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Gaussian estimates and L^p-boundedness of Riesz means

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Abstract. We show that suitable upper estimates of the heat kernel are sufficient to imply the L^p boundedness of several families of operators associated with the Schrödinger group in various situations. This generalizes results by Sjöstrand and others in the Euclidean case, and by Alexopoulos in the case of Lie groups and Riemannian manifolds.

1. Introduction

It is well-known ([28]) that the Schrödinger group $e^{i\Delta t}$ acts boundedly on $L^p(\mathbb{R}^N)$ only if $p = 2$.

For $p \neq 2$, several substitutes to the L^p boundedness have been found: Lanconelli ([32]) has shown that for $\alpha > 2N|\frac{1}{2} - \frac{1}{p}|$, $e^{i\Delta t}$ sends the Sobolev space $L^p_\alpha(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, in other words $(I + \Delta)^{-\alpha/2}e^{i\Delta t}$ acts boundedly on $L^p(\mathbb{R}^N)$, and that this is not the case if $0 < \alpha < 2N|\frac{1}{2} - \frac{1}{p}|$. Sjöstrand ([44]) has considered, for k an integer, the Riesz means

$$t^{-k}\int_0^t (t-s)^{k-1}e^{i\Delta s}\,ds,$$

and he has shown that they act on $L^p(\mathbb{R}^N)$ and are uniformly bounded in $t > 0$, if $k > N|\frac{1}{2} - \frac{1}{p}|$, and that this is not the case if $k < N|\frac{1}{2} - \frac{1}{p}|$.

These results and their generalizations were in fact results on multipliers and relied heavily on Fourier analysis. They can be formulated and generalized in the language of more or less recent theories such as distribution semigroups ([7]), C-semigroups ([9]), or integrated semigroups ([29]). See also the recent book [5], in particular Theorem 8.3.9.

On Lie groups with polynomial growth and manifolds with non-negative Ricci curvature, similar results have been first announced by Lohoué in [34], then Alexopoulos obtained sharp exponents in [1]. There, the method is to replace Fourier analysis by the finite propagation speed of the associated wave equation, as in [46]. The recent preprint [2] emphasizes even more that what is needed is simply an upper Gaussian estimate of the heat kernel.

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Here we propose a different approach. We consider an open subset Ω of a metric measured space X with regular volume growth (the so-called doubling volume property), and a non-negative self-adjoint operator A on $L^2(\Omega)$ whose heat kernel satisfies upper bounds which are typical of operators of order m (see (2.3) below). Using some techniques introduced by Davies, we show that the upper Gaussian estimate extends to a similar estimate where time ranges in the whole complex right half-plane. This pointwise estimate yields an estimate for the $L^p - L^p$ norm of the semigroup e^{-zA}, $z \in \mathbb{C}^+$, where $\mathbb{C}^+$ is the open right half-plane, namely

$$\|e^{-zA}\|_{p\to p} \le C_\varepsilon \left(\frac{|z|}{\mathrm{Re} z}\right)^{\nu|\frac{1}{2}-\frac{1}{p}|+\varepsilon}, \ \forall\, p \in [1, +\infty], \ \forall\, z \in \mathbb{C}^+$$

(Theorem 4.3 below). Here ν is the exponent naturally associated with the doubling condition (see (2.2) below) and is equal to N when $X = \mathbb{R}^N$. Of course this estimate explodes as the real part of z goes to zero, but a contour integration technique due to El-Mennaoui ([23]) enables one to show, in the limit, that the Riesz means

$$I_\alpha(t) := t^{-\alpha} \int_0^t (t-s)^{\alpha-1} e^{-iAs}\, ds$$

are uniformly bounded on $L^p(\Omega)$ for $t > 0$, $1 \le p \le +\infty$ and $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$ (Theorem 5.1). In other terms, iA generates an α-integrated group (Theorem 5.2), or a $(I + A)^{-\alpha/2}$-regularized group (Theorem 5.3), for $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$. Yet another reformulation of our results is that heat kernel upper estimates suffice to run the theory of boundary values of holomorphic (semi)groups ([9], [6]). Note that the condition on α is sharp already when $\Omega = X = \mathbb{R}^N$, and $-A$ is the Laplace operator (see [23], [9], [6]). Note also that the case $\alpha = N|\frac{1}{2} - \frac{1}{p}|$ has been treated, at least when α is an integer, in [5], Theorem 8.3.9, see also the closing remarks of [1] as well as [35] for the case of manifolds.

In the forthcoming paper [21], some of these ideas and techniques are pushed further and more general multipliers are treated. We feel however that it is worth emphasizing a simple and direct treatment for the Schrödinger group. In particular, we do not use singular integrals theory and we are able to include the case $p = 1$.

