NANTES' UNIVERSITY

MASTER THESIS

LINEARLY STABLE AND UNSTABLE TORI FOR NLS EQUATION

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Contents

1. Introduction

We consider the non linear Schrödinger equation on the torus

$$i\partial_t u + \partial_{xx} u = |u^4|u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{1.1}$$

This system is Hamiltonian on the phase space $(u, \bar{u}) \in L^2(\mathbb{T})$ endowed with the symplectic form $-idu \wedge d\bar{u}$. The Hamiltonian of the equation is given by

$$h = \int_{\mathbb{T}} |u_x|^2 + \frac{1}{3}|u|^6 dx.$$

Let us expand u and $\bar{u}$ in Fourier basis:

$$u(x) = \sum_{j\in\mathbb{Z}} a_j e^{ijx}, \quad \bar{u}(x) = \sum_{j\in\mathbb{Z}} b_j e^{ijx}.$$

In this variables, the symplectic structure becomes $-i\sum_{j\in\mathbb{Z}} da_j \wedge db_j$, and the Hamiltonian h of the system reads

$$h = N + P = \sum_{j\in\mathbb{Z}} j^2 a_j b_j + \frac{1}{3} \sum_{j,\ell\in\mathbb{Z}^3, \mathcal{M}(j,l)=0} a_{j_1} a_{j_2} a_{j_3} b_{\ell_1} b_{\ell_2} b_{\ell_3}$$

where $\mathcal{M}(j,l) = j_1 + j_2 + j_3 - \ell_1 - \ell_2 - \ell_3$ denotes the momentum of the multi-index $(j,\ell) \in \mathbb{Z}^6$ or equivalently the momentum of the monomial $a_{j_1} a_{j_2} a_{j_3} b_{\ell_1} b_{\ell_2} b_{\ell_3}$.

In this article, we are studying the persistence of two and three dimensional linear invariant tori. Precisely, given $p, q \in \mathbb{Z}$ and $a_p, a_q \in \mathbb{C}$, we are interesting in the persistence of torus $\mathbf{T}_c^{lin} = \{|a_p|^2 = c_1,\ |a_q|^2 = c_2\}$ under the flow of h for $c \in \mathbb{R}^2$. By KAM theorem 2.3, we prove that for ρ in a Cantor set of full measure in $[1,2]^2$ and for ν small enough, the torus $\mathbf{T}_{\nu\rho}^{lin}$ is linearly stable.

Theorem 1.1. *Fix p, $q \in \mathbb{Z}$, and $s > \frac{1}{2}$. There exists $\nu_0 > 0$, and for $0 < \nu < \nu_0$, there exists $\mathcal{D}_\nu \subset [1,2]^2$ asymptotically of full measure (i.e. meas$([1,2]^2 \setminus \mathcal{D}_\nu) \to 0$ when $\nu \to 0$) such that for $\rho \in \mathcal{D}_\nu$, equation* (1.1) *admits a solution of the form*

$$u(x) = \sum_{j\in\mathbb{Z}} a_j(t\omega) e^{ijx}$$

where $\{a_j\}_j$ is analytic function form $\mathbb{T}^2$ to ℓ_s^2 satisfying uniformly in $\theta \in \mathbb{T}^2$

$$|a_p - \sqrt{\nu\rho_1}|^2 + |a_q - \sqrt{\nu\rho_2}|^2 + \sum_{j\neq p,q} (1+j^2)^s |a_j|^2 = \mathcal{O}(\nu^3).$$

Here ω is a nonresonant vector in $\mathbb{R}^2$ that satisfies

$$\omega = (p^2, q^2) + \mathcal{O}(\nu^2).$$

Furthermore, this solution is linearly stable.

For there dimensional tori, the dynamic structure is complicated. It depends on three internal modes (p, q, m) of that torus and starting energy on each mode. In this paper, with KAM theorem 2.3, we just focus on the case there is no ℓ solving equation

$$\begin{cases} 2j_1 + j_2 & = j_3 + \ell \\ 2j_1^2 + j_2^2 & = j_3^2 + \ell^2 \end{cases} \tag{1.2}$$

where $\{j_1, j_2, j_3\} = \{p, q, m\}$. We prove that for $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{D}_1 = [1, 2]^3$, the torus $\mathbf{T}^{lin}_{\nu\rho} = \{|a_p|^2 = \nu\rho_1,\ |a_q|^2 = \nu\rho_2,\ |a_m|^2 = \nu\rho_3\}$ is linearly stable for all $p, q, m \in \mathbb{Z}$, while for $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{D}_2 = [1-\epsilon, 1+\epsilon] \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon] \times [\frac{9}{2} - \epsilon, \frac{9}{2} + \epsilon]$ with ϵ sufficiently small, that torus is linearly unstable if there exist s, t solving the following equation system

$$\begin{cases} 2p + q & = m + s + t \\ 2p^2 + q^2 & = m^2 + s^2 + t^2 \end{cases} \tag{1.3}$$

Theorem 1.2. *Fix p, q, $m \in \mathbb{Z}$, and $s > \frac{1}{2}$, assume that we are not in the case* (1.2). *There exists $\nu_0 > 0$, and for $0 < \nu < \nu_0$, there exists $\mathcal{D}_\nu \subset \mathcal{D}$ asymptotically of full measure (i.e. meas$(\mathcal{D} \setminus \mathcal{D}_\nu) \to 0$ when $\nu \to 0$) such that for $\rho \in \mathcal{D}_\nu$, equation* (1.1) *admits a solution of the form*

$$u(x) = \sum_{j \in \mathbb{Z}} a_j(t\omega) e^{ijx} \tag{1.4}$$

where $\{a_j\}_j$ is analytic function form $\mathbb{T}^3$ to ℓ^2_s satisfying uniformly in $\theta \in \mathbb{T}^3$

$$|a_p - \sqrt{\nu\rho_1}|^2 + |a_q - \sqrt{\nu\rho_2}|^2 + |a_m - \sqrt{\nu\rho_3}|^2 + \sum_{j \neq p,q,m} (1 + j^2)^s |a_j|^2 = \mathcal{O}(\nu^3). \tag{1.5}$$

Here ω is a non resonant vector in $\mathbb{R}^3$ that satisfies

$$\omega = (p^2, q^2, m^2) + \mathcal{O}(\nu^2).$$

Furthermore, this solution is linearly stable if $\mathcal{D} = \mathcal{D}_1$, linearly unstable if $\mathcal{D} = \mathcal{D}_2$ and there are $s, t \neq p, q, m$ solving (1.3).

The main theorem used in this article is KAM theorem 2.3 which is stated without proof in [5] to study a system of coupled nonlinear Schrödinger equations on the torus. We will recall the theorem in section 2 and some of results needed to prove it in section 3. The proof is presented in section 4.

2. KAM theorem

In this section, I recall the KAM theorem stated in [5] , which is proved in section 4. We consider a Hamiltonian $H = h_0 + f$, where h_0 is a quadratic Hamiltonian in normal form

$$h_0 = \Omega(\rho)\cdot r + \sum_{a\in\mathcal{Z}} \Lambda_a(\rho)|\zeta_a|^2. \tag{2.1}$$

Here

- ρ is a parameter in $\mathcal{D}$, which is a compact in the space $\mathbb{R}^n$;
- $r \in \mathbb{R}^n$ are the actions corresponding to the internal modes $(r,\theta) \in (\mathbb{R}^n \times \mathbb{T}^n, dr\wedge d\theta)$;
- $\mathcal{L}$ and $\mathcal{F}$ are respectively infinite and finite sets, $\mathcal{Z}$ is the disjoint uninon $\mathcal{L}\cup\mathcal{F}$;
- $\zeta = (\zeta_a)_{a\in\mathcal{Z}} \in \mathbb{C}^{\mathcal{Z}}$ are the external modes endowed with the standard complex symplectic structure $-id\zeta\wedge d\eta$. The external modes decomposes in a infinite part $\zeta_{\mathcal{L}} = (\zeta_a)_{a\in\mathcal{L}}$, corresponding to elliptic directions, which means $\Lambda_a \in \mathbb{R}$ for $a \in \mathcal{L}$, and a finite part $\zeta_{\mathcal{F}} = (\zeta_a)_{a\in\mathcal{F}}$, corresponding to hyperbolic directions, which means $\Im\Lambda_a \neq 0$ for $a\in\mathcal{F}$;
- $\mathcal{L}$ has a clustering structure $\mathcal{L} = \cup_{j\in\mathbb{N}}\mathcal{L}_j$, where $\mathcal{L}_j$ are finite sets of cardinality $d_j \le d < \infty$. If $a \in \mathcal{L}_j$, we denote $[a] = \mathcal{L}_j$ and $w_a = j$, for $a\in\mathcal{F}$ we set $w_a = 1$;
- the mappings

$$\Omega : \mathcal{D} \to \mathbb{R}^n, \tag{2.2}$$

$$\Lambda_a : \mathcal{D} \to \mathbb{C}, \quad a \in \mathcal{Z}, \tag{2.3}$$

  are smooth;
- $f = f(r,\theta,\zeta;\rho)$ is a perturbation, small compare to the integrable part h_0.

Linear space Let $s \ge 0$, we consider the complex weighted ℓ_2- space

$$Z_s = \{\zeta = (\zeta_a \in \mathbb{C},\, a\in\mathcal{Z}) \mid \|\zeta\|_s\} < \infty,$$

where

$$\|\zeta\|_s = \sum_{a\in\mathcal{Z}} |\zeta_a|^2 w_a^{2s}.$$

Similarly we difine

$$Y_s = \{\zeta_{\mathcal{L}} = (\zeta_a \in \mathbb{C},\, a\in\mathcal{L}) \mid \|\zeta_{\mathcal{L}}\|_s\} < \infty,$$

with the same norm. We endow Z_s and Y_s with the symplectic structure $-id\zeta\wedge d\eta$, with $\eta = \bar{\zeta}$.

A class of Hamiltonian functions. Denote $\omega = (\zeta,\eta)$. On the space

$$\mathbb{C}^n \times \mathbb{C}^n \times (Z_s \times Z_s)$$

we define the norm

$$\|(r,\theta,\omega)\|_s = \max\left(|r|, |\theta|, \|\zeta\|_s\right).$$

For $\sigma > 0$ we denote

$$\mathbb{T}^n_\sigma = \{\theta \in \mathbb{C}^n : |\Im\theta| < \sigma\}/2\pi\mathbb{Z}^n.$$

For $\sigma, \mu \in (0,1]$ and $s \geq 0$ we set

$$\mathcal{O}^s(\sigma,\mu) = \{r \in \mathbb{C}^n : |r| < \mu^2\} \times \mathbb{T}^n_s \times \{\omega \in Z_s \times Z_s : \|\zeta\|_s < \mu\}.$$

We will denote points in $\mathcal{O}^s(\sigma,\mu)$ as $x = (r,\theta,\omega)$. Let $f : \mathcal{O}^0(\sigma,\mu)\times\mathcal{D} \to \mathbb{C}$ be a C^1–function, real holomorphic in the first variable x, such that for all $\rho \in \mathcal{D}$, $x \in \mathcal{O}^s(\sigma,\mu)$:

$$\nabla_\omega f(x,\rho) \in Z_s \times Z_s$$

and

$$\nabla^2_{\omega_{\mathcal{L}}\omega_{\mathcal{L}}} f(x,\rho) \in \mathcal{L}(Y_s, Y_s)$$

are real holomorphic functions. We denote by $\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ this set of functions. For $f \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$, we define

$$|\partial^j_\rho f|_{\sigma,\mu,\mathcal{D}} = \sup_{x\in\mathcal{O}^s(\sigma,\mu);\,\rho\in\mathcal{D}} \max(|\partial^j_\rho f|, \mu\left\|\partial^j_\rho \nabla_\omega f(x,\rho)\right\|_s, \mu^2\left\|\nabla^2_{\omega_{\mathcal{L}}\omega_{\mathcal{L}}}\partial^j_\rho f(x,\rho)\right\|),$$

and

$$[f]^s_{\sigma,\mu,\mathcal{D}} = \max_j(|\partial^j_\rho f|_{\sigma,\mu,\mathcal{D}})$$

where $j = 0, 1$.

Jet functions For any $f \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$, we define its jet $f^T(x)$ as the following Taylor polynomial of f at $r = 0$ and $\omega = 0$

$$f^T(x) = f(0,\theta,0) + d_r f(0,\theta,0)\cdot r + d_\omega f(0,\theta,0)[\omega] + 1/2 d^2_\omega f(0,\theta,0)[\omega,\omega].$$

We say that $f \in \mathcal{T}^s_{res}(\sigma,\mu,\mathcal{D})$ if there exists a constant M such that for all $k \neq 0$ and all $a, b \in \mathcal{L}$ with $[a] = [b]$ then

$$e^{ik\cdot\theta}\zeta_a\eta_b \in f^T \Longrightarrow a = b \text{ or } |w_a| \leq M|k|.$$

Infinite matrices For the elliptic variables, we denote by $\mathcal{M}_s$ the set of infinite matrices $A : \mathcal{L}\times\mathcal{L} \to \mathbb{C}$ such that A maps linearly Y_s into Y_s. We provide $\mathcal{M}_s$ with the operator norm

$$|A|_s = \|A\|_{\mathcal{L}(Y_s,Y_s)}.$$

We say that a matrix $A \in \mathcal{M}_s$ is in normal form if it is block diagonal and Hermitian, i.e.

$$A^\beta_\alpha = 0 \quad \text{for } [\alpha] \neq [\beta] \quad \text{and } A^\beta_\alpha = \bar{A}^\alpha_\beta \quad \text{for } \alpha;\beta \in \mathcal{L}.$$

In particular, if $A \in \mathcal{M}_s$ is in normal form, its eigenvalues are real.

Normal form A quadratic Hamiltonian function is on normal form if it reads

$$h = \Omega(\rho) \cdot r + \langle \zeta_{\mathcal{L}}, Q\eta_{\mathcal{L}} \rangle + 1/2 \langle \omega_{\mathcal{F}}, K\omega_{\mathcal{F}} \rangle$$

for some vector function $\Omega(\rho) \in \mathbb{R}^n$, some matrix functions $Q(\rho) \in \mathcal{M}_s$ on normal form and $K(\rho)$ is a matrix $\mathcal{F} \times \mathcal{F} \to \mathbb{C}$ symmetric in the following sense:$K_\alpha^\beta = {}^t K_\beta^\alpha$.

Hypothesis A0. There exists a constant $C > 0$ such that

$$|\Lambda_a - |w_a|^2| \leq C, \, \forall a \in \mathcal{L}.$$

Hypothesis A1.

$$\begin{aligned} |\Lambda_a| &\geq \delta, \quad \forall a \in \mathcal{L}; \\ |\Im \Lambda_a| &\geq \delta, \quad \forall a \in \mathcal{F}; \\ |\Lambda_a - \Lambda_b| &\geq \delta, \quad \forall a, b \in \mathcal{Z}, \; [a] \neq [b]; \\ |\Lambda_a + \Lambda_b| &\geq \delta, \quad \forall a, b \in \mathcal{L}. \end{aligned}$$

Hypothesis A2. There exists $\delta > 0$ such that for all Ω δ–close to Ω_0 in C^1 norm and for all $k \in \mathbb{Z}^n \backslash \{0\}$:

(1) either

$$|\Omega(\rho) \cdot k| \geq \delta \quad \forall \rho \in \mathcal{D},$$

or there exists a unit vector $z = z(k) \in \mathbb{R}^n$ such that

$$(\nabla_\rho \cdot z) \left(\Omega(\rho) \cdot k \right) \geq \delta \quad \forall \rho \in \mathcal{D};$$

(2) for all $a \in \mathcal{L}$ either

$$|\Omega(\rho) \cdot k + \Lambda_a| \geq \delta \quad \forall \rho \in \mathcal{D},$$

or there exists a unit vector $z = z(k) \in \mathbb{R}^n$ such that

$$(\nabla_\rho \cdot z) \left(\Omega(\rho) \cdot k + \Lambda_a \right) \geq \delta \quad \forall \rho \in \mathcal{D};$$

(3) for all α, $\beta \in \mathcal{L}$ and $a \in [\alpha]$, $b \in [\beta]$ either

$$|\Omega(\rho) \cdot k + \Lambda_a \pm \Lambda_b| \geq \delta \quad \forall \rho \in \mathcal{D},$$

or there exists a unit vector $z = z(k) \in \mathbb{R}^n$ such that

$$(\nabla_\rho \cdot z) \left(\Omega(\rho) \cdot k + \Lambda_a \pm \Lambda_b \right) \geq \delta \quad \forall \rho \in \mathcal{D};$$

(4) for all $a, b \in \mathcal{F}$

$$|\Omega(\rho) \cdot k + \Lambda_a \pm \Lambda_b| \geq \delta.$$

Remark 2.1. Hypotheses A1, A2 are used to control the following monomials of the perturbation f

$$
\begin{array}{ll}
e^{ik\cdot\theta} & \forall k\neq 0;\\
e^{ik\cdot\theta}\zeta_a, e^{ik\cdot\theta}\eta_a & \forall a\in\mathcal{Z},\ k\in\mathbb{Z}^n;\\
e^{ik\cdot\theta}\zeta_a\zeta_b, e^{ik\cdot\theta}\eta_a\eta_b & \forall a,b\in\mathcal{Z},\ k\neq 0;\\
\zeta_a\zeta_b, \eta_a\eta_b & a,b\in\mathcal{L};\\
e^{ik\cdot\theta}\zeta_a\eta_b & a,b\in\mathcal{Z},\ k\neq 0;\\
\zeta_a\eta_b & a,b\in\mathcal{Z},\ [a]\neq[b].
\end{array}
$$

Remark 2.2. Even Hypothesis A2 is required for all $k\neq 0$, we will see that in KAM procedure we just need to control a small divisor in case the corresponding monomial appears in the jet of the perturbation terms. In Appendix, we use the preservation of mass and momentum in order to reduce the number of divisors we have to control.

