L^p cohomology and the boundness of the Riesz transform

This a report on a joint work with Thierry Coulhon (Cergy) and Andrew Hassell (Canberra) [4].

Let M be a complete Riemannian manifold with infinite measure. The Riesz transform T on M is the operator

$$f \to d\Delta^{-1/2} f,$$

where Δ is the positive Laplace operator on M. Thanks to the Green formula we have :

$$\forall f \in C_0^\infty(M),\ \|\Delta^{1/2} f\|_{L^2}^2 = \langle \Delta f, f\rangle = \|df\|_{L^2}^2$$

hence the Riesz transform is always a bounded map from $L^2(M)$ to $L^2(M; T^*M)$. It is of interest to figure out the range of p for which T extends to a bounded map $L^p(M) \to L^p(M; T^*M)$. Equivalently, we can ask whether

$$\| df \|_p \leq \|\Delta^{1/2} f\|_p \text{ for all } f \in C_c^\infty(M).$$

There is a lot of result in this direction. I will only mention few of such results :

i) On $\mathbb{R}^n$, for all $p > 1$, the Riesz transform is bounded on L^p.
ii) On manifold with non negative Ricci curvature, the Riesz transform is bounded on L^p for all $p > 1$ ([3]).
iii) On certain Cartan-Hadamard manifolds, the Riesz transform is bounded on L^p for all $p > 1$
([7]).
iv) On simply connected nilpotent Lie groups endowed with a left invariant metric, the Riesz transform is bounded on L^p for all $p > 1$ ([1]).
v)In [5], T. Coulhon and X.T. Duong have shown that if for some $C > 0$ and $\nu > 0$, (M, g) satisfies the the relative Faber-Krahn inequality :

$$\forall x \in M, R > 0 \text{ and } \Omega \subset B(x,R)\ \lambda_1(\Omega) \geq \frac{C}{R^2}\left(\frac{\mathrm{vol}\Omega}{\mathrm{vol}B(x,R)}\right)^{-2/\nu},$$

then for all $p \in]1, 2]$ the Riesz transform is bounded on L^p.

They also indicated that for $p > n$, the Riesz transform is not bounded on L^p on a connected sum of two Euclidean space.

Our result is the following :

Theorem *Let M be a complete Riemannian manifold of dimension $n \geq 3$ which is the union of a compact part and a finite number of Euclidean ends. Then the Riesz transform is bounded from $L^p(M)$ to $L^p(M; T^*M)$ for $1 < p < n$, and is unbounded on L^p for all other values of p if the number of ends is at least two.*

Let's describe a key point in the proof in the case of a connected sum of two Euclidean space $M = \mathbb{R}^n \# \mathbb{R}^n$. Such manifold is topologically the product $\mathbb{R} \times \mathbb{S}^{n-1}$ endowed with the metric

$$(dr)^2 + (1 + r^2)(d\theta)^2$$

where $(d\theta)^2$ is the standard round metric on the sphere $\mathbb{S}^{n-1}$ and r the radial variable. One of our argument is a precise description of the behavior for large x and y of the Schwarz kernel of the operator :

$$\Delta^{-1/2} = \frac{2}{\pi} \int_0^\infty (\Delta + k^2)^{-1} \, dk.$$

If $P(x, y)$ is the kernel of this operator, we show that when $x \in M$ and $y = (r, \theta)$ then for $r \to \pm\infty$

$$P(x, (r, \theta)) \simeq \frac{u_\pm(x)}{r^{n-1}}$$

where $u_\pm$ is the harmonic function such that

$$\lim_{r \to \pm\infty} u_\pm(x) = 1$$

$$\lim_{r \to \mp\infty} u_\pm(x) = 0$$

At infinity, we also get

$$|du_\pm(r, \omega)| = O(r^{1-n}).$$

With this estimate, we can show that the Riesz transform can be bounded only when $p < n$. To get the full result, we obtain sharper estimates by using the scattering calculus introduced by R. Melrose ([**?**]).