Our method enables us to treat a lot of examples, and to unify and extend the results by a number of authors. We refer to Sections 3 and 6 for details and credits. For instance, we obtain the correct range of α's, i.e. $\alpha > N|\frac{1}{2} - \frac{1}{p}|$, in the case of the Laplace operator with Dirichlet boundary conditions on arbitrary domains of $\mathbb{R}^N$, which is new.

On the other hand, our method (contrary to the one in [1], in its range of application) does not seem to cover wave operators, and more generally fractional powers of second order operators, in an optimal way; it would be interesting to try to by-pass singular integrals theory in that case also.

2. Notation and assumptions

Let (X, d, μ) be a metric measured space. Denote by $B(x, r)$ the open ball in X for the distance d, of center $x \in X$ and radius $r > 0$, and by $V(x, r)$ its measure $\mu\,(B(x, r))$, which we shall assume always finite.

Throughout this paper, we shall assume that X has regular volume growth (or satisfies the doubling volume property), i.e.

$$V(x, 2r) \leq C\, V(x, r),\ \forall\, x \in X,\ r > 0. \tag{2.1}$$

Note that (2.1) implies easily that there exists $\nu > 0$ such that

$$V(x, s) \leq C \left(\frac{s}{r}\right)^{\nu} V(x, r),\ \forall x \in X,\ s \geq r > 0. \tag{2.2}$$

If $X = \mathbb{R}^N$ endowed with the Euclidean distance and Lebesgue measure, (2.2) is satisfied with $\nu = N$. Our aim is to obtain on more general spaces some estimates similar to the ones in [44] for the Euclidean spaces, with the Euclidean dimension replaced by ν.

Let Ω be an open set in X, and let A be a non-negative self-adjoint operator on $L^2(\Omega, \mu)$. Assume that the semigroup e^{-tA} has a kernel p_t, i.e. a continuous fonction $\Omega \times \Omega :\to \mathbb{C}$ such that

$$e^{-tA} f(x) = \int_{\Omega} p_t(x, y) f(y)\, d\mu(y),\ \text{for every } f \in L^2(\Omega) \text{ and a.e. } x \in \Omega.$$

We shall call p_t the heat kernel associated with A, even though our setting includes higher order differential operators.

We shall assume that p_t satisfies a so-called Gaussian upper estimate or order $m \geq 2$, i.e. that there exists $C, c > 0$ such that

$$|p_t(x, y)| \leq \frac{C}{V(x, t^{1/m})} \exp\left(-c \left(\frac{d^m(x, y)}{t}\right)^{\frac{1}{m-1}}\right),$$
$$\text{for all } t > 0 \text{ and a.e. } x, y \in \Omega. \tag{2.3}$$

Notice that in (2.3), $V(x, t^{1/m})$ is the volume of the ball in X and not in Ω, and that Ω itself is **not** supposed to satisfy the doubling volume property. Observe also that, thanks to (2.1), one can replace in the above estimate $V(x, t^{1/m})$ by $\sqrt{V(x, t^{1/m}) V(y, t^{1/m})}$, provided one changes c, C.

Since A is self-adjoint, iA generates a group of isometries of $L^2(\Omega)$, e^{itA}, which we shall call the Schrödinger group associated with A. Our main interest will be in L^p mapping properties of families of operators derived from the Schrödinger group.

3. Examples

3.1. *Laplacians on Riemannian manifolds and Lie groups*

Consider the case where $X = \Omega = M$ is a Riemannian manifold and $A = -\Delta$, where Δ is the Laplace-Beltrami operator. Let d be the geodesic distance, and μ the Riemannian measure.

Assume that M satisfies (2.1) and that the heat kernel p_t satisfies the upper bound

$$p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})}, \ \forall x \in M, \ t > 0. \tag{3.1}$$

Then the results of Sections 5 and 6 below apply. Indeed, (3.1) together with (2.1) self-improves into (2.3) for $m = 2$: see [27], Theorem 1.1.

The conjunction of conditions (3.1) and (2.1) is relatively well understood. According to the work of A. Grigor'yan [26], it is equivalent to the following relative Faber-Krahn inequality

$$\lambda_1(\omega) \geq \frac{c}{R^2} \left(\frac{|\omega|}{V(x, R)} \right)^{-2/\nu},$$

for all $x \in M$, $R > 0$, ω smooth precompact subset of $B(x, R)$. Here $|\omega|$ is the volume of ω and $\lambda_1(\omega)$ is the first eigenvalue of the Laplace operator on ω with Dirichlet boundary conditions:

$$\lambda_1(\omega) = \inf \left\{ \frac{\int_\omega |\nabla u|^2}{\int_\omega u^2}, \ u \in C_0^\infty(\omega) \right\}.$$

The fact that conditions (3.1) and (2.1) hold true on manifolds with non-negative Ricci curvature is known since [33]. But they may hold in much more general situations, including situations where Poincaré inequalities, the parabolic Harnack principle, or the lower bound counterpart of (2.3) for $m = 2$ do not hold.