Theorem 2.3 (KAM theorem)**.** *Assume that hypothesis A0, A1, A2 are satisfied and that $f\in\mathcal{T}^s_{res}(\sigma,\mu,\mathcal{D})$ with $s>1/2$. Let $\gamma>0$, there exists a constant C_0 such that if*

$$[f]^s_{\sigma,\mu,\mathcal{D}}\leq C_0\delta,\quad \varepsilon:=[f^T]^s_{\sigma,\mu,\mathcal{D}}\leq C_0\delta^{1+\gamma}, \tag{2.4}$$

then there exists a Cantor set $\mathcal{D}'\subset\mathcal{D}$ asymptotically of full measure (i.e. meas$(\mathcal{D}\setminus\mathcal{D}')\to 0$ when $\varepsilon\to 0$) and there exists a symplectic change of variables $\Phi:\mathcal{O}^s(\sigma/2,\mu/2)\to\mathcal{O}^s(\sigma,\mu)$ such that for all $\rho\in\mathcal{D}'$

$$(h_0+f)\circ\Phi = h+g$$

with $h=\Omega(\rho)\cdot r+\langle\zeta_{\mathcal{L}},Q(\rho)\eta_{\mathcal{L}}\rangle+1/2\langle\omega_{\mathcal{F}},K(\rho)\omega_{\mathcal{F}}\rangle$ on normal form, and $g\in\mathcal{T}^s_{res}(\sigma/2,\mu/2,\mathcal{D}')$ with $g^T\equiv 0$. Furthermore there exists $C>0$ such that for all $\rho\in\mathcal{D}'$

$$|\Omega-\Omega_0|\leq C\varepsilon,\quad |Q-diag\,(\Lambda_a,\ a\in\mathcal{L})\,|\leq C\varepsilon,\quad |JK-diag\,(\Lambda_a, a\in\mathcal{F})\,|\leq C\varepsilon.$$

As a dynamic consequence $\Phi\left(\{0\}\times\mathbb{T}^n\times\{0\}\right)$ is an invariant torus for h_0+f and this torus is linearly stable if and only if $\mathcal{F}=\varnothing$ (see [5]*)*

Here, the matrix J is of the form,

$$\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$$

where I is identity matrix of size $\#F$. .

3. Preparation

In order to prove theorem 2.3, we need to recall some results which are proved in [4] (Lemma 3.3 and 3.6), [5] (Lemma 3.4 and Proposition 3.5), [2] (Lemma 3.7 and 3.11).

Definition 3.1. The **Poisson brackets** of two Hamiltonian functions is defined by

$$\{f, g\} = \nabla_\theta f \cdot \nabla_r g - \nabla_r f \cdot \nabla_\theta g - i\langle \nabla_\omega f, J\nabla_\omega g\rangle.$$

Lemma 3.2. *Let $f : \mathbb{T}^n_\sigma \longrightarrow \mathbb{C}$ be a periodic, analytic function on $\mathbb{T}^n_{\sigma'}$ and continuous on $\mathbb{T}^n_\sigma$ for $0 < \sigma' < \sigma$, then we have*

$$|\hat{f}(j)| \leq Ce^{-|j|\cdot\sigma}|f(x)|_\sigma. \tag{3.1}$$

Here $|j| = |j_1| + |j_2| + ... + |j_n|$, $j = (j_1, j_2, ..., j_n)$ and $|f(x)|_\sigma = \sup_{x\in\mathbb{T}^n_\sigma} |f(x)|$

Proof. We have

$$\begin{aligned}\hat{f}(j) &= \int_0^{2\pi} f(x)e^{ijx}dx = \int_{0+i\sigma'}^{2\pi+i\sigma'} f(x)e^{ijx}dx \\ &\leq \int_0^{2\pi} |f(x)|e^{-|j|\cdot\sigma'}dx \\ &\leq 2\pi e^{-|j|\cdot\sigma'}|f(x)|_{\sigma'}\end{aligned}$$

for $0 < \sigma' < \sigma$. Since this is true for all $\sigma' < \sigma$, f is continuous on $\mathbb{T}^n_\sigma$ and $\hat{f}(j)$ is independent of σ, we have (3.1).

□

Lemma 3.3. *Let $s > 1/2$. Let $f, g \in \mathcal{T}^s(\sigma, \mu, \mathcal{D})$ be two jet functions then for any $0 < \sigma' < \sigma$ we have $f, g \in \mathcal{T}^s(\phi, \mu, \mathcal{D})$ and*

$$[\{f, g\}]^s_{\sigma',\mu,\mathcal{D}} \leq C(\sigma - \sigma')^{-n}\mu^{-2}[f]^s_{\sigma,\mu,\mathcal{D}}[g]^s_{\sigma,\mu,\mathcal{D}}.$$

Furthermore if $f, g \in \mathcal{T}^s_{res}(\sigma, \mu, \mathcal{D})$ then $f, g \in \mathcal{T}^s_{res}(\phi, \mu, \mathcal{D})$.

See [4], Lemma 4.3.

Lemma 3.4. *Let I be an open interval and let $f : I \to \mathbb{R}$ be a $\mathcal{C}^1$-function satisfying*

$$|f'(x)| \geq \delta, \qquad \forall x \in I.$$

Then

$$meas\{x \in I : |f(x)| < \varepsilon\} \leq C\frac{\varepsilon}{\delta}.$$

See [5], Lemma 2.9.

Proposition 3.5. *Let $M, N \geq 1$ and $0 < \kappa \leq \delta$. Assume Hypothesis A0, A1 , A2. Then there exists a closed subset $\mathcal{D}' = \mathcal{D}'(\kappa, N) \subset \mathcal{D}$ satisfying*

$$meas \mathcal{D} \setminus \mathcal{D}' \leq C\delta^{-1}\kappa M^2 N^{n+2},$$

such that for all $\rho \in \mathcal{D}'$, for all $|k| \leq N$ and for all $a, b \in \mathcal{Z}$

$$|\Omega(\rho) \cdot k| \geq \kappa \qquad except\, if\; k = 0, \tag{3.2}$$

$$|\Omega(\rho) \cdot k + \Lambda_a(\rho)| \geq \kappa, \tag{3.3}$$

$$|\Omega(\rho) \cdot k + \Lambda_a(\rho) + \Lambda_b(\rho)| \geq \kappa, \tag{3.4}$$

and for all $\rho \in \mathcal{D}'$, for all $|k| \leq N$ and for all $a, b \in \mathcal{L}$ such that either $[a] \neq [b]$ or $[a] = [b]$ and $w_a \leq M|k|$

$$|\Omega(\rho) \cdot k + \lambda_a(\rho) - \lambda_b(\rho)| \geq \kappa.$$

See [5], Proposition 2.8. Notice that this proposition is true for all Ω δ–close in C^1 norm from Ω_0.

Lemma 3.6. *Let $f \in \mathcal{T}^s(\sigma, \mu, \mathcal{D})$ then $f^T \in \mathcal{T}^s(\sigma, \mu, \mathcal{D})$, and*

$$\left[f^T\right]^s_{\sigma,\mu,\mathcal{D}} \leq C\,[f]^s_{\sigma,\mu,\mathcal{D}} \tag{3.5}$$

$$\left[f - f^T\right]^s_{\sigma,\mu',\mathcal{D}} \leq C\left(\frac{\mu'}{\mu}\right)^3 [f]^s_{\sigma,\mu',\mathcal{D}} \tag{3.6}$$

where C is an absolute constant and $0 < \mu' < \mu$.

See [4], Proposition 4.2.

Lemma 3.7. *Let $s > 1/2$. Let $f, g \in \mathcal{T}^s(\sigma, \mu, \mathcal{D})$ then for any $0 < \sigma' < \sigma$, $0 < \mu' < \mu < 1$ we have $f, g \in \mathcal{T}^s(\sigma', \mu', \mathcal{D})$ and*

$$[\{f, g\}]^s_{\sigma',\mu',\mathcal{D}} \leq C\left(\frac{1}{(\sigma - \sigma')^n} + \frac{1}{(\mu - \mu')^2}\right)[f]^s_{\sigma,\mu,\mathcal{D}}\,[g]^s_{\sigma,\mu,\mathcal{D}}\,.$$

See [2], Proposition 2.9 which is not exactly stated for $\mathcal{T}^s(\sigma, \mu, \mathcal{D})$, but the proof can be applied directly for this lemma.

Let

$$h = \Omega \cdot r + 1/2\,\langle \omega, A\omega\rangle = \Omega \cdot r + \langle \zeta, Q\eta\rangle + 1/2\,\langle \omega_{\mathcal{F}}, K\omega_{\mathcal{F}}\rangle \tag{3.7}$$

is on normal form, when h_0 is of that form with

$$Q_0 = diag\{\Lambda_a(\rho) : a \in \mathcal{L}\},$$

$$JK_0 = diag\{\Lambda_a(\rho) : a \in \mathcal{F}\}.$$

Denote by $Q_{[a]}$ restriction of the matrix Q to $[a] \times [a]$ and let $Q_\varnothing = 0$. Let also $H_\varnothing = 0$.

For any $a, b \in \mathcal{L} \cup \varnothing$ and $\lambda \in \mathbb{R}$, denote

$L(\rho, k, a, b)_\pm : X \longrightarrow \langle k, \Omega(\rho)\rangle + Q_{[a]}(\rho)X \pm XQ_{[b]}$,

$L(\rho, k, a, \mathcal{F}) : X \longrightarrow \langle k, \Omega(\rho)\rangle + Q_{[a]}X + XJK(\rho)$,

$L(\rho, k, \mathcal{F})_\pm : X \longrightarrow \langle k, \Omega(\rho)\rangle + K(\rho)JX \pm XJK(\rho)$.

For simplicity, we write $L(\rho, k, \Omega)$ instead of $L(\rho, k, \varnothing, \varnothing)$, and $L(\rho, k)$ in the case we want to mention all these functions. We also denote $L^0(\rho, k, a, b)_\pm$, $L^0(\rho, k, a, \mathcal{F})$, $L^0(\rho, k, \mathcal{F})_\pm$ respectively for $L(\rho, k, a, b)_\pm$, $L(\rho, k, a, \mathcal{F})$, $L(\rho, k, \mathcal{F})_\pm$ in case Ω δ–close in C^1 norm from Ω_0 and $A = A_0$.

Then, we can rewrite the conditions A1, A2 into the following way:

Hypothesis A1

Since $|\Im\Lambda_a| \geq \delta$ for all $a \in \mathcal{F}$ and since $\Lambda_b \in \mathbb{R}$:

$$\left\|L^0(\rho, k, b, \mathcal{F})\right\| \geq \delta \quad \forall b \in \mathcal{L}.$$

Since $|\Lambda_a| \geq \delta$, $\forall a \in \mathcal{L}$:

$$\left\|L^0(\rho, 0, a, \varnothing)\right\| \geq \delta.$$

Since $|\Lambda_a + \Lambda_b| \geq \delta$, $\forall a, b \in \mathcal{L}$:

$$\left\|L^0(\rho, 0, a, b)_+\right\| \geq \delta,$$

and for $a, b \in \mathcal{L}$, $[a] \neq [b]$ since $|\Lambda_a - \Lambda_b| \geq \delta$

$$\left\|L^0(\rho, 0, a, b)_-\right\| \geq \delta.$$

Hypothesis A2

For all $k \in \mathbb{Z}^n\{0\}$:

a) for $a, b \in \mathcal{L} \cup \varnothing$ either

$$\left\|L^0(\rho, k, a, b)_\pm\right\| \geq \delta,$$

or there exists a unit vector $z = z(k) \in \mathbb{R}^n$ such that

$$\left\|(\partial_\rho \cdot z)L^0(\rho, k, a, b)_\pm\right\| \geq \delta;$$

b)

$$\left\|L^0(\rho, k, \mathcal{F})_\pm\right\| \geq \delta.$$

Hence, Proposition 3.5 becomes:

Proposition 3.8. *Let $M, N \geq 1$ and $0 < \kappa \leq \delta$. Assume Hypothesis A0, A1, A2. Then there exists a closed subset $\mathcal{D}' = \mathcal{D}'(\kappa, N) \subset \mathcal{D}$ satisfying*

$$meas(D \setminus \mathcal{D}') \leq C\frac{\kappa}{\delta}M^2N^{n+2},$$

such that for all $\rho \in \mathcal{D}'$, *for all* $|k| \le N$ *and for all* $a, b \in \mathcal{L} \cup \{\varnothing\}$

$$\left\| L^0(\rho, k, \Omega) \right\| \ge \kappa \qquad \textit{except if} k = 0, \tag{3.8}$$

$$\left\| L^0(\rho, k, a, b)_+ \right\| \ge \kappa, \tag{3.9}$$

$$\left\| L^0(\rho, k, a, \mathcal{F})_\pm \right\| \ge \kappa, \tag{3.10}$$

$$\left\| L^0(\rho, k, \mathcal{F}) \right\| \ge \kappa, \tag{3.11}$$

and for all $\rho \in \mathcal{D}'$, *for all* $|k| \le N$ *and for all* $a, b \in \mathcal{L}$ *such that either* $[a] \neq [b]$ *or* $[a] = [b]$ *and* $w_a \le M|k|$

$$\left\| L^0(\rho, k, a, b)_- \right\| \ge \kappa. \tag{3.12}$$

Although this proposition is crucial in our proof, we do not work exactly with h_0 but with other normal forms h_k sufficiently close to h_0. So we need to change it a little bit.

Proposition 3.9. *Let* $M, N \ge 1$ *and* $0 < \kappa \le \delta/2$. *Assume that the Hamiltonian normal form* h (3.7) *satisfies*

$$|\partial_\rho^j (A - A_0)|_s \le \frac{\delta}{4}, \quad |\partial_\rho^j (\Omega - \Omega_0)| < \delta \tag{3.13}$$

for $j = 0, 1$ *and* $\rho \in \mathcal{D}$. *Then there exists a closed subset* $\mathcal{D}' = \mathcal{D}'(\kappa, N) \subset \mathcal{D}$ *satisfying*

$$meas(D \setminus \mathcal{D}') \le C \frac{\kappa}{\delta} M^2 N^{n+2},$$

such that for all $\rho \in \mathcal{D}'$, *for all* $|k| \le N$ *and for all* $a, b \in \mathcal{L} \cup \{\varnothing\}$

$$\| L(\rho, k, \Omega) \| \ge \kappa \qquad \textit{except if} k = 0, \tag{3.14}$$

$$\| L(\rho, k, a, b)_+ \| \ge \kappa, \tag{3.15}$$

$$\| L(\rho, k, a, \mathcal{F})_\pm \| \ge \kappa, \tag{3.16}$$

$$\| L(\rho, k, \mathcal{F}) \| \ge \kappa, \tag{3.17}$$

and for all $\rho \in \mathcal{D}'$, *for all* $|k| \le N$ *and for all* $a, b \in \mathcal{L}$ *such that either* $[a] \neq [b]$ *or* $[a] = [b]$ *and* $w_a \le M|k|$

$$\| L(\rho, k, a, b)_- \| \ge \kappa. \tag{3.18}$$

To prove this, we recall a result proved in the appendix of [1]

Lemma 3.10. *Let* $A(t)$ *be a real diagonal* $N \times N-$*matrix with diagonal components* a_j *which are* C^1 *on* $I =]-1, 1[$, *satisfying for all* $j = 1, \ldots, N$ *and for all* $t \in I$

$$a'(j) \ge \delta.$$

Let $B(t)$ be a Hermitian $N \times N-$matrix of class C^1 on I such that

$$\|B'(t)\| \leq \delta/2,$$

for all $t \in I$. Then

$$meas\{t \in I : \min_{\lambda(t) \in \sigma(A(t)+B(t))} |\lambda(t)| \leq \kappa\} \leq CN\frac{\kappa}{\delta},$$

where C is a constant independent of N.

