{1}{\delta} < \|x\| < \delta$, we define

$$\begin{aligned} f_0 : P^{-1}(A) &\to \mathbb{R}^3 \\ (u, \theta, \phi) &\mapsto (u, \theta, \phi) \end{aligned}$$

as well as

$$\begin{aligned} \mathrm{inv} : P^{-1}(\mathbb{R}^3 \setminus \{0\}) &\to \mathbb{R}^3 \setminus \{0\} \\ (u, \theta, \phi) &\mapsto \left(\frac{1}{u}, \theta, \phi\right) \end{aligned}$$

$$\begin{aligned} r : \mathbb{R}^3 &\to \mathbb{R}^3 \\ (u, \theta, \phi) &\mapsto (u, \theta, -\phi) \end{aligned}$$

We define $f_1 = r \circ \mathrm{inv} \circ f_0$ which everts A outside in. In fact, while f_0 is just the inclusion $i_A : A \hookrightarrow \mathbb{R}^3$ (canonical coordinates), for any $u > 0$, inv maps $u \cdot S^2$ onto $\frac{1}{u} \cdot S^2$, and r is the reflection through the horizontal plane $\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \gamma = 0\}$.

Smale stated that f_0 and f_1 are regularly homotopic to each other, that is, they are homotopic to each other through a family of immersions $f_t : A \to \mathbb{R}^3$. First, we find out that, for any $x \in A$

$$\begin{aligned} f_1(u, \theta, \phi) &= r \circ \mathrm{inv}(u, \theta, \phi) \\ &= r\left(\frac{1}{u}, \theta, \phi\right) \\ &= \left(\frac{1}{u}, \theta, -\phi\right) \end{aligned}$$

and that, both maps $f_0,\ f_1$ are immersions, when looking at their Jacobian matrices:

$$Jf_0(x) = I_3 \; ; \; Jf_1(x) = \begin{pmatrix} -\frac{1}{u^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Since S^2 is a subset of A, any smooth map from A to $\mathbb{R}^3$ can be restricted to S^2 where its differential can be seen as an endomorphism of $\mathbb{R}^3$. Precisely, when restricted to S^2, both maps $df_0(x)$ and $df_1(x)$ belong to the group $SO(3)$ and it is admitted that $\pi_2(SO(3)) = \{0\}$, that is, any continuous map from S^2 to $SO(3)$ is homotopic to a constant map.

Remark. It is recalled that a $n \times n$ matrix M, to belong to $SO(n)$, must fulfill both conditions: that $M^t\, M = I_n$, and $\det M = 1$.

Since the above matrices are homotopic to the same constant map $x \mapsto 0$, they are homotopic to each other.

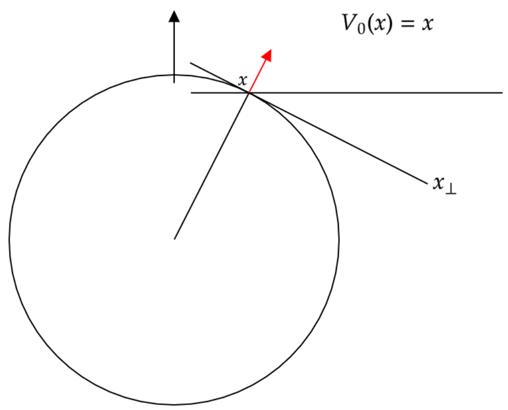

Figure 4: For $t = 0$, the red vector is x, and $\psi_0(x) = 0$.

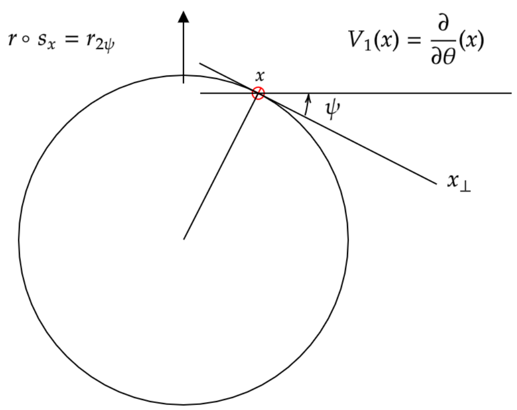

Figure 5: For $t = 1$, we denote $\frac{\partial}{\partial\theta}(x)$ the red vector of spherical coordinates $\left(1, \theta - \frac{\pi}{2}, 0\right)$, pointing outwards (i.e. it is tangent to the parallel of latitude x is located on). We have also $\psi = \psi_1(x) = \frac{\pi}{2} - \phi$, from the spherical coordinates stated above.

