Correction of my paper "Spectre du Laplacien et écrasement d'anses"
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(with the notations of this paper) I don't understand what I did at this time and I don't remember very well the context. In the following I shall concider only the problem for $X_1(\varepsilon) \cup NUX_2$. As it seems and as you have noticed, I don't prove that $\|C_\varepsilon\| \to 0$ but only that

$$\int_{\partial NUX_2} |C_\varepsilon(f)|^2 \leq \omega(\varepsilon)\|f\|_1^2$$

if $f \in H^1(X_1(\varepsilon))$ and $\omega(\varepsilon) = \varepsilon^d \int_\varepsilon^{r_0} r^{1-d} dr$.
Actually $\partial NUX_2 = NUY$ and in local coordinates (y, r, σ),

$$\begin{aligned}\int_{NUY} |C_\varepsilon(f)|^2 &= \int \varepsilon^d |f(\varepsilon,\sigma,y)|^2 d\sigma dy = \int \varepsilon^d |\int_\varepsilon^{r_0} \partial_r f(r,\sigma,y)\, dr|^2 d\sigma dy \\ &\leq \int \varepsilon^d \int_\varepsilon^{r_0} r^{1-d} dr \int_\varepsilon^{r_0} |\partial_r f(r,\sigma,y)|^2 r^{d-1} dr d\sigma dy\end{aligned}$$

This is sufficient because we don't need the convergence in norm of C_ε but only a strong convergence. I prefere to regard C_ε as an operator of $F_\varepsilon = (f_\varepsilon, h_\varepsilon)$.
Definition. — We say that C_ε converge strongly to 0 if for any bounded family $F_\varepsilon \in \mathcal{D}(Q_\varepsilon)$ (*i.e.* there exists $B > 0$ such that $\forall \varepsilon > 0\ \|F_\varepsilon\|_1 \leq B$) then

$$\lim_{\varepsilon \to 0} \|C_\varepsilon(F_\varepsilon)\| = 0.$$

1. If C_ε converge strongly to 0 the general theorem 1 holds. — This is quite obvious because in a finite dimension space all the norms are equivalent. Fix $n \in \mathbf{N}$ and $F_1(\varepsilon), \ldots, F_n(\varepsilon)$ an orthonormal base for the n first eigenvalues of Q_ε. These families are bounded in H^1 (because of the first control $\lambda_n(\varepsilon) \leq \lambda_n^0(\varepsilon)$) so by hypothesis

$$\lim_{\varepsilon \to 0} \sum_{1 \leq j \leq n} \|C_\varepsilon(F_j(\varepsilon))\|^2 = 0.$$

We test the Rayleigh-Ritz quotient of Q_ε^1 on $F^1(\varepsilon) = \Big(f(\varepsilon), h(\varepsilon) - C_\varepsilon(F(\varepsilon))\Big)$ for any $F(\varepsilon) = (f(\varepsilon), h(\varepsilon))$ in the n-eigenspace of Q_ε and we can write as in the paper

$$\frac{Q_\varepsilon^1(F^1(\varepsilon))}{\|F^1(\varepsilon)\|^2} \leq \frac{Q_\varepsilon(F(\varepsilon))}{\|F^1(\varepsilon)\|^2} \leq \lambda_n(\varepsilon) \frac{\|F(\varepsilon)\|^2}{\|F^1(\varepsilon)\|^2}$$

But if $F(\varepsilon) = \sum a_j F_j(\varepsilon)$ then $\|F(\varepsilon)\|^2 = \sum a_j^2$ and $\|F^1(\varepsilon)\| \geq \|F(\varepsilon)\| - \|C_\varepsilon(F(\varepsilon))\|$ and

$$\|C_\varepsilon(F(\varepsilon))\| = \|\sum a_j C_\varepsilon(F_j(\varepsilon))\| \leq \|F(\varepsilon)\| \sqrt{\sum_{1 \leq j \leq n} \|C_\varepsilon(F_j(\varepsilon))\|^2}.$$

We obtain finally

$$\Big(1 - \sqrt{\sum_{1 \leq j \leq n} \|C_\varepsilon(F_j(\varepsilon))\|^2}\Big)^2 \lambda_n^1(\varepsilon) \leq \lambda_n(\varepsilon).$$