Proof of Proposition 3.9. The estimate (3.14) is true by Proposition 3.5. For (3.16) and (3.17), by Hypothesis A1, A2, we already have

$$\left\|L^0(\rho, k, a, \mathcal{F})_\pm\right\| \geq \delta, \quad \left\|L^0(\rho, k, \mathcal{F})\right\| \geq \delta.$$

By assumption (3.13),

$$\left\|L^0(\rho, k, a, \mathcal{F})_\pm - L(\rho, k, a, \mathcal{F})_\pm\right\| \leq \delta/2, \quad \left\|L^0(\rho, k, \mathcal{F}) - L(\rho, k, \mathcal{F})\right\| \leq \delta/2.$$

Hence

$$\|L(\rho, k, a, \mathcal{F})_\pm\| \geq \delta/2, \quad \|L(\rho, k, \mathcal{F})\| \geq \delta/2.$$

For (3.15) (similar as (3.18)), for $a, b \in \mathcal{L}$ we have $L(\rho, k, a, b)_+$ is Hermitian matrix operator for all $k \in \mathbb{R}^n$. Eigenvalues of $L^0(\rho, k, a, b)_+$ are of form

$$\nu(\rho) = \langle k, \Omega \rangle + \Lambda_a + \Lambda_b.$$

By Hypothesis A2, we have either $|\nu(\rho)| \geq \delta$ or there exists a unit vector $z = z(k) \in \mathbb{R}^n$ such that $(\partial_\rho \cdot z)\nu(\rho) \geq \delta$. To apply Lemma 3.10 we consider Hermitian matrix operator $e^{\rho \cdot z}L(\rho, k, a, b)$. Then we have

$$(\partial_\rho \cdot z)(e^{\rho \cdot z}\nu) \geq e^{\rho \cdot z}\delta,$$

and

$$\left\|e^{\rho \cdot z}(L(\rho, k, a, b)_+ - L^0(\rho, k, a, b)_+)\right\| \leq e^{\rho \cdot z}\delta/2$$

for all $a, b \in \mathcal{L} \cup \{\varnothing\}$. Hence by Lemma 3.10,

$$meas\{t \in I : \min_{a,b} \min_{\lambda(t) \in \sigma(L(\rho,k,a,b)_+)} |\lambda(t)| \leq \kappa\} \leq CN\frac{\kappa}{\delta}$$

for $|k| \leq N$ fixed, i.e. there exists a subset $\mathcal{D}_k(\kappa, N) \subset \mathcal{D}$ such that for all $|k| \leq N$ and $a, b \in \mathcal{L} \cup \{\varnothing\}$:

$$meas(\mathcal{D} \setminus \mathcal{D}_k(\kappa, N)) \leq CN\frac{\kappa}{\delta}$$

and on $\mathcal{D}_k(\kappa, N)$,

$$\|L(\rho, k, a, b)_+\| \geq \kappa.$$

Now sum up all together, we get

$$meas(\mathcal{D} \setminus \bigcup_{|k|\leq N} \mathcal{D}_k) \leq CN^{n+2}\frac{\kappa}{\delta}.$$

□

Lemma 3.11. *Let $S \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$, and let $\sigma' < \sigma$ and $\mu' < \mu \leq 1$. If*

$$[S]^s_{\sigma,\mu,\mathcal{D}} \leq \frac{1}{C} min\left(\sigma - \sigma', \mu - \mu'\right)$$

then

- *the Hamiltonian flow map $\Phi^t = \Phi^t_S$, for $|t| \leq 1$ is a C^1-map*
$$\mathcal{O}^s(\sigma',\mu') \times \mathcal{D} \to \mathcal{O}^s(\sigma,\mu)$$
which is real holomorphic and symplectic for any fixed $\rho \in \mathcal{D}$. Moreover,
$$\left\|\partial^j_\rho \left(\Phi^t(x,\rho) - x\right)\right\| \leq C[S]^s_{\sigma,\mu,\mathcal{D}}$$
and
$$\left\|\partial^j_\rho \left(d\Phi^t(x,\rho) - I\right)\right\| \leq C[S]^s_{\sigma,\mu,\mathcal{D}}$$
for any $x \in \mathcal{O}^s(\sigma',\mu')$ and $j = 0,1$;
- *$f \circ \Phi^t \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$ for $|t| \leq 1$ and*

$$[f \circ \Phi^t]^s_{\sigma',\mu',\mathcal{D}} \leq C[f]^s_{\sigma,\mu,\mathcal{D}}, \tag{3.19}$$

See [2], Proposition 2.11 for the proof.

Scheme of the proof of KAM theorem. We would like to construct sequences of Hamiltonian normal forms h_k, perturbations f_k defined on domains $\mathcal{O}^s(\sigma_k,\mu_k)\times\mathcal{D}_k$ and symplectic changes of variables $\Phi_k : \mathcal{O}^s(\sigma_k,\mu_k) \to \mathcal{O}^s(\sigma_{k-1},\mu_{k-1})$ such that

- the normal form $h_k = \Omega_k \cdot r + \langle\omega, A_k\omega\rangle$ stays closed to h_0 when $k \to \infty$, i.e.
$$|\partial^j_\rho(\Omega_k - \Omega_0)| \leq C\delta^{1+\alpha} \qquad |\partial^j_\rho(A_k - A_0)| \leq C\delta^{1+\alpha} \quad j = 0,1;$$
- the perturbation f_k is in good class, i.e. $f_k \in \mathcal{T}^s_{\sigma_k,\mu_k,\mathcal{D}_k}$ and
$$[f^T_k]^s_{\sigma_k,\mu_k,\mathcal{D}_k} \leq \varepsilon_k \sim \varepsilon^\beta_{k-1}$$
with $\beta > 1$;
- for all $\rho \in \mathcal{D}_k$
$$(h_{k-1} + f_{k-1}) \circ \Phi_k = h_k + f_k;$$
- the symplectic change of variable Φ_k stays closed to identity map and $\Phi^k_N = \Phi_{k+1} \circ \ldots \Phi_N$ close to identity map too, for $N > j \geq 0$.

- $\sigma_k > \sigma/2$, $\mu_k > \mu/2$ and $\mathcal{D}_k$ stays closed to $\mathcal{D}$;
- h_k, f_k, Φ_k, σ_k, μ_k and $\mathcal{D}_k$ converge respectively to desired h, g, Φ, $\sigma/2$, $\mu/2$ and $\mathcal{D}'$.

At each step of this procedure, we need to solve equation

$$(h+f)\circ\Phi = h' + f'.$$

Normally, we will try to find a jet function S, such that $\Phi = \Phi_S^1$ and

$$\{h, S\} + f^T = h^+ + R \tag{3.20}$$

where h^+ is a normal form and R is a very small error term. However, by this way, the perturbation f' will be of form

$$f' = f - f^T + \{f - f^T, S\} + \{f^T, S\} + \int_0^1 \{(1-t)(h^+ + R) + tf^T, S\} \circ \Phi_S^t dt + R.$$

Its jet function $(f')^T$ is of order $O(\varepsilon)$ not $O(\varepsilon^\beta)$ with $\beta > 1$ as we want. The problem here is $\{f - f^T, S\}$. To deal with this, we shall solve non linear homological equation

$$\{h, S\} + f^T + \{f - f^T, S\}^T = h^+ + R.$$

This equation may look difficult because of its non-linearity, but luckily, it can be solved easily after solving 3.20. In both equations, we need to estimate S, h^+, R, f' and Φ_S^t up to $t = 1$. The estimation of h^+ and R are directly verified by the estimation of f. The Proposition 3.9 allows us to control S, and hence combine with Lemma 3.11 control Φ_S^t. As we will se that

$$(f')^T = \{f^T, S\} + \left(\int_0^1 \{(1-t)(h^+ + R) + tf^T, S\} \circ \Phi_S^t dt\right)^T + R^T.$$

By Lemma 3.3 we can control the first term, and by Lemma 3.11 we can control the second term. Finally, we need to study the limit when k rises to infinity.

In next section, we understand all functions as functions of ρ and we omit ρ in their presentations unless necessary.

4. Homological equation

Let h is a Hamiltonian normal form satisfying assumptions (3.13), i.e. $\forall \rho \in \mathcal{D}$

$$|\partial_\rho^j (A - A_0)| \leq \frac{\delta}{4}, \quad |\partial_\rho^j (\Omega - \Omega_0)| < \delta$$

for $j = 0, 1$. Let $f \in \mathcal{T}_{res}^s(\sigma, \mu, D)$, we will construct a jet function S that solves the non-linear homological equation

$$\{h, S\} + \{f - f^T, S\} + f^T = h^+ + R, \tag{4.1}$$

where h_+ is normal form and R plays as an error term.

In order to do this, we shall start by analysing the homological equation

$$\{h, S\} + f^T = h^+ + R. \tag{4.2}$$

Let us write

$$f^T(\theta, r, \omega) = f_r(r, \theta) + \langle f_\omega, \omega \rangle + \tfrac{1}{2} \langle f_{\omega\omega} \omega, \omega \rangle,$$

where

$$\begin{aligned}
f_r(r, \theta) &= f_\theta(\theta) + f_r(\theta) \cdot r, \\
\langle f_\omega, \omega \rangle &= \langle f_{\zeta_\mathcal{L}}(\theta), \zeta_\mathcal{L} \rangle + \langle f_\eta(\theta), \eta_\mathcal{L} \rangle + \langle f_{\omega_\mathcal{F}}(\theta), \omega_\mathcal{F} \rangle, \\
\tfrac{1}{2} \langle f_{\omega\omega} \omega, \omega \rangle &= \tfrac{1}{2} \langle f_{\eta_\mathcal{L}\eta_\mathcal{L}}(\theta) \eta_\mathcal{L}, \eta_\mathcal{L} \rangle + \tfrac{1}{2} \langle f_{\eta_\mathcal{L}\eta_\mathcal{L}}(\theta) \eta_\mathcal{L}, \eta_\mathcal{L} \rangle + \langle f_{\eta_\mathcal{L}\eta_\mathcal{L}}(\theta) \eta_\mathcal{L}, \eta_\mathcal{L} \rangle \\
&\quad + \tfrac{1}{2} \langle f_{\eta_\mathcal{L}\omega_\mathcal{F}}(\theta) \eta_\mathcal{L}, \omega_\mathcal{F} \rangle + \tfrac{1}{2} \langle f_{\eta_\mathcal{L}\omega_\mathcal{F}}(\theta) \eta_\mathcal{L}, \omega_\mathcal{F} \rangle + \tfrac{1}{2} \langle f_{\omega_\mathcal{F}\omega_\mathcal{F}}(\theta) \omega_\mathcal{F}, \omega_\mathcal{F} \rangle.
\end{aligned}$$

Let

$$S(\theta, r, \omega) = S^T(\theta, r, \omega) = S_r(r, \theta) + \langle S_\omega, \omega \rangle + \frac{1}{2} \langle S_{\omega\omega}(\theta) \omega, \omega \rangle,$$

then the Poisson bracket equals

$$\begin{aligned}
&-\Omega \cdot \partial_\theta S_r(\theta, r) - \Omega \cdot \partial_\theta \langle S_\omega, \omega \rangle - \tfrac{1}{2} \Omega \cdot \partial_\theta \langle S_{\omega\omega}(\theta) \omega, \omega \rangle + i \langle AJS_\omega, \omega \rangle + \\
&\qquad + i \tfrac{1}{2} \langle AJS_{\omega\omega}(\theta) \omega, \omega \rangle - i \tfrac{1}{2} \langle S_{\omega\omega}(\theta) JA\omega, \omega \rangle.
\end{aligned}$$

Accordingly, the homological equation decomposes into three linear equations

$$\Omega \cdot \partial_\theta S_r(\theta, r) = f_r(\theta, r) + h_r^+(r, \theta) - R_r, \tag{4.3}$$

$$\Omega \cdot \partial_\theta S_\omega(\theta) - iAJS_\omega(\theta,) = f_\omega(\theta) + h_\omega^+(\theta) - R_\omega(\theta), \tag{4.4}$$

$$\Omega \cdot \partial_\theta S_{\omega\omega}(\theta) + iAJS_{\omega\omega}(\theta) - iS_{\omega\omega}(\theta) JA = f_{\omega\omega}(\theta) + h_{\omega\omega}^+(\theta) - R_{\omega\omega}(\theta) \tag{4.5}$$

Denote

$$\chi := |\partial_\rho \Omega_0| + sup_{a \in \mathcal{L}} |\partial_\rho \Lambda_a| + sup_{b \in \mathcal{F}} |\partial_\rho \Lambda_b|.$$

Assume that $\chi \leq C\delta$, where C is an independent constant.

The first equation. Write it in the Fourier form so that we get

$$L(k,\Omega)\hat{S}_r(k) = \langle k,\Omega\rangle \hat{S}_r(k) = -i\hat{f}_r(k), \tag{4.6}$$

for $0 < |k| \le N$, and $h_r(\theta) = \hat{f}_r(0)$. The error term

$$R_r = \sum_{|k|\ge N} \hat{f}_r(k)e^{ik\cdot\theta}.$$

By Lemma 3.2 we get

$$|\partial_\rho^j R_r|_{\sigma'} \le \sum_{|k|\ge N} e^{-(\sigma-\sigma')|k|}|\partial_\rho^j f_r|_\sigma \le \frac{e^{-(\sigma-\sigma')N}}{(\sigma-\sigma')^n}|\partial_\rho^j f_r|_\sigma,$$

where $j = 0, 1$. All the error terms in other equations are treated in the same way, so we do not care about them again.

By Proposition 3.9, for any $\rho \in \mathcal{D}'(\kappa, N)$:

$$\|L(k,\Omega)\| \ge \kappa,$$

thus

$$|\hat{S}_r(k)| \le \frac{|\hat{f}_r(k)|}{\kappa}.$$

By Lemma 3.2, we get

$$|\hat{S}_r(k)| \le C\frac{e^{-|k|\sigma}}{\kappa}\sup_{\theta\in\mathbb{T}^n_\sigma}|f_r|.$$

Since

$$S_r = \sum_{|k|\le N} \hat{S}_r(k)e^{ik\cdot\theta}$$

we get

$$|S_r|_{\sigma'} \le C\sum_{|k|\le N} e^{-(\sigma-\sigma')|k|}\frac{|f_r|_\sigma}{\kappa} \le C\frac{1}{(\sigma-\sigma')^n\kappa}|f_r|_\sigma.$$

Take the derivative of the equation 4.6

$$L(k,\Omega)\cdot\partial_\rho\hat{S}_r(k) = -i\partial_\rho\hat{f}_r(k) - \partial_\rho L(k,\Omega)\cdot\hat{S}_r(k).$$

Since

$$\|\partial_\rho L(k,\Omega)\| \le \left\|\partial_\rho L^0(k,\Omega)\right\| + \left\|\partial_\rho(L^0(k,\Omega) - L(k,\Omega))\right\| \le N\chi + |k|\delta \le N(\chi+\delta) \le CN\delta,$$

then

$$|\partial_\rho\hat{S}_r(k)| \le \frac{1}{\kappa}|\partial_\rho\hat{f}_r(k)| + \frac{CN\delta}{\kappa^2}|\hat{f}_r(k)|.$$

Hence, using Lemma 3.2, we get

$$|\partial_\rho S_r|_{\sigma'} \leq \sum_{|k|\leq N} |\partial\rho \hat{S}(k)|e^{|k|\sigma'} \leq C\sum_{|k|\leq N} e^{-|k|(\sigma-\sigma')}\left(\frac{1}{\kappa}|\partial_\rho f_r|_\sigma + \frac{\delta N}{\kappa^2}|f_r|_\sigma\right)$$
$$\leq C\frac{1}{(\sigma-\sigma')^n}\left(\frac{1}{\kappa}|\partial_\rho f_r|_\sigma + \frac{\delta N}{\kappa^2}|f_r|_\sigma\right).$$

The second equation. The equation decomposes into

$$\Omega\cdot\partial_\theta S_{\zeta_\mathcal{L}} + iQS_{\zeta_\mathcal{L}} = f_{\zeta_\mathcal{L}} + h^+_{\zeta_\mathcal{L}} - R_{\zeta_\mathcal{L}}, \tag{4.7}$$
$$\Omega\cdot\partial_\theta S_{\zeta_\mathcal{L}} + iQS_{\eta_\mathcal{L}} = f_{\zeta_\mathcal{L}} + h^+_{\zeta_\mathcal{L}} - R_{\zeta_\mathcal{L}}, \tag{4.8}$$
$$\Omega\cdot\partial_\theta S_\mathcal{F} + iKJS_\mathcal{F} = f_\mathcal{F} + h^+_\mathcal{F} - R_\mathcal{F}. \tag{4.9}$$

Let consider the equation 4.7 (the others are similar). Write it in Fourier fomular, we get

$$\langle k,\Omega\rangle\hat{S}_{\zeta_\mathcal{L}}(k) + Q\hat{S}_{\zeta_\mathcal{L}}(k) = -i\left(\hat{f}_{\zeta_\mathcal{L}}(k) + \hat{h}^+_{\zeta_\mathcal{L}}(k) - \hat{R}_{\zeta_\mathcal{L}}(k)\right).$$

This equation decomposes into its components over the blocks $[a]$, which takes the form

$$L(k,a,\varnothing)_+\hat{S}_{[a]}(k) = \langle k,\Omega\rangle\hat{S}_{[a]}(k) + Q_{[a]}\hat{S}_{[a]}(k) = -i\hat{f}_{[a]}(k),$$

for $|k|\leq N$. Argument now is similar to the first equation. Thanks to Proposition 3.9, for any $\rho\in\mathcal{D}'(\kappa,N)$ we have estimate

$$\|L(k,a,\varnothing)_+\|\geq\kappa$$

i.e.