Surprisingly enough, the weak assumptions (3.1) and (2.1) are enough to carry out a part of what is usually called real analysis (see [10]). On the other hand, some more subtle phenomena or limit cases seem to require the stronger Poincaré inequalities (see [40], [41]). It is interesting to know that basic L^p properties of the Schrödinger group are on the easy side of the picture.

In the above situation, Theorem 5.1 below is contained in [1], Theorem 1, see also [2].

The case of polynomial growth Lie groups endowed with a sublaplacian is also covered in [1] as well as by our method. There $V(x, r) \simeq r^d$ for small r, $V(x, r) \simeq r^D$ for large r, where d, D are integers, D depending only on the Lie group and d depending also on the sublaplacian; one therefore finds $\nu = \max(d, D)$. For a precise statement we refer to [1].

3.2. *Operators acting on domains*

Assume that the Riemannian manifold M and its heat kernel $p_t(x, y)$ satisfy assumptions (2.1) and (3.1). Let Ω be any open set of M and denote by Δ_Ω the Laplacian acting on $L^2(\Omega)$ with Dirichlet boundary conditions. Let $h_t(x, y)$ denote the heat kernel associated with $-\Delta_\Omega$. From the well known inequality

$$h_t(x, y) \leq p_t(x, y), \ \forall t > 0, x, y \in \Omega,$$

we deduce that the Gaussian upper bound (2.3) with $m = 2$ is satisfied for the kernel $h_t(x, y)$. As a consequence of our results, we obtain the fact that the operator

$$I_{\alpha,\Omega}(t) = t^{-\alpha} \int_0^t (t-s)^{\alpha-1} e^{is\Delta_\Omega}\, ds$$

is bounded uniformly in t on $L^p(\Omega)$ whenever $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$. Moreover the operator $i\Delta_\Omega$ generates an α-integrated group on $L^p(\Omega)$ for any $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$. This result is new even in the case where $M = \mathbb{R}^N$ endowed with the flat metric. In this latter case, the previously known result asserts that, for Ω an open subset of $\mathbb{R}^N$, $i\Delta_\Omega$ generates an α-integrated group on $L^p(\Omega)$ for any $\alpha > 2N|\frac{1}{2} - \frac{1}{p}|$ ([23]). Our result shows that this is the case for $\alpha > N|\frac{1}{2} - \frac{1}{p}|$. Note however that if Ω has finite volume then one can show a better result ([3], [24]).

3.3. *Adding a potential*

Consider a potential $V : \mathbb{R}^N \to \mathbb{R}$ whose positive part V^+ belongs to L^1_{loc} and whose negative part V^- is in the Kato class (see [43]). Let $B = -\Delta + V$ acting on $L^2(\mathbb{R}^N)$; denote by $p_t(x, y)$ the associated heat kernel. It is known ([43], Prop. B.6.7) that for some $\omega \geq 0$, $p_t(x, y)e^{-\omega t}$ satisfies (2.3) with $m = 2$. Thus by our results the Riesz means associated with the operator $A = B + \omega I$ are bounded in $L^p(\mathbb{R}^N)$ provided $\alpha > N|\frac{1}{2} - \frac{1}{p}|$. For the same α's, $(B + (\omega + 1)I)^{-\alpha} e^{itB}$ is bounded on $L^p(\mathbb{R}^N)$ with norm estimated from above by $C(1 + |t|)^\alpha$. This improves a result in [9] (see also [39], [37]) where it is assumed that $\alpha > 2N|\frac{1}{2} - \frac{1}{p}|$. Note also that the class of potentials under consideration in [9] and in [39] is more restricted than ours.

One could also try to add potentials to higher order constant coefficients operator as in [8].

3.4. *Higher order operators*

Consider an elliptic self-adjoint differential operator on $L^2(\mathbb{R}^N)$ of the form

$$Hf(x) = \sum_{|\alpha|\leq n, |\beta|\leq n} (-1)^{|\alpha|} D^\alpha (a_{\alpha,\beta}(x) D^\beta f(x)).$$

In the case of smooth coefficients, it has been known for some time that (2.3) with $m = 2n$ holds for $A = H + \omega I$ ([25]). When the coefficients are only measurable, it is still the case if $N \leq 2n$, but false if $N > 2n \geq 4$. A lot of information on this and other recent developments in the field is contained in the survey [17].