In ([2]), authors have remarked that if the Riesz transform is bounded on $L^{p/(p-1)}$ and on L^p then the operator $P = d\Delta^{-1}d^*$ extend to a bounded operator on $L^p(T^*M)$. On $L^2(T^*M)$, this operator P is the orthogonal projection on the closure of the space $dC_0^\infty(M)$. But on (M, g), the space of L^2 differential one-forms admits the Hodge decomposition

$$L^2(T^*M) = \mathcal{H}^1(M) \oplus \overline{dC_0^\infty(M)} \oplus \overline{d^*C_0^\infty(\Lambda^2 T^*M)},$$

where $\mathcal{H}^1(M) = \{\alpha \in L^2(T^*M), d\alpha = 0 = d^*\alpha\}$ (see [6]). Let us recall now the definition of reduced L^p-cohomology: for $p \geq 1$, the first space of reduced L^p cohomology of (M, g) is

$$H_p^1(M) = \frac{\{\alpha \in L^p(T^*M), d\alpha = 0\}}{\overline{dC_0^\infty(M)}} \ ,$$

where we take the closure in L^p. On L^2, we have

$$\{\alpha \in L^2(T^*M), d\alpha = 0\} = \mathcal{H}^1(M) \oplus \overline{dC_0^\infty(M)},$$

hence the first space of reduced L^2 cohomology can be identified with $\mathcal{H}^1(M)$.

We assume now that the manifold (M, g) satisfies the following conditions

$$\left(\int_M |f|^{\frac{2\nu}{\nu-2}} \text{dvol} \right)^{1-2/\nu} \leq C \int_M |df|^2 \text{dvol}, \ \forall f \in C_0^\infty(M),$$

And that at one point $x_0 \in M$, we have a control on the growth of geodesic balls centered at x_0 :

$$\operatorname{vol} B(x_0, r) \leq C r^\nu, \forall r \geq 1.$$

Proposition *Under these two hypotheses, if the Riesz transform is bounded in L^p for some $p > 2$, then*

$$H^1_p(M) = \{\alpha \in L^p(M; T^*M) \mid d^*\alpha = d\alpha = 0\}.$$

If moreover the Ricci curvature of M is bounded from below then there is a natural map

$$H^1_2(M) \to H^1_p(M)$$

which is injective.

Corollary *If M has at least two ends, then the Riesz transform is not bounded on L^p for any $p \geq \nu$.*

Let's us describe two examples :

First example: The manifold $M = \mathbb{R}^n \# \mathbb{R}^n = \mathbb{R} \times \mathbb{S}^{n-1}$ endowed with a metric $(dr)^2 + (1+r^2)(d\theta)^2$. Then it is easy to show by direct computation that

$$\{\alpha \in L^p(M; T^*M) \mid d^*\alpha = d\alpha = 0\} = \mathbb{R} \frac{dr}{(1+r^2)^{\frac{n-1}{2}}}.$$

Where as

$$H^1_p(M) = \begin{cases} \mathbb{R} & \text{if} p < n \\ \{0\} & \text{if } p \geq n \end{cases}$$

Hence the map $H^1_2(M) \to H^1_p(M)$ is injective only for $p < n$. The proposition gives in this case the right range of p's where the Riesz transform is bounded on L^p.

Second example: Let M be a connected sum of several (say $l \geq 2$) copies of a simply connected nilpotent Lie group N endowed with a left invariant metric, let ν be the homogeneous dimension of N; that is

$$\nu = \lim_{R \to \infty} \frac{\log \operatorname{vol} B(o, R)}{\log R},$$

Then M satisfies the Sobolev inequality, the upper bound on the volume growth of geodesic balls and more over it's clear that its Ricci curvature is bounded from below. We known that the Riesz transform is bounded on L^p for $p \in]1, 2]$ and not bounded on L^p for any $p \geq \nu$. Moreover we can compute the L^p cohomology of M:

$$H^1_p(M) = \begin{cases} H^1_c(M) \simeq \mathbb{R}^{l-1} & \text{if } p < \nu \\ \{0\} & \text{if } p > \nu \end{cases}$$

Hence the map

$$H^1_2(M) \to H^1_p(M)$$

is injective when $p < \nu$. It is then tempting to propose the following conjecture :

On such a manifold M , is the Riesz transform bounded on any $p \in]1, \nu[$?

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