Let

$$\begin{aligned} G : [0,1] \times S^2 &\to SO(3) \\ (t,x) &\mapsto R_{V_t(x),\, 2\,\psi_t(x)} \end{aligned}$$

be a homotopy between the maps $R_{x,\,0} = \mathrm{id} = df_0(x)$, and $R_{\frac{\partial}{\partial\theta}(x),\, 2\,\psi} = df_1(x)$, where $x = (1,\theta,\phi) \in S^2$. Furthermore, we can continuously extend G_1 to the poles of S^1 : $r \circ s_{(1,0,\pm\,\pi/2)} = \mathrm{id}$.

We may figure both homotopies, the first resembling a "rotation" in t from $\frac{\partial}{\partial\theta}(x)$ to x, and the other just linear

$$\begin{aligned} V_t(x) &= R_{\vec{n}(x),\, \frac{\pi}{2}\,t}(x) \\ \psi_t(x) &= t\, \left(\frac{\pi}{2} - \phi\right) \end{aligned}$$

where $\vec{n}(x)$ is obtained by the Gram-Schmidt process such that $\left(x, \frac{\partial}{\partial\theta}(x), \vec{n}(x)\right)$ is a direct orthonormal basis of $\mathbb{R}^3$.

Furthermore, in Figure 4, s_x is simply the reflection orthogonal to $\mathrm{Vect}\{x\}$ defined by

$$\begin{aligned} s_x(\alpha,\beta,\gamma) &= (\mathrm{id} - 2p_x)(\alpha,\beta,\gamma) \\ &= (\alpha,\beta,\gamma) - 2\, \cdot\, (\alpha x_1 + \beta x_2 + \gamma x_3)\, x \end{aligned}$$

whose matrix, in linear algebra, is

$$[s_x] = \begin{pmatrix} 1-2x_1^2 & -2x_1x_2 & -2x_1x_3 \\ -2x_1x_2 & 1-2x_2^2 & -2x_2x_3 \\ -2x_1x_3 & -2x_2x_3 & 1-2x_3^2 \end{pmatrix} = d\,\mathrm{inv}(x)$$

We have found a homotopy between $df_0(x)$ and $df_1(x)$ for $x \in S^2$, and therefore f_0 and f_1 are homotopic to each other through a family of immersions $f_t,\ t \in [0,1]$. The projection

$$\begin{aligned} p : S^2 \times \left]\frac{1}{\delta},\, \delta\right[&\to S^2 \times \left]\frac{1}{\delta},\, \delta\right[\\ (x,u) &\mapsto (x,1) \end{aligned}$$

is obviously homotopic to the identity on $S^2 \times \left]\frac{1}{\delta},\, \delta\right[$. This homotopy equivalence $p \sim \mathrm{id}$, and the inclusion $S^2 \hookrightarrow S^2 \times \left]\frac{1}{\delta},\, \delta\right[$ induce a homotopy equivalence between the homotopies on $S^2 \to SO(3)$ and $S^2 \times \left]\frac{1}{\delta},\, \delta\right[\to SO(3)$.

Therefore, there is a homotopy $F : [0,1] \times \left(S^2 \times \left]\frac{1}{\delta},\, \delta\right[\right) \to SO(3) \subset GL(3)$ with $F(0) = df_0$ and $F(1) = df_1$, then inducing a homotopy $(h(t))_{0\le t\le 1}$ with $h(0) = f_0$ and $h(1) = f_1$, and such that $d(h(t))$ is C^0-close to $F(t)$, and $h(t) \in \Gamma_{\mathcal{I}}(S^2 \times \left]\frac{1}{\delta},\, \delta\right[)$ for any t.

Thus, using the Hirsch theorem, we obtain a homotopy $(j^1(h(t)))_{0\le t\le 1}$ with $j^1(h(t))(x) \overset{C^0}{\approx} (x, f_t(x), F(t)(x))$.

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