$$|\hat{S}_{[a]}(k)| \leq \frac{|\hat{f}_{[a]}(k)|}{\kappa} \leq C\frac{e^{-|k|\sigma}}{\kappa}|f_{[a]}|_\sigma$$

which leads us to estimate

$$|S_{[a]}|_{\sigma'} \leq C\frac{1}{(\sigma-\sigma')^n\kappa}|f_{[a]}|_\sigma$$

for any $0<\sigma'<\sigma$. For $\partial_\rho S_{[a]}$, again we have estimate

$$\|\partial_\rho L(k,a,\varnothing)_+\|\leq CN\delta$$

thus

$$|\partial_\rho\hat{S}_{[a]}(k)| \leq \frac{1}{\kappa}|\partial_\rho\hat{f}_{[a]}(k)| + \frac{CN\delta}{\kappa^2}|\hat{f}_{[a]}(k)|.$$

In the end

$$|\partial_\rho S_{[a]}|_{\sigma'} \leq C\frac{1}{(\sigma-\sigma')^n}(\frac{1}{\kappa}|\partial_\rho f_{[a]} + \frac{N\delta}{\kappa^2}|f_{[a]}|_\sigma).$$

The third equation. The equation decomposes into its components:

$$\Omega \cdot \partial_\theta S_{\zeta_\mathcal{L}\zeta_\mathcal{L}} + iQS_{\zeta_\mathcal{L}\zeta_\mathcal{L}} + iS_{\zeta_\mathcal{L}\zeta_\mathcal{L}}Q = f_{\zeta_\mathcal{L}\zeta_\mathcal{L}} + h^+_{\zeta_\mathcal{L}\zeta_\mathcal{L}} - R_{\zeta_\mathcal{L}\zeta_\mathcal{L}}, \tag{4.10}$$

$$\Omega \cdot \partial_\theta S_{\zeta_\mathcal{L}\eta_\mathcal{L}} + iQS_{\zeta_\mathcal{L}\eta_\mathcal{L}} - iS_{\zeta_\mathcal{L}\eta_\mathcal{L}}Q = f_{\zeta_\mathcal{L}\eta_\mathcal{L}} + h^+_{\zeta_\mathcal{L}\eta_\mathcal{L}} - R_{\zeta_\mathcal{L}\eta_\mathcal{L}}, \tag{4.11}$$

$$\Omega \cdot \partial_\theta S_{\zeta_\mathcal{L}\omega_\mathcal{F}} + iQS_{\zeta_\mathcal{L}\omega_\mathcal{F}} + S_{\zeta_\mathcal{L}\omega_\mathcal{F}}JK = f_{\zeta_\mathcal{L}\omega_\mathcal{F}} + h^+_{\zeta_\mathcal{L}\omega_\mathcal{F}} - R_{\zeta_\mathcal{L}\omega_\mathcal{F}}, \tag{4.12}$$

$$\Omega \cdot \partial_\theta S_{\omega_\mathcal{F}\omega_\mathcal{F}} + KJS_{\omega_\mathcal{F}\omega_\mathcal{F}} - S_{\omega_\mathcal{F}\omega_\mathcal{F}}JK = f_{\omega_\mathcal{F}\omega_\mathcal{F}} + h^+_{\omega_\mathcal{F}\omega_\mathcal{F}} - R_{\omega_\mathcal{F}\omega_\mathcal{F}}, \tag{4.13}$$

and the similar equations with $\zeta_\mathcal{L}$ is replaced by $\eta_\mathcal{L}$, which are treated in the same way.

Equation 4.10. Written in the Fourier variables, it becomes

$$\langle k, \Omega(\rho)\rangle \hat{S}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}(k) + Q\hat{S}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}(k) + \hat{S}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}(k)Q = -i\hat{f}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}(k)$$

for $|k| \le N$. Here we get $h^+_{\zeta_\mathcal{L}\zeta_\mathcal{L}} = 0$, $\hat{R}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}(k) = \hat{f}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}(k)$ for $|k| > N$ and $\hat{R}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}(k) = 0$ for $|k| \le N$. This equation decomposes into its components over the product blocks $[a] \times [b]$, which takes the form

$$L(k,a,b)_+\hat{S}^{[b]}_{[a]} = \langle k, \Omega(\rho)\rangle \hat{S}^{[b]}_{[a]}(k) + Q_{[a]}\hat{S}^{[a]}_{[b]}(k) + \hat{S}^{[a]}_{[b]}(k)Q_{[b]} = -i\hat{f}^{[b]}_{[a]}(k). \tag{4.14}$$

Since $\|L(k,a,b)_+\| \ge \kappa$, $\forall \rho \in \mathcal{D}$ we have

$$|\hat{S}^{[b]}_{[a]}(k)| \ge \frac{1}{\kappa}|\hat{f}^{[b]}_{[a]}(k)|.$$

Therefor we obtain a solution satisfying for any $|\Im\theta| \le \sigma'$

$$|S(\theta)_{\zeta_\mathcal{L}\zeta_\mathcal{L}}| \le C\frac{1}{(\sigma-\sigma')^n \cdot \kappa}|f(\theta)_{\zeta_\mathcal{L}\zeta_\mathcal{L}}|.$$

For the estimating of $\partial_\rho S_{\zeta_\mathcal{L}\zeta_\mathcal{L}}$, we have

$$L(k,a,b)_+\partial_\rho\hat{S}^{[b]}_{[a]}(k) = -i\partial_\rho\hat{f}^{[b]}_{[a]}(k) - \partial_\rho L(k,a,b)_+\hat{S}^{[b]}_{[a]}(k). \tag{4.15}$$

Since

$$\left\|\partial_\rho(L(k,a,b)_+ - L^0(k,a,b)_+)\right\| \le k\delta,$$

we have

$$\|\partial_\rho L(k,a,b)_+\| \le \left\|\partial_\rho(L(k,a,b)_+ - L^0(k,a,b)_+)\right\| + \left\|\partial_\rho L^0(k,a,b)_+\right\| \le N\chi + |k|\delta \le CN\delta.$$

Now we consider this equation as the equation 4 where $i\hat{f}^{[b]}_{[a]}$ is replaced by $-i\partial_\rho\hat{f}^{[b]}_{[a]}(k) - \partial_\rho L(k,a,b)\hat{S}^{[b]}_{[a]}(k)$, then we get the desired estimate

$$|\partial_\rho\hat{S}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}| \le C\frac{1}{(\sigma-\sigma')^n}\left(\frac{1}{\kappa}|\partial_\rho\hat{f}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}| + \frac{N\delta}{\kappa^2}|\hat{f}_{\zeta_\mathcal{L}\zeta_\mathcal{L}}|\right).$$

The others are solved in the same way and give us the same estimation except the case $k = 0$ and $[a] = [b]$ in the equation 4.11. In this case, we get $h_a^b = \hat{f}_a^b(0)$. In the end, we get the solution S the normal form h^+ and the error term R such that

$$|S|_{\sigma'} \leq C\frac{1}{(\sigma-\sigma')^n\kappa}|f^T|_\sigma, \tag{4.16}$$

$$|\partial_\rho S|_{\sigma'} \leq C\frac{1}{(\sigma-\sigma')^n}\left(\frac{1}{\kappa}|\partial_\rho f^T|_\sigma + \frac{\delta N}{\kappa^2}|f^T|_\sigma\right), \tag{4.17}$$

$$|\partial_\rho^j R|_{\sigma'} \leq C\frac{e^{-(\sigma-\sigma')N}}{(\sigma-\sigma')^n}|\partial_\rho^j f^T|_\sigma, \tag{4.18}$$

$$|\partial_\rho^j h^+|_\sigma \leq C|\partial_\rho^j f^T|_\sigma, \tag{4.19}$$

for $j = 0, 1$, and C is an absolute constant. We would like to have $\partial_\rho^j S$ and $\partial_\rho^j R$ small and controlled, which is dependent on the choice of κ, σ' and N. A specific choice of such parameters would be given later, but we can see that for $\partial_\rho^j R$ we just need to take N sufficiently large. Since $[f]^s_{\sigma,\mu,\mathcal{D}} \leq \varepsilon = \delta^{1+\gamma}$, the terms $\frac{|f^T|}{\kappa}$ and $\frac{|\partial_\rho f^T|}{\kappa}$ in estimates of S and $\partial_\rho S$ are small and controlled if $\delta > \kappa > \varepsilon$. The only remained problem is the term $\frac{\delta|f^T|_\sigma}{\kappa^2}$ which is small when $\delta > \kappa > \delta^{1+\gamma/2}$.

Proposition 4.1. *Let h is a Hamiltonian normal form satisfying* (3.13), *and $f \in \mathcal{T}^s_{res}(\sigma, \mu, \mathcal{D})$ with $s > 1/2$. Assume aslo that Hypothesis A0, A1, A2 are satisfied, then there exist a closed subset $\mathcal{D}' \subset \mathcal{D}$ such that*

$$meas(\mathcal{D} \setminus \mathcal{D}') \leq C\frac{\kappa}{\delta}M^2N^{n+2}$$

and there exist jet function S, R and h^+ verifying, for $\rho \in \mathcal{D}'$

$$\{h, S\} + f^T = h_+ + R$$

and

$$[S]^s_{\sigma',\mu,\mathcal{D}'} \leq C\frac{1}{(\sigma-\sigma')^n}\frac{N\delta}{\kappa}[f^T]^s_{\sigma,\mu,\mathcal{D}} \tag{4.20}$$

$$[R]^s_{\sigma',\mu,\mathcal{D}'} \leq C\frac{e^{-(\sigma-\sigma')N}}{(\sigma-\sigma')^n}[f^T]^s_{\sigma,\mu,\mathcal{D}} \tag{4.21}$$

$$[h^+]^s_{\sigma',\mu,\mathcal{D}'} \leq [f^T]^s_{\sigma,\mu,\mathcal{D}}. \tag{4.22}$$

for any $0 < \sigma' < \sigma$.

Here we use $\frac{1}{\kappa} < \frac{1}{\kappa} + \frac{N\delta}{\kappa^2} < C\frac{N\delta}{\kappa^2}$, which is satisfied by the choice of parameters.

Now, we turn back to the non linear homological equation

The non linear homological equation

$$\{h, S\} + f^T + \{f - f^T, S\}^T = h^+ + R.$$

Let $S = S_0 + S_1 + S_2$, with S_0, S_1, S_2 are jet functions start with oder $0, 1, 2$ of r, ω. The equation now decomposes into three equations:

$$\{h, S_0\} + f^T = h_1^+ + R_1, \tag{4.23}$$

$$\{h, S_1\} + f_1^T = h_2^+ + R_2, \qquad \{f - f^T, S_0\} = f_1, \tag{4.24}$$

$$\{h, S_2\} + f_2^T = h_3^+ + R_3, \qquad \{f_1 - f_1^T, S_1\} = f_2. \tag{4.25}$$

Let

$$X = C\left(\frac{1}{(\sigma-\sigma')^n} + \frac{1}{(\mu-\mu')^2}\right), \qquad Y = \frac{1}{(\sigma-\sigma')^n \mu^2}.$$

Let $\sigma' < \sigma_3 < \sigma_2 < \sigma_1 < \sigma$, $\mu' < \mu_3 < \mu_2 < \mu_1 < \mu$ and $\mathcal{D}' \subset \mathcal{D}_1 \subset \mathcal{D}$. Denote $\varepsilon = [f^T]^s_{\sigma,\mu,\mathcal{D}}$ by Lemma 3.7 and Proposition 4.1, we have

$$[f_1]^s_{\sigma_2,\mu_2,\mathcal{D}_1} \le X[f]^s_{\sigma_1,\mu_1,\mathcal{D}_1}[S]^s_{\sigma_1,\mu_1,\mathcal{D}_1} \le XY\delta\frac{\delta N}{\kappa^2}[f^T]^s_{\sigma,\nu,\mathcal{D}} = XYN\frac{\delta^2}{\kappa^2}\varepsilon.$$

By Lemma 3.6, $[f_1^T]^s_{\sigma_2,\mu_2,\mathcal{D}_1}$ have the same bound as $[f_1]^s_{\sigma_2,\mu_2,\mathcal{D}_1}$, hence

$$\varepsilon_1 = [f_1^T]^s_{\sigma_2,\mu_2,\mathcal{D}_1} \le XYN\frac{\delta^2}{\kappa^2}\varepsilon$$

Similar, we get

$$\varepsilon_2 = [f_2^T]^s_{\sigma',\mu',\mathcal{D}'} \le XYN\frac{\delta^2}{\kappa^2}\varepsilon_1$$

Hence

$$[S_i]^s_{\sigma',\mu',\mathcal{D}'} \le CY\frac{N\delta}{\kappa^2}\varepsilon_i, \tag{4.26}$$

$$[R_i]^s_{\sigma',\mu',\mathcal{D}'} \le C\frac{e^{-(\sigma-\sigma')N}}{(\sigma-\sigma')^n}\varepsilon_i, \tag{4.27}$$

$$[h_{+,i}]^s_{\sigma',\mu',\mathcal{D}'} \le C\varepsilon_i \tag{4.28}$$

for $i = 0, 1, 2$. Putting each term respectively together, we find that

$$\varepsilon + \varepsilon_1 + \varepsilon_2 \le \left(1 + XYN\frac{\delta^2}{\kappa^2}\right)^3 \varepsilon \le CX^3Y^3N^3\frac{\delta^6\varepsilon}{\kappa^6}, \tag{4.29}$$

and

$$[S]^s_{\sigma',\mu',\mathcal{D}'} \le CX^3Y^4N^4\frac{\delta^7\varepsilon}{\kappa^8} \tag{4.30}$$

$$[R]^s_{\sigma',\mu',\mathcal{D}'} \le C\frac{e^{-(\sigma-\sigma')N}}{(\sigma-\sigma')^n}X^3Y^3N^3\frac{\delta^6\varepsilon}{\kappa^6} \tag{4.31}$$

$$[h^+]^s_{\sigma',\mu',\mathcal{D}'} \le CX^3Y^3N^3\frac{\delta^6\varepsilon}{\kappa^6}. \tag{4.32}$$

Proposition 4.2. *Let h is a Hamiltonian normal form satisfying* (3.13)*, and $f \in \mathcal{T}^s_{res}(\sigma, \mu, \mathcal{D})$ with $s > 1/2$. Assume aslo that Hypothesis A0, A1, A2 are satisfied, then there exist a closed subset $\mathcal{D}' \subset \mathcal{D}$ such that*

$$meas(\mathcal{D} \setminus \mathcal{D}') \leq C \frac{\kappa}{\delta} M^2 N^{n+2}$$

and there exist jet functions S, R and h^+ verifying, for $\rho \in \mathcal{D}'$

$$\{h, S\} + f^T + \{f - f^T, S\} = h_+ + R$$

and S, R, h^+ satisfies estimates (4.30) *to* (4.32) *for any $0 < \sigma' < \sigma$, $0 < \nu' < \nu$.*

5. Proof of the KAM theorem

The theorem 2.3 is proved by an interactive KAM procedure. We first describe the general step of this KAM procedure.

The KAM step. Let h be a Hamiltonian normal form

$$h = \Omega \cdot r + \frac{1}{2}\langle \omega, A\omega\rangle$$

such that

$$|\partial_\rho^j (A - A_0)| \leq \frac{\delta}{4}, \quad |\partial_\rho (\Omega - \Omega_0)| \leq \delta$$

for $j = 0, 1$. Let $f \in \mathcal{T}^s_{res}(\sigma, \mu, \mathcal{D})$ be a (small) Hamiltonian perturbation. Let $S = S^T \in \mathcal{T}^s_{res}(\sigma', \mu', \mathcal{D}')$ be the solution of the homological equation

$$\{h, S\} + f^T + \{f - f^T, S\}^T = h^+ + R.$$

Then defining

$$\tilde{h} := h + h^+,$$

we get

$$h \circ \Phi_S = \tilde{h} + \tilde{f}$$

with

$$\tilde{f} = f - f^T - \{f - f^T, S\}^T + \{f^T, S\} + \int_0^1 \{(1-t)(h^+ + R) + tf^T, S\} \circ \Phi_S^t dt + R.$$

We first estimate the new perturbation.

Esmating $\tilde{f}^T$.

$$\tilde{f}^T = \{f^T, S\} + \left(\int_0^1 \{(1-t)(h^+ + R) + tf^T, S\} \circ \Phi_S^t dt\right)^T + R.$$

For the first term, thank to (4.30) and Lemma 3.3, we get

$$[\{f^T, S\}]^s_{\sigma', \mu', \mathcal{D}} \leq C\frac{1}{(\sigma - \sigma')^n \cdot \mu^2}[f^T]^s_{\sigma'', \mu', \mathcal{D}'}|S|_{\sigma'', \mu', \mathcal{D}'} \leq CX^3Y^5N^4\frac{\delta^7\varepsilon^2}{\kappa^8},$$

here we choose $0 < \sigma' < \sigma'' = \frac{\sigma+\sigma'}{2} < \sigma$.