4. Heat kernel upper bounds for complex time

From now on, we shall consider a metric measured space (X, d, μ), an open subset Ω of x and a non-negative self-adjoint operator A on $L^2(\Omega, \mu)$ satisfying all the assumptions of Section 2.

4.1. *Pointwise bounds*

Let $\mathbb{C}^+ := \{z \in \mathbb{C}; \mathrm{Re} z > 0\}$. The semigroup e^{-zA} acting on $L^2(\Omega, \mu)$ is holomorphic on $\mathbb{C}^+$ and, for fixed $x, y \in \Omega$, its kernel $p_z(x, y)$ extends to a holomorphic function of $z \in \mathbb{C}^+$. This is a very general fact, see [16], Lemma 2, and by the same token one obtains the estimate

$$|p_z(x, y)| \leq \sqrt{p_t(x, x) p_t(y, y)} \leq \frac{C}{(V(x, t^{1/m}) V(y, t^{1/m}))^{1/2}}, \tag{4.1}$$

for all $x, y \in \Omega$, $z \in \mathbb{C}^+$, where $t = \mathrm{Re}\, z$.

To estimate the $L^p - L^p$ norm of some regularized forms of the Schrödinger group e^{itA}, $t \in \mathbb{R}$, we shall need an estimate of the $L^p - L^p$ norm of the heat semigroup e^{-zA}, valid for $z \in \mathbb{C}^+$. This requires an estimate of $p_z(x, y)$ valid for the same values of z, but with a Gaussian term involving $d(x, y)$ similar to the one in (2.3).

A technique to obtain such estimates was introduced by Davies and consists in using analytic function theory to interpolate between (2.3) and (4.1).

In the case where $V(x, t^{1/m})$ can be replaced by a power function $t^{\frac{N}{m}}$, this was done in [12], Theorem 3.4.8 (for $m = 2$ but the general case is similar), see also [22], p. 118–119.

In the non-uniform case, i.e. when $V(x, r)$ depends on x, this technique was pushed further in [22] and [11], to obtain estimates of the form

$$|p_z(x, y)| \leq \frac{C}{(V(x, t^{1/m})) V(y, t^{1/m}))^{1/2}} \exp\left(-c \left(\frac{d^m(x, y)}{t}\right)^{\frac{1}{m-1}}\right)$$

(see [11], Lemmas 2.3 and 2.4). However, these estimates are valid in a sector which is strictly included in $\mathbb{C}^+$, and, by keeping x, y fixed and distinct and letting $t = \mathrm{Re} z$ go to zero, one sees that they cannot hold on the whole of $\mathbb{C}^+$.

The following version of the same bounds does extend to the whole right half-plane, which is crucial for our purpose.

PROPOSITION 4.1. *One has*

$$|p_z(x,y)| \leq \frac{C}{(V(x,(\frac{|z|}{(\cos\theta)^{m-1}})^{1/m})V(y,(\frac{|z|}{(\cos\theta)^{m-1}})^{1/m}))^{1/2}} \exp\left(-c\left(\frac{d^m(x,y)}{|z|}\right)^{\frac{1}{m-1}}\cos\theta\right)\frac{1}{(\cos\theta)^{\nu}}$$

for all $z \in \mathbb{C}^+$ *and all* $x, y \in \Omega$, *where* $\theta = \arg z$.

Proof. It is enough to prove that there exists C, c such that.

$$|p_z(x,y)| \leq \frac{Ce^{\frac{\mathrm{Re}z}{\lambda}}}{(V(x,\lambda^{1/m})V(y,\lambda^{1/m}))^{1/2}} \exp\left(-c\left(\frac{d^m(x,y)}{|z|}\right)^{\frac{1}{m-1}}\cos\theta\right)\left(\frac{\lambda}{\mathrm{Re}z}\right)^{\nu/m}, \tag{4.2}$$

for all $\lambda > 0$, all $z \in \mathbb{C}^+$ and a.e. $x, y \in \Omega$.

Indeed, if (4.2) holds, choosing

$$\lambda = \frac{|z|}{(\cos\theta)^{m-1}} = \frac{|z|^m}{(\mathrm{Re}z)^{m-1}} = \frac{\mathrm{Re}z}{(\cos\theta)^m}$$

yields the proposition.

Let us now prove (4.2). Fix $\lambda \in]0, +\infty[$.