2. C_ε **converge strongly to 0.** — Let $F_\varepsilon = (f_\varepsilon, h_\varepsilon)$ be a family such that for all $\varepsilon > 0$

$$\|F_\varepsilon\|^2 + \int_{X_1(\varepsilon)} |df_\varepsilon|^2 + q_\varepsilon(h_\varepsilon) \leq B.$$

We write $h_\varepsilon = h^0_\varepsilon + r_\varepsilon$ where $r_\varepsilon = C_\varepsilon(F_\varepsilon)$ and $h^0_\varepsilon \in H^1_0(NUX_2)$. We have to show that $\lim_{\varepsilon\to 0} \|r_\varepsilon\| = 0$.
Let α be the first eigenvalue of the Dirichlet problem of NUX_2 (for q_1!) then $\alpha > 0$ and

$$\alpha\|h^0_\varepsilon\|^2 \leq q_1(h^0_\varepsilon) \leq q_\varepsilon(h^0_\varepsilon) \leq q_\varepsilon(h_\varepsilon) \leq B$$

because $1 \leq 1/\varepsilon^2$ and $q_\varepsilon(h_\varepsilon) = q_\varepsilon(h^0_\varepsilon) + q_\varepsilon(r_\varepsilon)$. Then

$$\|r_\varepsilon\| \leq \|h^0_\varepsilon\| + \|h_\varepsilon\| \leq 2\sqrt{B}$$

We have also

$$q_1(r_\varepsilon) \leq q_\varepsilon(r_\varepsilon) \leq q_\varepsilon(h_\varepsilon) \leq B$$

then the family r_ε is bounded in $H^1(NUX_2)$.
Let $r_n = r_{\varepsilon_n}$, $\varepsilon_n \to 0$ be a subsequence which converges to r weakly in H^1 and in L_2 norm (and there exists at least one).

2.1 $\Pi_1(r) = 0$ if Π_1 is the projection on the orthogonal of the functions constants on the fibers of NUX_2. There exists $0 < \beta \leq \lambda_2(NU_yX_2)$ as explained in the "prépublication 67, p.6". Then

$$\|\Pi_1(r_\varepsilon)\|^2 \leq \int \frac{1}{\beta}|d_V(r_\varepsilon)| \leq \frac{\varepsilon^2}{\beta} q_\varepsilon(r_\varepsilon).$$

2.2 r **is harmonic on** X_2**.** (for the general case r satisfies $A(r) = 0$ here I don't take care of the volume of the fiber which I take equal to 1). Let $f \in C^\infty_0(X_2)$. I identify f with $f \circ \Pi$ constant on the fiber.

$$< dr, df >= \lim < dr_n, df >= 0$$

because f is 0 on the boundary. Then $< \Delta r, f >= 0$, $\forall f \in C^\infty_0(X_2)$ which is dense in $L_2(X_2)$.

2.3 $r \in H^1_0(X_2)$. I have to show that $r_{|\partial X_2} = 0$. Let $f \in H^2_0(X_2)$

$$\begin{aligned}
\int_{\partial X_2} r\partial_{\vec{n}} f + \int_{X_2} r\Delta f &= \int_{NUX_2} (dr, df) \\
&= \lim_{n\to\infty} q_1(r_n, f) = \lim_{n\to\infty} \Big(\int_{\partial NUX_2} r_n \partial_{\vec{n}} f + \int_{NUX_2} r_n \Delta_1 f \Big) \\
&= 0 + \int_{X_2} r\Delta f
\end{aligned}$$

because $\|r_n\|^2_{L_2(\partial NUX_2)} \leq \omega(\varepsilon_n)B$ as recalled at the beginning and $\Delta_1 f = \Delta f$ on a base function.

2.4 Conclusion. r is equal to 0, that is true for any subsequence of the family r_ε so

$$\lim_{\varepsilon\to 0} r_\varepsilon = 0.$$

Last comment. — I think that all this work would be rewriten ; for instance I use also this wrong control in the Lemme 3 of the "prepublication 67, p.3". But this is not necessary : it is always true for a family who has a limit in L_2 that $\|P_\varepsilon(\varphi_\varepsilon)_{|TUB^\varepsilon}\| \to 0$ with ε.
Unfortunately I have no time for the moment to do that. I thank you very much for taking such attention to my work.