For the second term, let $g^t = (1-t)(h^+ + R) + tf^T$. Thank to all estimate of h^+ and R we have done:

$$[\{g^t, S\}]^s_{\sigma', \mu', \mathcal{D}'} \leq CX^3Y^3N^3\frac{\delta^6}{\kappa^6}[\{f^T, S\}]^s_{\sigma', \mu', \mathcal{D}'},$$

By Lemma 3.11,

$$[\{g^t, S\} \circ \Phi_S^t]^s_{\sigma', \mu', \mathcal{D}'} \leq C[\{g^t, S\}]^s_{\sigma, \mu, \mathcal{D}} \leq CX^6Y^8N^7\frac{\delta^{13}\varepsilon^2}{\kappa^{14}},$$

For estimating R, we choose $N = -8(\sigma - \sigma')^{-1} ln\varepsilon$ then for ε sufficiently small we get a good estimate

$$[R]^s_{\sigma',\mu',\mathcal{D}'} \leq \varepsilon^2.$$

In the end, since X, Y, $N > 1$ and $\delta > \kappa$, we get

$$\varepsilon_+ = [\tilde{f}^T]^s_{\sigma',\mu',\mathcal{D}'} \leq CX^6Y^8N^7\frac{\delta^{13}\varepsilon^2}{\kappa^{14}}.$$

Here we use C as a constant which is the maximum of constants should appear in each estimate. Sine δ is very small compare to σ, μ, and 1 , the parameters relate to X, Y, N appear in each estimate are very small compare to $\frac{1}{\delta}$ and easy to deal with. By the definitions of X, Y and N, these parameters rise again slowly compare to decreasing of ε.
Let $\kappa = \delta^{1+\alpha}$, and $\varepsilon = \delta^{1+\gamma}$, assume that

$$CX^6Y^8N^7 < \frac{1}{\delta^\alpha}$$

we get:

$$\varepsilon_+ \leq \delta^{1+2\gamma-15\alpha} \tag{5.1}$$

Choose α small enough compare to γ, ($\alpha = \gamma/100$ for example) we get:

$$\varepsilon_+ = [\tilde{f}^T]^s_{\sigma',\mu',\mathcal{D}'} \leq \delta^{1+\frac{4}{3}\gamma}$$

when $[f^T]^s_{\sigma,\mu,\mathcal{D}} \leq \delta^{1+\gamma}$.

Choice of parameters. We shall construct a transformation Φ as the composition of infinite many transformations Φ_{S_k} :

$$(h_k + f_k) \circ \Phi_{S_k} = h_{k+1} + f_{k+1}.$$

At each step the domain is $\mathcal{O}^s(\sigma_k, \mu_k) \times \mathcal{D}_k$, with $\mathcal{D}_k = D(\kappa_k, N_k) \cap \mathcal{D}_{k-1} \subset D_{k-1}$. The normal form $h_k = \Omega_k \cdot r + \frac{1}{2}\langle \omega, \mathcal{A}_k\omega\rangle$ is closed to h_0, and its Fourier series are truncated at order N_k. We now give here a specific choice of all the parameters for $k \geq 1$.
Let $\varepsilon_k = \delta^{1+\gamma_k}$, $\kappa_k = \delta^{1+\alpha_k}$ such that

$$\gamma_k = \frac{4}{3}\gamma_{k-1}, \quad \alpha_k = \frac{4}{3}\alpha_{k-1},$$

with $\gamma_0 = \gamma$, $\alpha_0 = \frac{1}{100}\gamma$. We also choose

$$\sigma_k = \frac{1}{2}\sigma + \frac{1}{2^{k+1}}\sigma, \quad \mu_k = \frac{1}{2}\mu + \frac{1}{2^{k+1}}\mu \quad N_k = -8(\sigma_k - \sigma_{k+1})^{-1} ln\varepsilon_k.$$

Iterative lemma. We have

$$h_0 = \Omega_0 \cdot r + \frac{1}{2}\langle \omega, A_0\omega\rangle$$

and $f_0 = f \in \mathcal{T}^s_{\sigma_0,\mu_0,\mathcal{D}_0}$ satisfying

$$[f_0]_{\sigma_0,\mu_0,\mathcal{D}_0} \leq \delta, \qquad [f_0^T]_{\sigma_0,\mu_0,\mathcal{D}_0} \leq \varepsilon = \delta^{1+\gamma}.$$

Let us denote $\mathcal{D}_0 = \mathcal{D}$ and $O_k = \mathcal{O}^s_{\sigma_k,\mu_k}$.

Lemma 5.1. *For δ sufficiently small compare to σ_0, μ_0 and 1, assume that $\delta \leq \chi \leq \delta^{1-\frac{\gamma_0}{2}}$. Then for all $k \geq 1$ there exist $\mathcal{D}_k \subset \mathcal{D}_{k-1}$, $S_k \in \mathcal{T}^s_{\sigma_k,\mu_k,\mathcal{D}_k}$, $h_k = \Omega_k \cdot r + \frac{1}{2}\langle \omega, A_k\omega\rangle$ on normal form and $f_k \mathcal{T}^s_{\sigma_k,\mu_k,\mathcal{D}_k}$ such that*

- *The mapping*

$$\Phi_k = \Phi^t_{S_k} : O_{k+1} \to O_k, \quad \rho \in \mathcal{D}_k,\ k = 1, 2, \dots$$

is an analytic symplectomorphism verifying

$$(h_k + f_k) \circ \Phi_k = h_{k+1} + f_{k+1}.$$

- *we have the estimates*

$$meas(\mathcal{D}_{k-1} \setminus \mathcal{D}_k) \leq \delta^{\frac{\alpha_k}{2}}, \tag{5.2}$$

$$[h_k - h_{k-1}]^s_{\sigma_k,\mu_k,\mathcal{D}_k} \leq \delta^{1+\frac{\gamma_k}{3}}, \tag{5.3}$$

$$[f_k^T]^s_{\sigma_k,\mu_k,\mathcal{D}_k} \leq \varepsilon_k, \tag{5.4}$$

$$\left\|\partial^j_\rho \left(\Phi_k(x,\rho) - x\right)\right\| \leq C\delta^{\frac{\gamma_k}{3}} \quad x \in O_{k+1},\ \rho \in \mathcal{D}_{k+1}, \tag{5.5}$$

for $j = 0, 1$.

Here C is an absolute constant.

Proof. At step 1, $h_0 = \Omega_0 \cdot r + \frac{1}{2}\langle \omega, A_0\omega\rangle$ satisfies condition 3.13 trivially, so by Proposition 4.2 and the choices of parameters, we can construct S_0, R_0, h_0^+ verifying, for $\rho \in \mathcal{D}_1$

$$\{h_0, S_0\} + f_0^T + \{f_0 - f_0^T, S_0\}^T = h_0^+ + R_0$$

such that

$$meas(\mathcal{D}_0 \setminus \mathcal{D}_1) \leq C\frac{\kappa_0}{\delta} M^2 N_0^{n+2} \leq \delta^{\frac{\kappa_0}{2}}$$

and

$$[S_0]^s_{\sigma_1,\mu_1,\mathcal{D}_1} \leq CX_0^3 Y_0^4 N_0^4 \frac{\delta^7 \varepsilon_0}{\kappa_0^8} \leq \delta^{\gamma_0 - 9\alpha_0} \leq \delta^{\frac{\gamma_0}{3}} \tag{5.6}$$

$$[h_0^+]^s_{\sigma_1,\mu_1,\mathcal{D}_1} \leq CX_0^3 Y_0^3 N_0^3 \frac{\delta^6 \varepsilon_0}{\kappa_0^6} \leq \delta^{1+\gamma_0 - 7\alpha_0} \leq \delta^{1+\frac{\gamma_0}{3}}. \tag{5.7}$$

By Lemma 3.11, for any $\rho \in \mathcal{D}_1$, $\Phi_0 = \Phi^1_{S_0} : O_1 \to O_0$ is an analytic symplectomorphism such that

$$(h_0 + f_0) \circ \Phi_1 = h_1 + f_1$$

with $h_1 = h_0 + h_0^+$ and

$$\left\|\partial^j_\rho \left(\Phi_0(x,\rho) - x\right)\right\| \le C[S_0]^s_{\sigma_1,\mu_1,\mathcal{D}_1} \le C\delta^{\frac{\gamma_0}{3}} \quad x \in O_1,\ \rho \in \mathcal{D}_1,$$

for $j = 0, 1$. The estimate of f_1^T is already done before.

Assume that the iteration is true up to step ℓ. We want to prove it for step $\ell + 1$. By construction

$$h_\ell = h_0 + h_0^+ + h_1^+ + \dots h_{\ell-1}^+$$

satisfying

$$[h_\ell - h_0]^s_{\sigma_\ell,\mu_\ell,\mathcal{D}_\ell} \le \delta(\delta^{\frac{\gamma_0}{3}} + \delta^{\frac{\gamma_1}{3}} + \dots + \delta^{\frac{\gamma_{\ell-1}}{3}}) \le 2\delta^{1+\frac{\gamma_0}{3}} \le \frac{\delta}{4}.$$

So that

$$|\partial^j_\rho(A_\ell - A_0)| \le \frac{\delta}{4}, \quad |\partial^j_\rho(\Omega_\ell - \Omega_0)| \le \delta$$

for $j = 0, 1$. Therefore condition (3.13) is satisfied at rank ℓ and by Proposition 4.2 we can construct S_ℓ, h_ℓ^+, R_ℓ verifying the non linear homological equation on $\mathcal{D}_{\ell+1}$ such that

$$meas(\mathcal{D}_\ell \setminus \mathcal{D}_{\ell+1}) \le C\frac{\kappa_\ell}{\delta} M^2 N_\ell^{n+2} \le \delta^{\frac{\kappa_\ell}{2}}$$

and

$$[S_\ell]^s_{\sigma_{\ell+1},\mu_{\ell+1},\mathcal{D}_{\ell+1}} \le C X_\ell^3 Y_\ell^4 N_\ell^4 \frac{\delta^7 \varepsilon_\ell}{\kappa_\ell^8} \le \delta^{\gamma_\ell - 9\alpha_\ell} \le \delta^{\frac{\gamma_\ell}{3}} \tag{5.8}$$

$$[h_\ell^+]^s_{\sigma_{\ell+1},\mu_{\ell+1},\mathcal{D}_{\ell+1}} \le C X_\ell^3 Y_\ell^3 N_\ell^3 \frac{\delta^6 \varepsilon_\ell}{\kappa_\ell^6} \le \delta^{1+\gamma_\ell - 7\alpha_\ell} \le \delta^{1+\frac{\gamma_\ell}{3}}. \tag{5.9}$$

By Lemma 3.11, for any $\rho \in \mathcal{D}_{\ell+1}$, $\Phi_\ell = \Phi^1_{S_\ell} : O_{\ell+1} \to O_\ell$ is an analytic symplectomorphism such that

$$(h_0 + f_0) \circ \Phi_1 = h_1 + f_1$$

with $h_{\ell+1} = h_\ell + h_\ell^+$ and

$$\left\|\partial^j_\rho \left(\Phi_\ell(x,\rho) - x\right)\right\| \le C[S_\ell]^s_{\sigma_{\ell+1},\mu_{\ell+1},\mathcal{D}_{\ell+1}} \le C\delta^{\frac{\gamma_0}{3}} \quad x \in O_{\ell+1},\ \rho \in \mathcal{D}_{\ell+1},$$

for $j = 0, 1$. Finally, we have

$$f_{\ell+1} = f_\ell - f_\ell^T - \{f_\ell - f_\ell^T, S_\ell\}^T + \{f_\ell^T, S_\ell\} + \int_0^1 \{(1-t)(h_\ell^+ + R_\ell) + t f_\ell^T, S_\ell\} \circ \Phi^t_{S_\ell} dt + R_\ell,$$

satisfying

$$[f_{\ell+1}^T]^s_{\sigma_{\ell+1},\mu_{\ell+1},\mathcal{D}_{\ell+1}} \le \varepsilon_{\ell+1}.$$

$\square$

Proof of KAM theorem. Let

$$\mathcal{D}' = \cap_{k\geq 0}\mathcal{D}_k.$$

By Lemma 5.1:

$$meas(\mathcal{D}\backslash\mathcal{D}') \leq \delta^{\frac{\alpha_0}{2}} + \delta^{\frac{\alpha_1}{2}} + \ldots \leq 2\delta^{\frac{\alpha_0}{2}}.$$

Notice that $\sigma_k > \sigma/2$ and $\mu_k > \mu/2$ and

$$\mathcal{O}^s(\sigma/2,\mu/2) = \cap_{k\geq 0}\mathcal{O}^s(\sigma_k,\mu_k) = \lim_{k\to\infty}\mathcal{O}^s(\sigma_k,\mu_k).$$

Let us denote $\Phi_N^\ell = \Phi_{\ell+1}\circ\ldots\circ\Phi_N$ for $N\geq\ell\geq 0$. By Lemma 5.1, Φ_j^N is an analytic symplectomorphism from O_N to O_j satisfying

$$(h_\ell + f_\ell)\circ\Phi_N^\ell = h_N + f_N$$

and

$$\left\|\partial_\rho^j\left(\Phi_N^\ell(x,\rho) - Id\right)\right\| \leq C(\delta^{\frac{\gamma_\ell}{3}} + \ldots + \delta^{\frac{\gamma_N}{3}}) \leq 2C\delta^{\frac{\gamma_\ell}{3}} \quad x\in O_N,\ \rho\in\mathcal{D}_N,\ j=0,1.$$

We also have for $M > N > \ell$

$$\left\|\partial_\rho^j\left(\Phi_N^\ell - \Phi_M^\ell\right)\right\| \leq C\delta^{\frac{\gamma_N}{3}}, \qquad j=0,1;$$

i.e. $(\Phi_N^\ell)_N$ is a Cauchy sequence, which converge to the analytic symplectomorphism $\Phi_\infty^j :$ $\mathcal{O}^s(\sigma/2,\mu/2)\to O_\ell$. By Lemma 5.1, we also have h_k and f_k are Cauchy sequences. Let us denote $h = lim_{k\to\infty}h_k$, $g = lim_{k\to\infty}f_k$ and $\Phi = \Phi_\infty^0$. By construction, we have for $\rho\in\mathcal{D}'$

$$(h_0 + f)\circ\Phi = h + g$$

with $g\in\mathcal{T}^s_{\sigma/2,\mu/2,\mathcal{D}'}$ and $g^T\equiv 0$. The normal form

$$h = \Omega\cdot r + 1/2\langle\omega, A\omega\rangle = \Omega(\rho)\cdot r + \langle\zeta_{\mathcal{L}}, Q(\rho)\eta_{\mathcal{L}}\rangle + 1/2\langle\omega_{\mathcal{F}}, K(\rho)\omega_{\mathcal{F}}\rangle$$

satisfies

$$[h-h_0]^s_{\sigma/2,\mu/2,\mathcal{D}'} \leq \delta(\delta^{\frac{\gamma_0}{3}} + \delta^{\frac{\gamma_1}{3}} + \ldots) \leq 2\delta^{1+\frac{\gamma_0}{3}}$$

that is

$$|\partial_\rho^j(\Omega-\Omega_0)| \leq 2\delta^{1+\frac{\gamma_0}{3}}, \quad |\partial_\rho(Q-Q_0)| \leq 2\delta^{1+\frac{\gamma_0}{3}}, \quad |\partial_\rho(K-K_0)| \leq 2\delta^{1+\frac{\gamma_0}{3}}$$

for $j=0,1$.

6. Applications

Consider the non linear Schrödinger equation on the torus

$$i\partial_t u + \partial_{xx} u = |u^4|u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}. \tag{6.1}$$

The Hamiltonian of the equation is given by

$$h = \int_{\mathbb{T}} |u_x|^2 + \frac{1}{3}|u|^6 dx.$$

Let us expand u and $\bar{u}$ in Fourier basis:

$$u(x) = \sum_{j\in\mathbb{Z}} a_j e^{ijx}, \quad \bar{u}(x) = \sum_{j\in\mathbb{Z}} b_j e^{ijx}.$$

Define

$$P(a,b) = \frac{1}{3}\int_{\mathbb{S}^1} |u|^6 dx = \frac{1}{3} \sum_{j,\ell\in\mathbb{Z}^3, \mathcal{M}(j,\ell)=0} a_{j_1}a_{j_2}a_{j_3}b_{\ell_1}b_{\ell_2}b_{\ell_3}, \tag{6.2}$$

$$N(a,b) = \sum_{j\in\mathbb{Z}} j^2 a_j b_j, \tag{6.3}$$

where $\mathcal{M}(j,\ell) = j_1+j_2+j_3-\ell_1-\ell_2-\ell_3$ denotes the momentum of the milti-index $(j,l) \in \mathbb{Z}^6$ or equivalently the momentum of the monomial $a_{j_1}a_{j_2}a_{j_3}b_{\ell_1}b_{\ell_2}b_{\ell_3}$. In this Fourier formular, the equation 6.1 reads as an infinte Hamiltonian system

$$\begin{cases} i\dot{a}_j & = j^2 a_j + \frac{\partial P}{\partial b_j} \quad j \in \mathbb{Z}, \\ -i\dot{b}_j & = j^2 b_j + \frac{\partial P}{\partial a_j} \quad j \in \mathbb{Z}, \end{cases}$$

and the Hamiltonian:

$$h = N + P = \sum_{j\in\mathbb{Z}} j^2 a_j b_j + \frac{1}{3} \sum_{j,\ell\in\mathbb{Z}^3, \mathcal{M}(j,\ell)=0} a_{j_1}a_{j_2}a_{j_3}b_{\ell_1}b_{\ell_2}b_{\ell_3}.$$

We also introduce the mass and momentum Hamiltonians:

$$\mathbb{L} = \sum_{j\in\mathbb{Z}} a_j b_j, \quad \mathbb{M} = \sum_{j\in\mathbb{Z}} j a_j b_j.$$

Notice that the Hamiltonian flow preserves the mass and the momentum, or equivalently h commutes with both $\mathbb{L}$ and $\mathbb{M}$:

$$\{h, \mathbb{L}\} = \{h, \mathbb{M}\} = 0.$$

The Birkhoff normal form procedure. We first recall a result proved in [3].