If z is such that $\mathrm{Re}z \in]0, \lambda]$, apply (4.1) and write

$$V(x,\lambda^{1/m}) \leq C\left(\frac{\lambda}{\mathrm{Re}z}\right)^{\nu/m} V(x,(\mathrm{Re}z)^{1/m}),$$

which gives

$$\begin{aligned} |p_z(x,y)| &\leq \frac{C}{(V(x,(\mathrm{Re}z)^{1/m})V(y,(\mathrm{Re}z)^{1/m}))^{1/2}} \\ &\leq \frac{C'}{(V(x,\lambda^{1/m})V(y,\lambda^{1/m}))^{1/2}}\left(\frac{\lambda}{\mathrm{Re}z}\right)^{\nu/m}. \end{aligned}$$

If moreover $z = t$ belongs to $]0, \lambda[$, use (2.3) and write

$$|p_t(x,y)| \leq \frac{C}{(V(x,\lambda^{1/m})V(y,\lambda^{1/m}))^{1/2}}\left(\frac{\lambda}{t}\right)^{\nu/m}\exp\left(-c\left(\frac{d^m(x,y)}{t}\right)^{\frac{1}{m-1}}\right).$$

If on the contrary $\mathrm{Re}z > \lambda$, write

$$\begin{aligned}|p_z(x,y)| &\leq \frac{C}{(V(x,(\mathrm{Re}z)^{1/m})V(y,(\mathrm{Re}z)^{1/m}))^{1/2}}\\ &\leq \frac{C'}{(V(x,\lambda^{1/m})V(y,\lambda^{1/m}))^{1/2}}\left(\frac{\lambda}{\mathrm{Re}z}\right)^{\nu/m} e^{\frac{\mathrm{Re}z}{\lambda}},\end{aligned}$$

and if moreover $z = t$ belongs to $]\lambda, +\infty[$,

$$\begin{aligned}|p_t(x,y)| &\leq \frac{C}{(V(x,t^{1/m})V(y,t^{1/m}))^{1/2}}\exp\left(-c\left(\frac{d^m(x,y)}{t}\right)^{\frac{1}{m-1}}\right)\\ &\leq \frac{C'}{(V(x,\lambda^{1/m})V(y,\lambda^{1/m}))^{1/2}}\left(\frac{\lambda}{t}\right)^{\nu/m} e^{t/\lambda}\exp\left(-c\left(\frac{d^m(x,y)}{t}\right)^{\frac{1}{m-1}}\right).\end{aligned}$$

Finally, if $\mathrm{Re}z, \lambda > 0$ one always has

$$|p_z(x,y)| \leq \frac{C}{(V(x,\lambda^{1/m})V(y,\lambda^{1/m}))^{1/2}}\left(\frac{\lambda}{\mathrm{Re}z}\right)^{\nu/m} e^{\frac{\mathrm{Re}z}{\lambda}},$$

and if moreover $z = t$ belongs to $]0, +\infty[$,

$$|p_t(x,y)| \leq \frac{C}{(V(x,\lambda^{1/m})V(y,\lambda^{1/m}))^{1/2}}\left(\frac{\lambda}{t}\right)^{\nu/m} e^{t/\lambda}\exp\left(-c\left(\frac{d^m(x,y)}{t}\right)^{\frac{1}{m-1}}\right).$$

Recall the following very useful consequence of the three lines lemma, which was worked out and used in a similar context by E. -B. Davies.

LEMMA 4.2. ([13] Lemma 9) *Let F be analytic on $\mathbb{C}^+$. Suppose that it satisfies the bounds*

$$|F(re^{i\theta})| \leq a_1(r\cos\theta)^{-\beta}$$

$$|F(r)| \leq a_1 r^{-\beta}\exp(-a_2 r^{-\alpha})$$

for some $a_1, a_2 > 0$, $\beta \geq 0$, $\alpha \in]0,1]$, all $r > 0$ and all $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Then

$$|F(re^{i\theta})| \leq a_1 2^{\beta}(r\cos\theta)^{-\beta}\exp\left(-\frac{a_2\alpha}{2}r^{-\alpha}\cos\theta\right)$$

for all $r > 0$ and all $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

Note that the statement of Lemma 9 in [13] contains a misprint: it should read $0 < \alpha \leq 1$ instead of $0 < \alpha < 1$. Only the former inequality is used in the proof.

One can now apply Lemma 4.2 to $F(z) = p_z(x,y)e^{-\frac{z}{\lambda}}$, with $a_1 = \frac{C\lambda^{\nu/m}}{(V(x,\lambda^{1/m})V(y,\lambda^{1/m}))^{1/2}}$, $\beta = \nu/m$, $a_2 = c[d(x,y)]^{\frac{m}{m-1}}$ and $\alpha = \frac{1}{m-1}$. This yields (4.2) with constants independent of λ, and finishes the proof of Proposition 4.1. □

4.2. *L^p bound for the heat operator*

The pointwise bound in Proposition 4.1 has the following integrated form. One denotes by $\|.\|_{p\to p}$ the operator norm from $L^p(\Omega)$ to itself.

