SOME COMPLEMENTS TO "ON THE DISTRIBUTION AND ASYMPTOTIC RESULTS FOR EXPONENTIAL FUNCTIONALS OF LÉVY PROCESSES"

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Abstract. The purpose of this note is to give a unified treatment of the following problem: identifiy the distribution of the stochastic integral

$$Z = \int_0^\infty e^{\xi_{s-}}\, d\eta_s \tag{1}$$

where ξ and η are independent Lévy processes. In Theorems 1 and 2 we show that the density and the characteristic function of the distribution of Z are solutions of integro-differential equations, generalizing thus results of Paulsen [3], Nilsen and Paulsen [2], Carmona, Petit and Yor [1].

For a Markov process X, let L^X denote its infinitesimal generator, with domain $\mathcal{D}(L^X)$. We assume in Theorems 1 and 2 that $\mathbb{E}[|\xi_1|] < \infty$ and $\mathbb{E}[\xi_1] < 0$.

Theorem 1. *Assume that:*

- *the law of Z admits a density k;*
- *$k \in \mathcal{D}(L^{-\eta})$;*
- *$\tilde{k} \in \mathcal{D}(L^{-\xi})$, where $\tilde{k}(x) = k(e^x)e^x$.*

Then, k is a solution of the equation

$$L^{-\eta}(k)(x) + \frac{1}{x} L^{-\xi}(\tilde{k})(\log x) = 0 \qquad (x > 0). \tag{2}$$

Let ψ be the Lévy-Khintchine exponent of the Lévy process η, that is

$$\mathbb{E}\left[e^{iu\eta_t}\right] = e^{t\psi(u)} \qquad (t \geq 0, u \in \mathbb{R}).$$

Theorem 2. *Assume that:*

- *for all $t > 0$, the law of e^{ξ_t} admits a density strictly positive on $(0, r_t)$, with $r_t \to +\infty$;*
- *the characteristic function $f(u) = \mathbb{E}\left[e^{iuZ}\right]$ is such that for all real u, the function $f_u(x) = f(ue^x)$ is in the domain $\mathcal{D}(L^\eta)$.*

Then f satisfies the equation

$$L^\xi f_u(0) + f(u)\psi(u) = 0 \qquad (u \in \mathbb{R}). \tag{3}$$

Remark 3. One can state the analog of Theorem 2, by considering Laplace transforms instead of Fourier transforms.

Date: September 17, 1996.

Let us explain the reason why we shall only give sketch of proofs for Theorems 1 and 2. Suppose that you want to identify the distribution of Z for processes ξ and η not considered in the papers [3], [2], or [1]. The sensible thing to do is to: either look for a density function k satisfying (2), either look for a function f satisfying (3) and take the inverse Fourier transform. Then, you should properly define the function spaces on which are defined the semi-groups and infinitesimal generators, and check the domain conditions for f or k.

Proof of Theorem 1. Let us recall that the process

$$V_t = e^{\xi_t}\left(x + \int_0^t e^{-\xi_{s-}}\, d\eta_s \right)$$

is a Markov process (see Corollary 5.2 of [1]). We claim that its infinitesimal generator is

$$L^V f(x) = L^\eta f(x) + L^\xi(f_x)(0)$$

for $f \in \mathcal{D}(L^\eta)$ such that the function $f_x : u \to f(xe^u)$ is in the domain $\mathcal{D}(L^\xi)$. Observe that the invariance of L^ξ under the action of translations entails that for positive x

$$L^V f(x) = L^\eta f(x) + L^\xi(f \circ \exp)(\log x)\,.$$

We now use the identity in law (for the Lévy processes ξ and η)

$$(X_t - X_{(t-s)-}\,;\, s \le t) \stackrel{\mathcal{L}}{=} (X_s\,;\, s \le t)$$

to obtain that for all x:

$$\begin{aligned} V_t &= e^{\xi_t}\, x + \int_0^t e^{\xi_t - \xi_{(t-s)-}}\, d\eta_{t-s} \\ &\stackrel{\mathcal{L}}{=} e^{\xi_t}\, x + \int_0^t e^{\xi_{s-}}\, d\eta_s\,. \end{aligned}$$