Proposition 6.1. *There exist a canonical change of variable τ from $\mathcal{O}^s(\sigma,\mu)$ into $\mathcal{O}^s(2\sigma,2\mu)$ such that*

$$\bar{h} = h\circ\tau = N + Z_6 + R_{10},$$

where

- N *is the term* $N(I)=\sum_{j\in\mathbb{Z}} j^2 I_j$;
- Z_6 *is the homogeneous polynomial of degree 6*

$$Z_6 = \sum_{\mathcal{R}} a_{j_1}a_{j_2}a_{j_3}b_{\ell_1}b_{\ell_2}b_{\ell_3}$$

  where
  $\mathcal{R} = \{(j,\ell)\in\mathbb{Z}^3\times\mathbb{Z}^3 \; s.t\; j_1+j_2+j_3=\ell_1+\ell_2+\ell_3, \quad j_1^2+j_2^2+j_3^2=\ell_1^2+\ell_2^2+\ell_3^2\}$;
- R_{10} *is the remainder of order 10, i.e a Hamiltonian satisfying*

$$\|X_{R_{10}}(x)\|_s \le C\,\|x\|_s^9$$

  for all $x\in\mathcal{O}^s(\sigma,\mu)$;
- τ *is close to the identity: there exists a constant* C *such that*

$$\|\tau(x)-x\| \le C\,\|x\|^2$$

  for all $x\in\mathcal{O}^s(\sigma,\mu)$.

Start with two modes.

Firstly, we want to study the persistence of a two dimensional invariant torus for equation (6.1) around the original point. Assume that

$$\begin{cases} a_p &= (\nu\rho_1 + r_1(t))^{\frac{1}{2}}\, e^{i\theta_1(t)} =: \sqrt{I_p}e^{i\theta_1(t)} \\ a_q &= (\nu\rho_2 + r_2(t))^{\frac{1}{2}}\, e^{i\theta_2(t)} =: \sqrt{I_q}e^{i\theta_2(t)} \\ a_j &= \zeta_j \qquad j\neq p,q, \end{cases}$$

where $\{\rho_1,\rho_2\}\in[1,2]^2=\mathcal{D}$ and ν is a small parameter such that

$$|a_p-\sqrt{\nu\rho_1}|^2 + |a_q-\sqrt{\nu\rho_2}|^2 + \sum_{j\neq p,q}(1+j^2)^s|a_j|^2 = \mathcal{O}(\nu^3).$$

The canonical symplectic structure now becomes

$$-id\zeta\wedge d\eta - dI\wedge d\theta$$

with $I=(I_1,I_2)$, $\theta=(\theta_1,\theta_2)$, $\zeta=(\zeta_j)_j$ and $\eta=(\eta_j)_j=(\bar{\zeta}_j)_j$.

Let

$$\mathbf{T}_\rho^{lin} := \{(I,\theta,\zeta) | |I-\nu\rho|=0,\; |\Im\theta|<\sigma,\; \|\zeta\|_s=0\}$$

and its neighborhood

$$\mathbf{T}_\rho(\nu,\sigma,\mu,s) := \{(I,\theta,\zeta) | |I-\nu\rho| < \nu\mu^2,\ |\Im\theta| < \sigma,\ \|\zeta\|_s < \nu^{1/2}\mu\}.$$

We want to study the persistence of torus $\mathbf{T}_\rho(\nu,\sigma,\mu,s)$. Indeed we have

$$\mathbf{T}_\rho(\nu,\sigma,\mu,s) \approx \mathcal{O}^s(\sigma,\nu^{1/2}\mu) = \{(r,\theta,\zeta) | |r| < \nu\mu,\ |\Im\theta| < \sigma,\ \|\zeta\|_s < \nu^{1/2}\mu\}.$$

By theorem 6.1 we have

$$h \circ \tau = N + Z_6 + R_{10}.$$

We see that the term N contributes the effective Hamiltonian and the term R_{10} contributes the remainder term f. So we just need to focus on the term Z_6. Let us split it:

$$Z_6 = Z_{0,6} + Z_{1,6} + Z_{2,6} + Z_{3,6}.$$

Here $Z_{0,6}$, $Z_{1,6}$, $Z_{2,6}$ are homogeneous polynomial of degree 6 which contains respectively external modes of order $0,1,2$. $Z_{3,6}$ is an homogeneous polynomial of degree 6 contains external modes of at least order 3,this term contributes the remainder term.
Thank to Lemma 2.2 on [3], the term $Z_{1,6} = 0$. We have

$$\begin{aligned}
Z_{0,6} &= |a_p|^6 + |a_q|^6 + 9\left(|a_p|^4|a_q|^2 + |a_p|^2|a_q|^4\right) \\
&= (\nu\rho_1 + r_1)^3 + (\nu\rho_2 + r_2)^3 + 9\left(\nu\rho_1 + r_1\right)\left(\nu\rho_2 + r_2\right)\left(\nu\rho_1 + r_1 + \nu\rho_2 + r_2\right) \\
&= \nu^3(\rho_1^3 + \rho_2^3 + 9\rho_1^2\rho_2 + 9\rho_2^2\rho_1) + 3\nu^2\left(r_1(\rho_1^2 + 6\rho_1\rho_2 + 3\rho_2^2) + r_2(\rho_2^2 + 6\rho_1\rho_2 + 3\rho_1^2)\right) \\
&\quad + remainder.
\end{aligned}$$

For the term $Z_{2,6}$, there are two cases that can happen.

The first case

There are not $s, t \neq p, q$ such that

$$\begin{cases} 2p + s &= 2q + t \\ 2p^2 + s^2 &= 2q^2 + t^2. \end{cases} \tag{6.4}$$

Hence

$$Z_{2,6} = Z_{2,6}^1 = 9\left(|a_p|^4 + |a_q|^4 + 4|a_p|^2|a_q|^2\right)\sum_{j\neq p,q}|a_j|^2 = 9\nu^2\left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right)\sum_{j\neq p,q}|\zeta_j|^2 + remainder.$$

Hence the effective Hamiltonian h^e reads

$$\begin{aligned}
h^e = &\left(p^2 + 3\nu^2\left(\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_2\right)\right)r_1 + \left(q^2 + 3\nu^2\left(\rho_2^2 + 3\rho_1^2 + 6\rho_1\rho_2\right)\right)r_2 \\
&+ \sum_j\left(j^2 + 9\nu^2\left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right)\right)|\zeta_j|^2
\end{aligned}$$

It is on normal form

$$\Omega(\rho)\cdot r+\sum_{j\neq p,q}\Lambda_j|\zeta_j|^2 \tag{6.5}$$

where

$$\Omega(\rho)=\begin{pmatrix} p^2+3\nu^2\left(\rho_1^2+3\rho_2^2+6\rho_1\rho_2\right)\\ q^2+3\nu^2\left(\rho_2^2+3\rho_1^2+6\rho_1\rho_2\right)\end{pmatrix}$$

and

$$\Lambda_j=j^2+9\nu^2\left(\rho_1^2+\rho_2^2+4\rho_1\rho_2\right).$$

The remainder term R reads

$$\begin{aligned}R&=R_{10}+Z_{3,6}+3\nu\rho_1r_1^2+r_1^3+3\nu\rho_2r_2^2+r_2^3+9r_1r_2(r_1+r_2)\\&\quad+\left(r_1^2+r_2^2+2\nu(\rho_1+2\rho_2)r_1+2\nu(\rho_2+2\rho_1)r_2\right)\sum_{j\neq p,q}|\zeta_j|^2.\end{aligned}$$

In order to work on $\mathcal{O}^s(\sigma,\mu)$ we use the rescaling $\Psi:r\mapsto\nu r,\ \zeta\mapsto\nu^{1/2}\zeta$. The symplectic structure now becomes

$$-\nu dr\wedge d\theta-i\nu d\zeta\wedge d\eta.$$

By definition, this change of variables send $\mathcal{O}^s(\sigma,\mu)$ to the neighborhood of T_ρ^{lin}. By this rescaling, we get

$$(h^e+R)\circ\Psi=\nu h_0+\nu f$$

where h_0 and f are defined by

$$h_0=\frac{1}{\nu}h^e\circ\Psi\qquad f=\frac{1}{\nu}R\circ\Psi.$$

By theorem 6.1, $R_{10}\in\mathcal{T}^s(\sigma,\nu^{1/2}\mu,\mathcal{D})$. It is straightforward to prove that the rest part of R is in $\mathcal{T}^s(\sigma,\nu^{1/2}\mu,\mathcal{D})$. By construction, all of these terms commute with $\mathbb{L}$ and $\mathbb{M}$, hence thank to Lemma 4.3 on [5] they are all in $\mathcal{T}^s_{res}(\sigma,\nu^{1/2}\mu,\mathcal{D})$, so that $R\in\mathcal{T}^s_{res}(\sigma,\nu^{1/2}\mu,\mathcal{D})$. After rescaling, we get $f\in\mathcal{T}^s_{res}(\sigma,\mu,\mathcal{D})$. For estimating the norm of f, notice that R contains only term of order at least 3 in ν and $R^T=R^T_{10}$ is of order 9 in ν, so that

$$[f]^s_{\sigma,\mu,\mathcal{D}}\lesssim\nu^2$$

and

$$[f^T]^s_{\sigma,\mu,\mathcal{D}}\lesssim\nu^{7/2}.$$

Theorem 6.2. *Assume that for $p,q\in\mathbb{Z}$ there are not s,t solving the equation 6.4. The change of variables $\Phi_\rho=\tau\circ\Psi$ is a real holomorphic transformations, symplectic and analytically depending on ρ satisfying*

- $\Phi_\rho:\mathcal{O}^s(\sigma,\mu)\to\mathbf{T}_\rho(\nu,2\sigma,2\nu,s)$;

- *Φ_ρ puts the Hamiltonian h in normal form in the following sense:*

$$\frac{1}{\nu}(h \circ \Phi_\rho - C) = h_0 + f$$

where C is a constant and the effective part h_0 of the Hamiltonian reads

$$h_0 = \Omega(\rho) \cdot r + \sum_{j \neq p,q} \Lambda_j |\zeta_j|^2$$

where

$$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2 \left(\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_2\right) \\ q^2 + 3\nu^2 \left(\rho_2^2 + 3\rho_1^2 + 6\rho_1\rho_2\right) \end{pmatrix}$$

and

$$\Lambda_j = j^2 + 9\nu^2 \left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right);$$

- *The remainder term f belongs to $\mathcal{T}^s(\sigma, \mu, \mathcal{D})$ and satisfies*

$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^2$$

and

$$[f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{7/2}.$$

The second case

There are $s, t \neq p, q$ solving 6.4, so

$$Z_{2,6} = Z^1_{2,6} + (a_p^2 a_s b_q^2 b_t + b_p^2 b_s a_q^2 a_t) = Z^1_{2,6} + Z_{s,t}$$

For the second term, let us rewrite it

$$\sum_{s,t} (\nu\rho_1 + r_1)(\nu\rho_2 + r_2) \left(e^{2i(\theta_1 - \theta_2)} \zeta_s \eta_t + e^{-2i(\theta_1 - \theta_2)} \eta_s \zeta_t\right)$$

The effective part of this term is just given by

$$\nu^2 \rho_1 \rho_2 \sum_{s,t} \left(e^{2i(\theta_1 - \theta_2)} \zeta_s \eta_t + e^{-2i(\theta_1 - \theta_2)} \eta_s \zeta_t\right).$$

Notice that

$$\{I_s, \zeta_s\eta_t + \eta_s\zeta_t\} = \{I_t, \zeta_s\eta_t + \eta_s\zeta_t\} = 0.$$

This gives us a clue that the above term does not effect to the stability of the solution. In order to kill the angles, we introduce the symplectic change of variables

$$\Psi_{angles}(r_1, r_2, \theta, \zeta) = (r_1', r_2', \theta, \zeta'),$$

defined by

$$\begin{cases} \zeta'_s & = e^{2i(\theta_1-\theta_2)}\zeta_s \\ \zeta'_t & = \zeta_t \\ \zeta'_j & = \zeta_j, \quad j \neq s,t,p,q \\ r'_1 & = r_1 - 2|\zeta_s|^2 \\ r'_2 & = r_2 + 2|\zeta_s|^2. \end{cases}$$

By this change of variables

$$\tilde{h} = \bar{h} \circ \Psi_{angles} = C + h^e + R.$$

Here C is a constant given by

$$C = \nu^3(\rho_1^3 + \rho_2^3 + 9\rho_1^2\rho_2 + 9\rho_2^2\rho_1) + \nu p^2\rho_1 + \nu q^2\rho_2.$$

The effective Hamiltonian h^e reads

$$\begin{aligned} h^e = & \left(p^2 + 3\nu^2\left(\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_2\right)\right) r'_1 + \left(q^2 + 3\nu^2\left(\rho_2^2 + 3\rho_1^2 + 6\rho_1\rho_2\right)\right) r'_2 \\ & + \sum_{j\neq p,q,s,t} \left(j^2 + 9\nu^2\left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right)\right)|\zeta'_j|^2 + \left(t^2 + 9\nu^2\left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right)\right)|\zeta'_t|^2 \\ & + \left(s^2 + 2p^2 - 2q^2 + \nu^2\left(21\rho_2^2 - 3\rho_1^2 + 36\rho_1\rho_2\right)\right)|\zeta'_s|^2 + \nu^2\rho_1\rho_2(\zeta'_s\eta'_t + \eta'_s\zeta'_t). \end{aligned}$$

It is on normal form

$$\Omega(\rho)\cdot r + \sum_{j\neq p,q,s,t} \Lambda_j|\zeta'_j|^2 + \Lambda_s|\zeta'_s|^2 + \Lambda_t|\zeta'_t|^2 + \nu^2\rho_1\rho_2(\zeta'_s\eta'_t + \eta'_s\zeta'_t)$$

where $\Omega(\rho)$ and Λ_j are defined as in the first case except

$$\Lambda_s = t^2 + \nu^2\left(21\rho_2^2 - 3\rho_1^2 + 36\rho_1\rho_2\right).$$

We would like to diagonalize it into the normal form as in KAM theorem. In order to do that, we use a change of variables

$$\begin{cases} \zeta_{t,+} & = \frac{1}{\sqrt{1+\alpha^2}}(\zeta'_t + \alpha\zeta'_s) \\ \zeta_{t,-} & = \frac{1}{\sqrt{1+\alpha^2}}(\zeta'_s - \alpha\zeta'_t) \end{cases}$$

Then h^e can be rewritten in normal form

$$\Omega(\rho)\cdot r + \sum_{j\neq p,q,s,t} \Lambda_j|\zeta_j|^2 + \Lambda_{t,+}|\zeta_{t,+}|^2 + \Lambda_{t,-}|\zeta_{t,-}|^2$$

Here α, $\Lambda_{t,+}$, $\Lambda_{t,-}$ are chosen by solving

$$\begin{cases} \alpha(\Lambda_{t,+} - \Lambda_{t,-}) & = (1+\alpha^2)\nu^2\rho_1\rho_2 \\ \Lambda_{t,+} + \alpha^2\Lambda_{t,-} & = (1+\alpha^2)\Lambda_t \\ \Lambda_{t,-} + \alpha^2\Lambda_{t,+} & = (1+\alpha^2)\Lambda_s \end{cases}$$

After solving this equation system, we get $\alpha \geq 1$ and $\Lambda_{t,+} = \Lambda_t - \frac{\nu^2\rho_1\rho_2}{\alpha}$, $\Lambda_{t,-} = \Lambda_s + \frac{\nu^2\rho_1\rho_2}{\alpha}$. The remainder term R reads

$$\begin{aligned} R = &R_{10} \circ \Psi_{angles} + Z_{3,6} \circ \Psi_{angles} + 3\nu\rho_1 r_1^2 + r_1^3 + 3\nu\rho_2 r_2^2 + r_2^3 \\ &+ 9r_1 r_2(r_1 + r_2) + \left(r_1^2 + r_2^2 + 2\nu(\rho_1 + 2\rho_2)r_1 + 2\nu(\rho_2 + 2\rho_1)r_2\right) \sum_{j \neq p,q} |\zeta_j|^2 \end{aligned}$$

with $r_1 = r_1' + 2|\zeta_s|^2,\ r_2 = r_2' - 2|\zeta_s|^2$.
By rescaling

$$(h^e + R) \circ \Psi = \nu h_0 + \nu f.$$

The study of f is the same as in the previous case.