THEOREM 4.3. *The following bound holds for all $\varepsilon > 0$:*

$$\|e^{-zA}\|_{p\to p} \le C_\varepsilon \left(\frac{|z|}{\mathrm{Re} z}\right)^{\nu|\frac{1}{2}-\frac{1}{p}|+\varepsilon}, \ \forall\, p \in [1, +\infty], \ \forall\, z \in \mathbb{C}^+. \tag{4.3}$$

Proof. It follows easily from the doubling volume property that

$$\frac{e^{-c(\frac{d^m(x,y)}{t})^{\frac{1}{m-1}}}}{(V(x,t^{1/m})V(y,t^{1/m}))^{1/2}} \le C' \frac{e^{-\frac{c}{2}(\frac{d^m(x,y)}{t})^{\frac{1}{m-1}}}}{V(y,t^{1/m})}, \ \forall\, t > 0, \ x, y \in \Omega.$$

This inequality and the doubling volume property imply the estimate

$$\sup_{y\in\Omega, t>0} \int_\Omega \frac{e^{-c(\frac{d^m(x,y)}{t})^{\frac{1}{m-1}}}}{(V(x,t^{1/m})V(y,t^{1/m}))^{1/2}}\, d\mu(x) < +\infty. \tag{4.4}$$

Indeed,

$$\begin{aligned}
\int_\Omega \frac{e^{-c(\frac{d^m(x,y)}{t})^{\frac{1}{m-1}}}}{V(y,t^{1/m})}\, d\mu(x) &= \sum_{k\ge 0} \int_{\{kt^{1/m}\le d(x,y)\le (k+1)t^{1/m}\}} \frac{e^{-c(\frac{d^m(x,y)}{t})^{\frac{1}{m-1}}}}{V(y,t^{1/m})}\, d\mu(x) \\
&\le \sum_{k\ge 0} e^{-ck^{\frac{m}{m-1}}} \frac{V(y,(k+1)t^{1/m})}{V(y,t^{1/m})} \\
&\le \sum_{k\ge 0} e^{-ck^{\frac{m}{m-1}}} (k+1)^\nu < +\infty
\end{aligned}$$

which shows (4.4).

By the semigroup property one has

$$\int_\Omega |p_{re^{i\theta}}(x,y)|^2\, d\mu(x) = \int_\Omega p_{re^{i\theta}}(x,y) p_{re^{-i\theta}}(y,x)\, d\mu(x) = p_{2r\cos\theta}(y,y).$$

We then obtain from assumption (2.3) that

$$\int_\Omega |p_{re^{i\theta}}(x,y)|^2\, d\mu(x) \le \frac{C}{V(y,(r\cos\theta)^{1/m})}. \tag{4.5}$$

In the rest of the proof we will follow an idea which was used in [22] and [20]. Let $\varepsilon \in]0, 1[$. Let $p_\varepsilon \in]0, 1[$ be such that $1 = \frac{1-\varepsilon}{2} + \frac{\varepsilon}{p_\varepsilon}$ and write

$$\int_\Omega |p_{re^{i\theta}}(x, y)|\, d\mu(x) \leq \left(\int_\Omega |p_{re^{i\theta}}(x, y)|^2\, d\mu(x)\right)^{\frac{1-\varepsilon}{2}} \left(\int_\Omega |p_{re^{i\theta}}(x, y)|^{p_\varepsilon}\, d\mu(x)\right)^{\frac{1+\varepsilon}{2}}. \tag{4.6}$$

From Proposition 4.1 it follows that

$$\int_\Omega |p_{re^{i\theta}}(x, y)|^{p_\varepsilon}\, d\mu(x) \leq \frac{C}{V(y, (\frac{r}{(\cos\theta)^{m-1}})^{1/m})^{p_\varepsilon}(\cos\theta)^{p_\varepsilon \nu}} \int_\Omega \exp\left(-c'\left(\frac{d^m(x, y)}{r}\right)^{\frac{1}{m-1}} \cos\theta\right) d\mu(x).$$

From (4.4) with $t = \frac{r}{(\cos\theta)^{m-1}}$ we deduce the estimate

$$\int_\Omega |p_{re^{i\theta}}(x, y)|^{p_\varepsilon}\, d\mu(x) \leq \frac{C'}{(\cos\theta)^{p_\varepsilon \nu}} V\left(y, \left(\frac{r}{(\cos\theta)^{m-1}}\right)^{1/m}\right)^{1-p_\varepsilon}.$$