The law of large numbers imply that almost surely: $\frac{\xi_t}{t} \underset{t\to\infty}{\longmapsto} \mathbb{E}[\xi_1]$. Hence, for all x:

$$V_t \overset{\mathcal{L}}{\underset{t\to\infty}{\longmapsto}} Z\,. \tag{4}$$

Consequently, the law of Z is the unique stationary law of the Markov process V. Since this law is absolutely continuous with density k, we have:

$$\left\langle L^V f, k \right\rangle = 0 \qquad (\forall f \in \mathcal{D}(L^V))\,,$$

with $\langle f, g\rangle = \int f(x)\, g(x)\, dx$.
We now use the fact that if X is a Lévy process, then the adjoint of L^X, with respect to the Lebesgue measure, is L^{-X}. Indeed, if $(P_t^X\,; t \ge 0)$

denotes the semi-group of X, then, for every positive measurable f and g, we have:

$$\begin{aligned}
\langle P_t f, g\rangle &= \int P_t^X f(x)\, g(x)\, dx \\
&= \int \mathbb{E}\,[f(X_t + x)] g(x)\, dx \\
&= \mathbb{E}\left[\int f(X_t + x)\, g(x)\, dx\right] \\
&= \mathbb{E}\left[\int f(y)\, g(y - X_t)\, dy\right] \\
&= \left\langle f, P_t^{-X} g\right\rangle
\end{aligned}$$

Eventually, taking into account the fact that k has support in $\mathbb{R}_+$, we have, for every "nice" f,

$$\begin{aligned}
0 = \left\langle L^V f, k\right\rangle &= \langle L^\eta f, k\rangle + \left\langle L^\xi (f \circ \exp) \circ \log, k\right\rangle \\
&= \left\langle f, L^{-\eta} k\right\rangle + \int L^\xi (f \circ \exp)(y)\, k(e^y) e^y\, dy \\
&= \left\langle f, L^{-\eta} k\right\rangle + \int f(x)\, L^{-\xi}(k(e^\bullet) e^\bullet)(\log x)\, \frac{dx}{x}
\end{aligned}$$

and we may conclude that k is a solution of (2). □

Proof of Theorem 2. Combining the convergence in distribution (4) with Proposition 5.3 of [1] that says that if $x = 0$:

$$\mathbb{E}\left[e^{iuV_t}\right] = \mathbb{E}\left[\exp\left(\int_0^t \psi(u e^{\xi_s})\, ds\right)\right]$$

we obtain that:

$$f(u) = \mathbb{E}\left[e^{iuZ}\right] = \mathbb{E}_u\left[\exp\left(\int_0^\infty \psi(U_s)\, ds\right)\right]$$

where U denotes the Markov process $U_t = U_0\, e^{\xi_t}$, and $\mathbb{E}_u\,[\bullet]$ the expectation when $U_0 = u$. Observe that the infinitesimal generator of U is $L^U g(u) = L^\xi (g_u)(0)$.

We now consider the martingale

$$M_t = \mathbb{E}_u\left[\exp\left(\int_0^\infty \psi(U_s)\, ds\right) \mid \mathcal{F}_t^\xi\right].$$

By the Markov property of U, we have

$$M_t = e^{S_t} f(U_t) \quad \text{with,} \quad S_t = \int_0^t \psi(U_s)\, ds\,.$$

By assumption, f is "nice" enough so that

$$N_t = f(U_t) - \int_0^t L^U f(U_s)\, ds$$

is a martingale. But Ito's formula yields that

$$dM_t = e^{S_t}\big(dN_t + dt\big(L^U f(U_t) + f(U_t)\psi(U_t)\big)\big)$$

M and N being martingales, we find that for all x in the interval $(0, ur_t)$ if $u > 0$, $(ur_t, 0)$ if $u < 0$, we have:

$$L^U f(x) + f(x)\psi(x) = 0$$

We conclude by letting t go to infinity. □

References

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