Theorem 6.3. *Assume that p, q, s, t satisfy the equation 6.4. The change of variables $\Phi_\rho = \tau \circ \Psi_{angles} \circ \Psi$ is a real holomorphic transformations, analytically depending on ρ satisfying*

- $\Phi_\rho : \mathcal{O}^s(\frac{\sigma}{2}, \frac{e^{\frac{-1}{2}\mu}}{2}) \to \mathbf{T}_\rho(\nu, \sigma, \mu, s)$;
- *Φ_ρ puts the Hamiltonian h in normal form in the following sense:*

  $$\frac{1}{\nu}(h \circ \Phi_\rho - C) = h_0 + f$$

  where C is a constant and the effective part h_0 of the Hamiltonian reads

  $$h_0 = \Omega(\rho) \cdot r + \sum_{j \neq p,q,s,t} \Lambda_j |\zeta_j|^2 + \Lambda_{t,+}|\zeta_{t,+}|^2 + \Lambda_{t,-}|\zeta_{t,-}|^2$$

  where

  $$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2\left(\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_2\right) \\ q^2 + 3\nu^2\left(\rho_2^2 + 3\rho_1^2 + 6\rho_1\rho_2\right) \end{pmatrix}$$

  and

  $$\Lambda_j = j^2 + 9\nu^2\left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right) \qquad j \neq s,$$
- *The remainder term f belongs to $\mathcal{T}^s(1, 1, \mathcal{D})$ and satisfies*

  $$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^2$$

  and

  $$[f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{7/2}.$$

Proof of Theorem 1.1. By Theorem 6.2 and 6.3, there exists a symplectic change of variables Φ_1, on $\mathcal{D} = [1, 2]^2$, puts the Hamiltonian $h = N + P$ in normal form $h_0 + f$, that satisfy assumption of KAM theorem 2.3 for $\delta = \nu^2$, $\varepsilon = \nu^{7/2} = \delta^{7/4}$ and $\Omega_0 = \omega = (p^2, q^2) + O(\nu^2)$. So by KAM theorem, since the hyperbolic set $\mathcal{F}$ is empty, the torus

$$\mathbf{T}^{lin}_\rho := \{(I, \theta, \zeta) | |I - \nu\rho| = 0,\ |\Im\theta| < \sigma,\ \|\zeta\|_s = 0\}$$

or equivalently, its neighborhood

$$\mathbf{T}_\rho(\nu,1,1,s) := \{(I,\theta,\zeta) | |I-\nu\rho| < \nu,\ |\Im\theta| < 1,\ \|\zeta\|_s < \nu^{1/2}\}.$$

is linear stable. Here we denote $I = (I_p, I_q)$.

$\square$

3 modes Assume that

$$\begin{cases} a_p &= (\nu\rho_1 + r_1(t))^{\frac{1}{2}}\, e^{i\theta_1(t)} =: \sqrt{I_p} e^{i\theta_1(t)} \\ a_q &= (\nu\rho_2 + r_2(t))^{\frac{1}{2}}\, e^{i\theta_2(t)} =: \sqrt{I_q} e^{i\theta_2(t)} \\ a_m &= (\nu\rho_3 + r_3(t))^{\frac{1}{2}}\, e^{i\theta_3(t)} =: \sqrt{I_m} e^{i\theta_3(t)} \\ a_j &= \zeta_j \qquad j \neq p,q,m \end{cases}$$

where $\rho = (\rho_1,\rho_2,\rho_3) \in \mathcal{D} \subset \mathbb{R}^3$ and ν is a small parameter such that

$$|a_p - \sqrt{\nu\rho_1}|^2 + |a_q - \sqrt{\nu\rho_2}|^2 + |a_m - \sqrt{\nu\rho_1}|^2 + \sum_{j\neq p,q} (1+j^2)|a_j|^2 = \mathcal{O}(\nu^3).$$

The canonical symplectic structure now becomes

$$-id\zeta \wedge d\eta - dI \wedge d\theta$$

with $I = (I_1, I_2, I_3)$, $\theta = (\theta_1,\theta_2,\theta_3)\, \zeta = (\zeta_j)_j$ and $\eta = (\eta_j) = (\bar{\zeta}_j)$.
The same as the two-modes case, we have

$$\bar{h} := h \circ \tau = N + Z_6 + R_{10}.$$

We see that as the previous case, the term N contributes the effective Hamiltonian h_0 and the term R_{10} contributes the remainder term f. So we just need to focus on the term Z_6. Let us split it:

$$Z_6 = Z_{0,6} + Z_{1,6} + Z_{2,6} + Z_{3,6}.$$

Here, $Z_{0,6}$ is homogeneous polynomial of degree 6 which just contains inner modes (p,q,m); $Z_{1,6}$, $Z_{2,6}$ are homogeneous polynomials of degree 6 which contain outer modes of order 1 and 2. $Z_{3,6}$ is an homogeneous polynomial of degree 6 contains outer modes of at least order 3, this term contributes the remainder term. We have:

$$Z_{0,6} = |a_p|^6 + |a_q|^6 + |a_m|^6 + 9 \sum_{j,\ell\in\{p,q,m\}} |a_j|^4 |a_\ell|^2 + 36 |a_p|^2 |a_q|^2 |a_m|^2$$

Even it looks a bit more complicated, we deal with $Z_{0,6}$ as in the previous case.

For $Z_{1,6}$ there are two case:
The first case, there is no s solving the equation

$$\begin{cases} 2j_1 + j_2 & = 2\ell + s \\ 2j_1^2 + j_2^2 & = 2\ell^2 + s^2 \end{cases} \tag{6.6}$$

with $\{j_1, j_2, \ell\} = \{p, q, m\}$. In this case, $Z_{1,6} = 0$.

The second case, there exist s solving the above equation, then $Z_{1,6}$ contains monomials of forms

$$a_{j_1}^2 a_{j_2} b_\ell^2 b_s \quad \text{and} \quad b_{j_1}^2 b_{j_2} a_\ell^2 a_s.$$

In this case we are not in KAM theorem. So we just assume that we are in the first case.
For $Z_{2,6}$, we have

$$\begin{aligned} Z_{2,6} = \sum_{j_1,j_2,\ell} |a_{j_1}|^2 |a_{j_2}|^2 |a_\ell|^2 &+ \sum_{j_3,j_4,s_1,t_1 \in \mathcal{A}} \left(a_{j_3}^2 a_{s_1} b_{j_4}^2 b_{t_1} + b_{j_3}^2 b_{s_1} a_{j_4}^2 a_{t_1} \right) \\ &+ \sum_{j_5,j_6,j_7,s_2,t_2 \in \mathcal{B}} \left(a_{j_5}^2 a_{j_6} b_{j_7} b_{s_2} b_{t_2} + b_{j_5}^2 b_{j_6} a_{j_7} a_{s_2} a_{t_2} \right) \\ &+ \sum_{j_8,j_9,j_{10},s_3,t_3 \in \mathcal{C}} \left(a_{j_9}^2 a_{s_3} b_{j_8} b_{j_{10}} b_{t_3} + b_{j_9}^2 b_{s_3} a_{j_8} a_{j_{10}} a_{t_3} \right) \\ &+ \sum_{j_{11},j_{12},j_{13},s_4 \in \mathcal{E}} \left(a_{j_{11}}^2 a_{j_{12}} b_{j_{13}} b_{s_4}^2 + b_{j_{11}}^2 b_{j_{12}} a_{j_{13}} a_{s_4}^2 \right) \end{aligned}$$

with $j_i \in \{p, q, m\}$, $s \neq t$. The sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$ are finite and possibly empty such that

$$\begin{cases} 2j_3 + s_1 & = 2j_4 + t_1 \\ 2j_3^2 + s_1^2 & = 2j_4^2 + t_1^2 \end{cases} \qquad \begin{cases} 2j_5 + j_6 & = j_7 + s_2 + t_2 \\ 2j_5^2 + j_6^2 & = j_7^2 + s_2^2 + t_2^2 \end{cases}$$

$$\begin{cases} 2j_9 + s_3 & = j_8 + j_{10} + t_3 \\ 2j_9^2 + s_3^2 & = j_8^2 + j_{10}^2 + t_3^2 \end{cases} \qquad \begin{cases} 2j_{11} + j_{12} & = j_{13} + 2s_4 \\ 2j_{11}^2 + j_{12}^2 & = j_{13}^2 + 2s_4^2 \end{cases}$$

We shall deal with each term one by one (in case it's not empty). The first term is just depends on the actions, and we have

$$|a_{j_1}|^2 |a_{j_2}|^2 |a_\ell|^2 = \nu^2 \rho_{j_1} \rho_{j_2} |\zeta_\ell|^2 + remainder.$$

The second and the fourth term are similar, since their effective parts are all of the form

$$e^{i\alpha} \zeta_s \eta_t + e^{-i\alpha} \eta_s \zeta_t.$$

The idea to deal with these two terms is the same as that in the two-modes case. Since

$$\{I_s + I_t, \zeta_s \eta_t\} = \{I_s + I_t, \zeta_t \eta_s\} = 0,$$

these terms do not affect the stability of the flow. We see that $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are disjointed, and as in the two-modes case, a change of variables that used to deal with a pair s,t only affect that modes, i.e the changes of variables commute. We call Φ_1 the composition of all changes of variables used to deal with the sets $\mathcal{A}$ and $\mathcal{C}$.
For the third term, its effective parts are of the form

$$\nu^2\rho_{j_5}\sqrt{\rho_{j_6}\rho_{j_7}}(e^{i\alpha}\zeta_s\zeta_t+e^{-i\alpha}\eta_s\eta_t)$$

where $\alpha=\theta_{j_7}-\theta_{j_6}-2\theta_{j_5}$. For explicitness, assuming that we are dealing with the case $j_5=p,\ j_6=q,\ j_7=m$, and $s,\ t$ solve the following equation

$$\begin{cases} 2p+q & =m+s+t\\ 2p^2+q^2 & =m^2+s^2+t^2,\end{cases} \tag{6.7}$$

then $\alpha=\theta_3-\theta_2-2\theta_1$. An example for this could be $(p,q,m,s,t)=(3,10,9,1,6)$. In order to kill the angles, we introduce the symplectic change of variables

$$\Psi_{ang,1}(r,\theta,\zeta)=(r',\theta,\zeta')\,,$$

defined by

$$\begin{cases} \zeta_s' & =ie^{-i\alpha}\eta_s \qquad \eta_s'=ie^{i\alpha}\zeta_s\\ \zeta_t' & =\zeta_t \qquad\qquad \eta_t'=\eta_t\\ \zeta_j' & =\zeta_j, \qquad\qquad \eta_j'=\eta_j \quad j\neq s,t,p,q\\ r_1' & =r_1+2|\zeta_s|^2\\ r_2' & =r_2+|\zeta_s|^2,\\ r_3' & =r_3-|\zeta_s|^2.\end{cases}$$

The effective part related to s,t is of form

$$\Lambda_s|\zeta_s'|^2+\Lambda_t|\zeta_t'|^2-i\nu^2\rho_1\sqrt{\rho_2\rho_3}(\zeta_s'\eta_t'+\eta_s'\zeta_t') \tag{6.8}$$

where

$$\Lambda_t=t^2+9\nu^2(\rho_1^2+\rho_2^2+\rho_3^2+4\rho_1\rho_2+4\rho_2\rho_3+4\rho_3\rho_1)$$

and

$$\Lambda_s=t^2+3\nu^2(-\rho_1^2+\rho_2^2+5\rho_3^2-6\rho_1\rho_2+12\rho_2\rho_3+6\rho_3\rho_1).$$

Denoting $a=\frac{\Lambda_t-\Lambda_s}{2}$ and $b=\frac{\Lambda_t+\Lambda_s}{2}$, we diagonalize (6.8) by the change of variables

$$\begin{cases} \zeta_{t,-} & =\frac{1}{\sqrt{1-\alpha^2}}(\zeta_s'-i\alpha\zeta_t') \quad \eta_{t,-}=\frac{1}{\sqrt{1-\alpha^2}}(\eta_s'-i\alpha\eta_t')\\ \zeta_{t,+} & =\frac{1}{\sqrt{1-\alpha^2}}(\zeta_t'+i\alpha\zeta_s') \quad \eta_{t,+}=\frac{1}{\sqrt{1-\alpha^2}}(\eta_t'+i\alpha\eta_s')\end{cases}$$

where

$$\alpha=-\frac{a-\sqrt{a^2-\nu^4\rho_1^2\rho_2\rho_3}}{\nu^2\rho_1\sqrt{\rho_2\rho_3}}.$$

Then (6.8) becomes

$$\Lambda_{t,+}|\zeta_{t,+}|^2 + \Lambda_{t,-}|\zeta_{t,-}|^2$$

where $\Lambda_{t,\pm} = b \pm \sqrt{a^2 - \nu^4\rho_1^2\rho_2\rho_3}$. We see that $\Lambda_{t,\pm}$ is real or not depends on sign of $a^2 - \nu^4\rho_1^2\rho_2\rho_3$, which is dependent on choice of parameter ρ. Precisely, for $\rho \in \mathcal{D}_1 = [1,2]^3$, we have $\Lambda_{t,\pm} \in \mathbb{R}$ while there is a neighborhood of $(1, \frac{1}{2}, \frac{9}{2}) : \mathcal{D}_2 = \mathcal{D}_\epsilon = [1-\epsilon, 1+\epsilon] \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon] \times [\frac{9}{2} - \epsilon, \frac{9}{2} + \epsilon]$ such that $|\Im\Lambda_{t,\pm}| > \nu^2$ for all $\rho \in \mathcal{D}$.

We call Φ_2 the composition of all changes of variables related to $\mathcal{B}$.

For the set $\mathcal{E}$, assume that we are dealing with the case

$$\begin{cases} 2p + q & = m + 2s \\ 2p^2 + q^2 & = m^2 + 2s^2. \end{cases} \tag{6.9}$$

Then, using the change of variables

$$\Psi_{ang,2}(r, \theta, \zeta) = (r', \theta, \zeta')\,,$$

defined by

$$\begin{cases} \zeta'_s & = e^{i\alpha/2}\zeta_s \quad \eta'_s = e^{-i\alpha/2}\eta_s \\ \zeta'_j & = \zeta_j, \qquad\quad \eta'_j = \eta_j \quad j \neq s, p, q \\ r'_1 & = r_1 + |\zeta_s|^2 \\ r'_2 & = r_2 + \frac{1}{2}|\zeta_s|^2 \\ r'_3 & = r_3 - \frac{1}{2}|\zeta_s|^2. \end{cases}$$

The effective part related to s becomes

$$\Lambda_s|\zeta'_s|^2 + \nu^2\rho_1\sqrt{\rho_2\rho_3}(\zeta_s'^2 + \eta_s'^2) \tag{6.10}$$

where

$$\Lambda_s = 3\nu^2(2\rho_1^2 + \rho_2^2 - \rho_3^2 + 9\rho_1\rho_2 + 3\rho_3\rho_1)$$

If $\Lambda_s \neq 0$, we can rewrite (6.10) into $\frac{1-\beta^2}{1+\beta^2}\Lambda_s|\frac{\zeta'_s + \beta\eta'_s}{\sqrt{1-\beta^2}}|^2$ with β satisfying $\Lambda_s\beta = (1 - \beta^2)\nu^2\rho_1\sqrt{\rho_2\rho_3}$, otherwise we rewrite it into $i\nu^2\rho_1\sqrt{\rho_2\rho_3}(\frac{\zeta'_s + i\eta'_s}{\sqrt{2}}\frac{\eta'_s + i\zeta'_s}{\sqrt{2}})$. However, $meas\{\rho \in \mathbb{R}^3 : \Lambda_s = 0\} = 0$, so we do not focus on this case. We call Φ_3 the composition of all changes of variables related to $\mathcal{E}$. Using the rescaling Ψ introduced in the two-modes case, we get

Theorem 6.4. *Assume that we are not in case of* (6.6)*. The change of variables $\Phi_\rho := \Psi \circ \Phi_3 \circ \Phi_2 \circ \Phi_1 \circ \tau$ is a holomorphic, symplectic transformation, and analytically depending on $\rho \in \mathcal{D}$, satisfying*

- $\Phi_\rho : \mathcal{O}^s(\frac{\sigma}{2}, \frac{e^{-\frac{1}{2}}\mu}{2}) \to \mathbf{T}_\rho(\nu, \sigma, \mu, s)$;

- *Φ_ρ puts the Hamiltonian h in normal form in the following sense:*
$$\frac{1}{\nu}(h\circ\Phi_\rho - C) = h_0 + f$$
where C is a constant and the effective part h_0 of the Hamiltonian reads
$$h_0 = \Omega(\rho)\cdot r + \sum_{a\in\mathcal{Z}} \Lambda_a|\zeta_a|^2$$
where
$$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2\left(\rho_1^2 + 3\rho_2^2 + 3\rho_3^2 + 6\rho_1\rho_2 + 6\rho_1\rho_3 + 12\rho_2\rho_3\right) \\ q^2 + 3\nu^2\left(\rho_2^2 + 3\rho_1^2 + 3\rho_3^2 + 6\rho_1\rho_2 + 6\rho_2\rho_3 + 12\rho_1\rho_3\right) \\ m^2 + 3\nu^2\left(\rho_3^2 + 3\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_3 + 6\rho_3\rho_2 + 12\rho_2\rho_1\right) \end{pmatrix}$$
- *$\mathcal{Z}$ is the disjoint union $\mathcal{L}\cup\mathcal{F}$; $\mathcal{L}$ corresponds to elliptic part, and $\mathcal{F}$ corresponds to hyperbolic part;*
- *for $\mathcal{D} = \mathcal{D}_1$, then $\mathcal{F} = \{\varnothing\}$;*
- *for $\mathcal{D} = \mathcal{D}_2$, then $\mathcal{F} = \{\varnothing\}$ if and only if $\mathcal{B} = \{\varnothing\}$;*
- *Λ_a satisfies the Hypothesis A0, A1, A2;*
- *the remainder term f belongs to $\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ and satisfies*
$$[f]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^2$$
and
$$[f^T]^s_{\sigma,\mu,\mathcal{D}} \lesssim \nu^{7/2}.$$

The remainder term is dealt as in the two-modes case. It remains to verify the Hypothesis A0, A1, A2, which requires explicit and careful calculus, but the idea is similar as in the two-modes case.