But the doubling property (2.2) implies that

$$V\left(y, \left(\frac{r}{(\cos\theta)^{m-1}}\right)^{1/m}\right) = V\left(y, (r\cos\theta)^{1/m}\frac{1}{\cos\theta}\right) \leq C(\cos\theta)^{-\nu} V(y, (r\cos\theta)^{1/m}),$$

therefore

$$\int_\Omega |p_{re^{i\theta}}(x, y)|^{p_\varepsilon}\, d\mu(x) \leq \frac{C}{(\cos\theta)^\nu} V(y, (r\cos\theta)^{1/m})^{1-p_\varepsilon}. \tag{4.7}$$

Inserting (4.5) and (4.7) in (4.6) we obtain

$$\int_\Omega |p_{re^{i\theta}}(x, y)|\, d\mu(x) \leq \frac{C''}{(\cos\theta)^{\frac{\nu(1+\varepsilon)}{2}}},$$

i.e.

$$\|e^{-zA}\|_{1\to 1} = \sup_{y\in\Omega} \int_\Omega |p_z(x, y)|\, d\mu(x) \leq C''\left(\frac{|z|}{\mathrm{Re}z}\right)^{\frac{\nu(1+\varepsilon)}{2}}.$$

The estimate of $\|e^{-zA}\|_{p\to p}$ follows now for $p \in [1, 2]$ by interpolation with the fact that, since A is self-adjoint,

$$\|e^{-zA}\|_{2\to 2} \leq 1$$

and for $p \in [2, +\infty]$ by duality. □

REMARK. Theorem 4.3 implies that the semigroup e^{-zA} acts as a holomorphic semigroup of angle $\frac{\pi}{2}$ on $L^p(\Omega)$, $1 \leq p < +\infty$. This was shown for domains of $\mathbb{R}^N$ in [38] and in [15] for manifolds with polynomial volume growth. These proofs extend verbatim to our setting.

Another application of Theorem 4.3 is that the spectrum of A in $L^p(\Omega)$ is independent of $p \in [1, +\infty]$ (see [13] and [15]). In the case where A is the Laplace operator on a Riemannian manifold, a better result in this direction is shown in [45] (see also [42]) where the only assumption is that the manifold has a sub-exponential volume growth.

5. L^p mapping properties for the Schrödinger group

5.1. *Riesz means for the Schrödinger group*

We can now assert the following abstract version of Sjöstrand's result ([44]). Note that the cases $p = 1, \infty$ are included.

THEOREM 5.1. *For all $p \in [1, +\infty]$ and every $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$, the Riesz mean operator defined by*

$$I_\alpha(t) = t^{-\alpha} \int_0^t (t-s)^{\alpha-1} e^{-isA}\, ds$$

for $t > 0$, and $I_\alpha(t) = \bar{I}_\alpha(-t)$ for $t < 0$, acts continously on $L^p(\Omega)$ and one has

$$\|I_\alpha(t)\|_{p\to p} \leq C_\alpha\,, \ \forall t \in \mathbb{R}^*. \tag{5.1}$$

Proof. It relies on the estimate (4.3). We follow closely an argument by El Mennaoui ([23], §2.4). For $z \in \mathbb{C}^+$, consider the operator

$$J_\alpha(z) = \int_{[0,z]} (z-\zeta)^{\alpha-1} e^{-\zeta A}\, d\zeta.$$

We are going to show that, if $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$,

$$\|J_\alpha(z)\|_{p\to p} \leq C_\alpha |z|^\alpha. \tag{5.2}$$

This will imply by continuity that, for $z = it$, $t \in \mathbb{R}$, $i^\alpha t^\alpha I_\alpha(t) = J_\alpha(it)$, which is well defined on $L^2 \cap L^p$, extends continously to a bounded operator on L^p. Finally, (5.2) yields (5.1).

Because $\zeta \mapsto e^{-\zeta A}$ is an holomorphic semigroup on $\mathbb{C}^+$, $J_\alpha(z)$ doesn't depend on the path chosen to integrate from 0 to z. We choose the following path: first, follow the straight

line $[0, |z|]$ and then follow the circle with radius $|z|$ and center 0. Let $J^1_\alpha(z)$ denote the first integral, $J^2_\alpha(z)$ the second one. One has

$$J^1_\alpha(z) = \int_0^{|z|} (z-s)^{\alpha-1} e^{-sA}\, ds.$$

Now it is well-known (and easy to check) that the assumptions (2.1) and (2.3) imply

$$\|e^{-tA}\|_{p\to p} \le M, \ \forall\, t > 0,$$

for some $M > 0$ (in any case this follows from Theorem 4.3!). Therefore

$$\|J^1_\alpha(z)\|_{p\to p} \le M \int_0^{|z|} |z-s|^{\alpha-1}\, ds,$$

and if $z = |z|e^{i\theta}$,

$$\|J^1_\alpha(z)\|_{p\to p} \le M \int_0^{|z|} ||z|e^{i\theta} - s|^{\alpha-1}\, ds = M|z|^\alpha \int_0^1 |e^{i\theta} - u|^{\alpha-1}\, du,$$