Proof of Theorem 1.2 By Theorem 6.4, there exists a symplectic change of variables Φ_1, on $\mathcal{D}_2$ puts the Hamiltonian $h = N + P$ in normal form $h_0 + f$, that satisfies assumptions of KAM theorem 2.3 for $\delta = \nu^2$, $\varepsilon = \nu^{7/2} = \delta^{7/4}$ and $\Omega_0 = \omega = (p^2, q^2, m^2) + \mathcal{O}(\nu^2)$. So by KAM theorem, the hyperbolic set $\mathcal{F}$ is not empty if and only if $\mathcal{D} = \mathcal{D}_2$ and there are $s, t \neq p, q, m$ solving the equation

$$\begin{cases} 2p + q & = m + s + t \\ 2p^2 + q^2 & = m^2 + s^2 + t^2 \end{cases} \tag{6.11}$$

Hence, for $\rho\in\mathcal{D}_1$, the torus
$$\mathbf{T}^{lin}_\rho := \{(I,\theta,\zeta) | |I - \nu\rho| = 0,\ |\Im\theta| < \sigma,\ \|\zeta\|_s = 0\}$$
or equivalently, its neighborhood
$$\mathbf{T}_\rho(\nu,1,1,s) := \{(I,\theta,\zeta) | |I - \nu\rho| < \nu,\ |\Im\theta| < 1,\ \|\zeta\|_s < \nu^{1/2}\}$$

is linearly stable, while for $\rho \in \mathcal{D}_2$ and p, q, m satisfying (6.11), that torus is linearly unstable.

$\square$

7. Appendix

In this appendix, we will verify the hypothesis A0, A1, A2 for the Hamiltonian in our applications. The hypothesis A0 is trivial, so we focus on A1 and A2.

The two-modes case. The first case In this case, we have $\mathcal{F} = \emptyset$ and the other estimates are trivial. For the hypothesis A2, we recall that

$$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2 \left(\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_2\right) \\ q^2 + 3\nu^2 \left(\rho_2^2 + 3\rho_1^2 + 6\rho_1\rho_2\right) \end{pmatrix}$$

and

$$\Lambda_j = j^2 + 9\nu^2 \left(\rho_1^2 + \rho_2^2 + 4\rho_1\rho_2\right).$$

Let $k = (k_1, k_2) \in \mathbb{Z}^2/\{0\}$ and $z = z(k) = \frac{(k_2,k_1)}{|k|}$, then we have

$$\begin{aligned} (\nabla_\rho \cdot z)(\Omega(\rho) \cdot k) &= 6\nu^2 \left(3(\rho_1 + \rho_2)k_2^2 + 3(\rho_2 + 3\rho_1)k_1^2 + 4(\rho_1 + \rho_2)k_1k_2\right) |k|^{-1} \\ &\geq \frac{6}{\sqrt{2}}\nu^2|k| \end{aligned}$$

and

$$(\nabla_\rho \cdot z)\Lambda_j = 18\nu^2((\rho_1 + 2\rho_2)k_2 + (\rho_2 + 2\rho_1)k_1)|k|^{-1}.$$

Choosing $\delta = 4\nu^2$, we get the hypothesis A2 (1). Since $(\nabla_\rho \cdot z)(\Lambda_j - \Lambda_\ell) = 0$, the estimate of small divisor $\Omega \cdot k + \Lambda_j - \Lambda_\ell$ is followed. To estimate the small divisors $\Omega \cdot k + \Lambda_j$ and $\Omega \cdot k + \Lambda_j + \Lambda_\ell$ we use the fact that f commute with both the mass $\mathbb{L}$ and momentum $\mathbb{M}$. We just need to control small divisors $\Omega \cdot k + \Lambda_j$ and $\Omega \cdot k + \Lambda_j + \Lambda_\ell$ whenever $e^{ik\cdot\theta}\eta_j \in f$ and $e^{ik\cdot\theta}\eta_j\eta_\ell \in f$, respectively. We have for the mass and momentum:

$$\mathbb{L} = \nu(\rho_1 + \rho_2) + r_1 + r_2 + \sum_j |\zeta_j|^2$$

and

$$\mathbb{M} = \nu(p\rho_1 + q\rho_2) + pr_1 + qr_2 + \sum_j j|\zeta_j|^2.$$

By conservation of $\mathbb{L}$, we have

$$\{e^{ik\cdot\theta}\eta_j, \mathbb{L}\} = ie^{ik\cdot\theta}\eta_j(k_1 + k_2 + 1) = 0.$$

Therefore, for A2 (2) we just have to study the case $k_1+k_2=-1$. In this situation

$$\begin{aligned}(\nabla_\rho\cdot z)(\Omega(\rho)\cdot k+\Lambda_j) &= 6\nu^2|k|^{-1}\left(3(\rho_1+\rho_2)k_2^2+3(\rho_2+\rho_1)k_1^2+4(\rho_1+\rho_2)k_1k_2\right)\\ &\quad+6\nu^2|k|^{-1}\left(3(\rho_1+2\rho_2)k_2+3(\rho_2+2\rho_1)k_1\right)\\ &=6\nu^2|k|^{-1}\left((\rho_1+\rho_2)k_2^2+(\rho_2+\rho_1)k_1^2+2(\rho_1+\rho_2)\right)\\ &\quad+6\nu^2|k|^{-1}\left(3\rho_2k_2+3\rho_1k_1-3(\rho_1+\rho_2)\right)\\ &=6\nu^2|k|^{-1}\left(2(\rho_1+\rho_2)k_1^2+(5\rho_1-\rho_2)k_1-3\rho_2\right).\end{aligned}$$

This term is bigger than δ except the cases $k=(-1,0)$ and $(0,-1)$. The conservation of $\mathbb{M}$ gives us

$$\{e^{ik\cdot\theta}\eta_j,\mathbb{M}\}=ie^{ik\cdot\theta}\eta_j(pk_1+qk_2+j)=0.$$

For $k\in\{(-1,0),\,(0,-1)\}$, this implies $j\in\{p,q\}$, which is excluded.
We consider the small divisor $\Omega\cdot k+\Lambda_j+\Lambda_\ell$ in the same way. The conservation of the mass $\mathbb{L}$ gives us $k_1+k_2=-2$ and then by computation we get $k\in\{(0,-2),(-2,0),(-1,-1),(-3,1),(1,-3)\}$. The conservation of the momentum gives us $pk_1+qk_2+j+\ell=0$. We have

$$\Omega\cdot k+\Lambda_j+\Lambda_\ell=N(p,q,j,\ell)+\mu(\rho,k,)$$

where $N(p,q,j,\ell)=p^2k_1+q^2k_2+j^2+\ell^2$ and $\mu(\rho)$ very small for $|k|\le 4$. We see that $N(p,q,j,\ell)\in\mathbb{Z}$, so $N(p,q,j,\ell)\le\delta$ if and only if $p^2k_1+q^2k_2+j^2+\ell^2=0$. Combined with conservation of the momentum, this gives
for the case $k=(-1,-1)$

$$p+q=j+\ell\quad\text{and}\quad p^2+q^2=j^2+\ell^2$$

for the case $k=(-2,0)$

$$2p=j+\ell\quad\text{and}\quad 2p^2=j^2+\ell^2$$

for the case $k=(0,-2)$

$$2q=j+\ell\quad\text{and}\quad 2q^2=j^2+\ell^2$$

for the case $k=(-3,1)$

$$3p=q+j+\ell\quad\text{and}\quad 3p^2=q^2+j^2+\ell^2$$

for the case $k=(1,-3)$

$$3q=p+j+\ell\quad\text{and}\quad 3q^2=p^2+j^2+\ell^2$$

In all these cases, we get $j,\ell\in\{p,q\}$ which is excluded.

The second case We see that Ω and $\{\Lambda_j\}_{j\neq p,q,s,t}$ are all the same as the previous case except $\Lambda_{t,+}$ and $\Lambda_{t,-}$. We remind that $\Lambda_{t,+}=\Lambda_t-\frac{\nu^2\rho_1\rho_2}{\alpha}$ and $\Lambda_{t,-}=\Lambda_s+\frac{\nu^2\rho_1\rho_2}{\alpha}$ with

$\alpha \geq 1$. It is easy to see that $|(\nabla_\rho \cdot z)^j \frac{\nu^2 \rho_1 \rho_2}{\alpha}| \leq 2\nu^2 = \delta/2$, so instead of estimating directly with $\Lambda_{t,+}, \Lambda_{t,-}$, we can estimate with Λ_t and Λ_s. Since Λ_t has the same form as the other Λ_j we just need to focus on estimating divisors relating to Λ_s, which are $\Omega \cdot k + \Lambda_s$ and $\Omega \cdot k + \Lambda_s \pm \Lambda_\ell$. For simplicity, we omit $\frac{\nu^4 \rho_1^2 \rho_2^2}{2\Lambda_t}$ since it is very small. For $\Omega \cdot k + \Lambda_s$, using the conservation of the mass, we get $k_1 + k_2 = -1$ and

$$(\nabla_\rho \cdot z)(\Omega(\rho) \cdot k + \Lambda_s) = 6\nu^2 |k|^{-1} \left(2(\rho_1 + \rho_2)k_1^2 + (9\rho_1 + 3\rho_2)k_1 + 4\rho_1 - 3\rho_2\right).$$

Then we get the estimate except for $k \in \{(0,1), (-1,0), (-2,1), (-3,2)\}$. By the conservation of the momentum, combining with 6.4, we need

$$\begin{cases} pk_1 + qk_2 + t & = 0 \\ 2p + s & = 2q + t \\ 2p^2 + s^2 & = 2q^2 + t^2 \end{cases}$$

For $k \in \{(0,1), (-1,0), (-2,1), (-3,2)\}$, this gives either $\{s,t\} = \{p,q\}$ or $p = q$ which are all excluded. For $\Omega \cdot k + \Lambda_s - \Lambda_\ell$, by the conservation of the mass, we have $k_1 + k_2 = 0$ and then

$$(\nabla_\rho \cdot z)(\Omega \cdot k + \Lambda_s - \Lambda_\ell) = 6\nu^2 |k|^{-1} \left(2(\rho_1 + \rho_2)k_1^2 + 8(\rho_2 + \rho_1)k_1 + 12\rho_1 + 16\rho_2\right) \geq \nu^2$$

we get the estimate. For $\Omega \cdot k + \Lambda_s + \Lambda_\ell$, the conservation of the mass $\mathbb{L}$ gives us $k_1 + k_2 = -2$ and then by computation we get the estimate except for $k \in \{(-1,-1), (-2,0), (-3,1), \}$. Combining with the conservation of the momentum and the equation 6.4, we get that all these cases are excluded.

The three modes case. Let us start with simple probability assuming that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are all empty. In this case, we have

$$\Omega(\rho) = \begin{pmatrix} p^2 + 3\nu^2 \left(\rho_1^2 + 3\rho_2^2 + 3\rho_3^2 + 6\rho_1\rho_2 + 6\rho_1\rho_3 + 12\rho_2\rho_3\right) \\ q^2 + 3\nu^2 \left(\rho_2^2 + 3\rho_1^2 + 3\rho_3^2 + 6\rho_1\rho_2 + 6\rho_2\rho_3 + 12\rho_1\rho_3\right) \\ m^2 + 3\nu^2 \left(\rho_3^2 + 3\rho_1^2 + 3\rho_2^2 + 6\rho_1\rho_3 + 6\rho_3\rho_2 + 12\rho_2\rho_1\right) \end{pmatrix}$$

and

$$\Lambda_j = j^2 + 9\nu^2(\rho_1^2 + \rho_2^2 + \rho_3^2 + 4\rho_1\rho_2 + 4\rho_2\rho_3 + 4\rho_3\rho_1).$$

Let $k = (k_1, k_2, k_3) \in \mathbb{Z}^3/\{0\}$, $k' = (k_2 + k_3, k_1 + k_3, k_1 + k_2)$ and $z = z(k) = \frac{k'}{|k'|}$, then we have

$$\begin{aligned}(\nabla_\rho \cdot z)(\Omega(\rho) \cdot k) = 6\nu^2 |k'|^{-1} (&3(\rho_2 + \rho_3)k_1^2 + 3(\rho_1 + \rho_3)k_2^2 + 3(\rho_2 + \rho_1)k_3^2 + \\ &6(\rho_1 + \rho_2 + \rho_3)(k_1 + k_2 + k_3)^2 + (\rho_1 + \rho_2)k_1k_2 + (\rho_3 + \rho_2)k_3k_2 + (\rho_1 + \rho_3)k_1k_3)\end{aligned}$$

and

$$(\nabla_\rho \cdot z)\Lambda_j = 18\nu^2 |k'|^{-1} ((4\rho_1 + 3\rho_2 + 3\rho_3)k_1 + (4\rho_2 + 3\rho_1 + 3\rho_3)k_2 + (4\rho_3 + 3\rho_2 + 3\rho_1)k_3).$$

Choosing $\delta = \nu^2$, we get the hypothesis A2 (1). Since $(\nabla_\rho \cdot z)(\Lambda_j - \Lambda_\ell) = 0$, the estimate of small divisor $\Omega \cdot k + \Lambda_j - \Lambda_\ell$ is followed. For $\Omega \cdot k + \Lambda_j$, by conservation of the mass, we just need to estimate this divisor in the case $k_1 + k_2 + k_3 = -1$, then by computation we have estimate except for $k \in \{(0, 0, -1); (0, -1, 0); (-1, 0, 0)\}$. By conservation of the momentum, we have $pk_1 + qk_2 + mk_3 + j = 0$, so that $j \in p, q, m$ which is excluded. For $\Omega \cdot k + \Lambda_j + \Lambda_\ell$, again we have $k_1 + k_2 + k_3 = -2$ by conservation of the mass, which leads us to consider $|k| = |k_1| + |k_2| + |k_3| \leq 2$. By conservation of the momentum, we have $pk_1 + qk_2 + mk_3 + j + \ell = 0$, and the term $\Omega \cdot k + \Lambda_j + \Lambda_\ell$ is small if only if

$$pk_1^2 + qk_2^2 + mk_3^2 + j^2 + \ell^2 = 0.$$

After all, we will get $j, \ell \in \{p, q, m\}$ which is excluded. Now, we will estimate divisors in case $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are not empty.

The set $\mathcal{B}$ For $\rho \in \mathcal{D}_2$: we have

$$|\Im \Lambda_{t,\pm}| > \nu^2$$

and so that

$$|\Omega \cdot k + \Lambda_{t,+} - \Lambda_{t,-}| \geq 2\nu^2.$$

For $\Omega \cdot k + \Lambda_{t,+} + \Lambda_{t,-}$, we see that $\Lambda_{t,+} + \Lambda_{t,-} = \Lambda_t + \Lambda_s = 2\Lambda_t + 2\Omega_1 + \Omega_2 - \Omega_3$, so argument as in the trivial case above for $k_1' = k_1 - 2$, $k_2' = k_2 - 1, k_3' = k_3 + 1$ we get desired estimates. Nevertheless, by the conservation of the mass and the momentum, we just need to estimate this small divisor if $k_1 + k_2 + k_3 = 2t$ and $p^2k_1 + q^2k_2 + m^2k_3 = 2t^2$, combining with (6.11), this is never the case.

For $\rho \in \mathcal{D}_1$: the importance of this domain is $\sqrt{a^2 - \nu^4\rho_1^2\rho_2\rho_3} \in \mathbb{R}$, so we can forget $\sqrt{a^2 - \nu^4\rho_1^2\rho_2\rho_3}$ in estimating (if it is not small we can change the domain). Besides, we have $b = \frac{\Lambda_s + \Lambda_t}{2} = \Lambda_t + \frac{2\Omega_1 + \Omega_2 - \Omega_3}{2}$, so by checking estimates in trivial case above with a change $k_1' = k_1 - 1$, $k_2' = k_2 - \frac{1}{2}, k_3' = k_3 + \frac{1}{2}$, we get desired estimates.

The set $\mathcal{A}$ Assume that s, t solve the following equation

$$\begin{cases} 2p + s & = 2q + t \\ 2p^2 + s^2 & = 2q^2 + t^2 \end{cases}$$

Then by change of variables, we have $\Lambda_s = \Lambda_t + 2\Omega_1 - 2\Omega_2 + s^2 - t^2 + O(\nu^4)$. Using results in the trivial case for $k_1' = k_1 - 2$, $k_2' = k_2 + 2$, we get desired estimates.

The set $\mathcal{C}$ Assume that s, t solve the following equation

$$\begin{cases} 2p + s & = q + m + t \\ 2p^2 + s^2 & = q^2 + m^2 + t^2 \end{cases}$$

Then by change of variables, we have $\Lambda_s = \Lambda_t + 2\Omega_1 - \Omega_2 - \Omega_3 + s^2 - t^2 + O(\nu^4)$. Using results in the trivial case for $k_1' = k_1 - 2$, $k_2' = k_2 + 1, k_3' = k_3 + 1$ we get desired estimates.

The set $\mathcal{E}$ is just a special case of the set $\mathcal{B}$ when $s = t$.

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