Finally

$$\|J^1_\alpha(z)\|_{p\to p} \le C_\alpha |z|^\alpha,$$

where

$$C_\alpha = M \sup_{\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]} \int_0^1 |e^{i\theta} - u|^{\alpha-1}\, du.$$

Now

$$J^2_\alpha(z) = \int_0^{\arg z} (z - |z|e^{i\varphi})^{\alpha-1} e^{-|z|e^{i\varphi}A} i|z|e^{i\varphi}\, d\varphi.$$

To fix ideas, assume that $\arg z \ge 0$. Using the estimate (4.3), and setting $\beta = \nu|\frac{1}{2} - \frac{1}{p}| + \varepsilon$, one obtains

$$\begin{aligned}\|J^2_\alpha(z)\|_{p\to p} &\le \int_0^{\arg z} |z - |z|e^{i\varphi}|^{\alpha-1} \frac{|z|}{(\cos\varphi)^\beta}\, d\varphi \\ &\le C'_\alpha |z|^\alpha \int_0^{\arg z} \frac{[\sin(\frac{\arg z-\varphi}{2})]^{\alpha-1}}{(\cos\varphi)^\beta}\, d\varphi,\end{aligned}$$

but the function sin is increasing on $[0, \pi/2]$ so that

$$\sin\left(\frac{\arg z - \varphi}{2}\right) \le \sin\left(\frac{\pi}{2} - \varphi\right) = \cos\varphi.$$

Hence,

$$\int_0^{\arg z} \frac{[\sin(\frac{\arg z-\varphi}{2})]^{\alpha-1}}{(\cos\varphi)^\beta}\, d\varphi \;\le\; \int_0^{\arg z} \left[\sin\left(\frac{\arg z-\varphi}{2}\right)\right]^{\alpha-\beta-1} d\varphi$$
$$\le \int_0^{\frac{\pi}{4}} (\sin\theta)^{\alpha-\beta-1}\; d\theta$$

The last term being finite if ε is chosen such that $\beta < \alpha$, which is possible if $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$. This proves that $\|J_\alpha(z)\|_{p\to p} \le C''_\alpha |z|^\alpha$ which finishes the proof of Theorem 5.1. □

5.2. *Regularisation of the Schrödinger group*

From the bound (4.3) on the norm of the heat operator acting on L^p, one can also deduce the following general version of Lanconelli's result ([32]). Recall that if C is a bounded injective operator on a Banach space E, a C-regularized group on E is a strongly continuous family $W(t)$, $t \in \mathbb{R}$, of bounded operators on E such that $W(0) = C$, $W(t)W(s) = CW(t+s)$, $t, s \in \mathbb{R}$. Its generator is defined by $Bx = C^{-1}\frac{dW}{dt}(0)$, with maximal domain. See for instance [19], [18] for details.

Using Theorem 4.3 and Theorem 2.1 in [9], which applies thanks to the remark after Theorem 4.3, we can state the following. We give a proof for the sake of completeness.

THEOREM 5.2. *For all $p \in [1, +\infty]$ and every $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$, iA generates a $(I + A)^{-\alpha}$-regularised group on $L^p(\Omega)$, and*

$$\|(I + A)^{-\alpha} e^{itA}\|_{p\to p} \le C_\alpha (1 + |t|)^\alpha, \forall\, t \in \mathbb{R}. \tag{5.3}$$

Proof. Let us start by proving the estimate (5.3). If $f \in L^2 \cap L^p$ and $\alpha > 0$, one has

$$(I + A)^{-\alpha} e^{itA} f = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} s^{\alpha-1} e^{(-s+it)A} f\, ds,$$

where Γ is the Euler Gamma function. Using (4.3), for $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$, one can bound the L^p norm of the right hand side by

$$C_{\alpha,\varepsilon} \|f\|_p \int_0^{+\infty} e^{-s} s^{\alpha-1} \left(\sqrt{\frac{s^2+t^2}{s^2}}\right)^{\nu|\frac{1}{2}-\frac{1}{p}|+\varepsilon} ds,$$

for every $\varepsilon > 0$. Then, cutting the integral at $s = |t|$, we can estimate it, for ε small enough, by

$$|t|^\alpha \int_0^1 u^{\alpha-1-\nu|\frac{1}{2}-\frac{1}{p}|-\varepsilon}\, du + 2^{\frac{\nu}{2}|\frac{1}{2}-\frac{1}{p}|+\frac{\varepsilon}{2}} \int_{|t|}^{+\infty} e^{-s} s^{\alpha-1}\, ds \le C'|t|^\alpha + C'',$$

which yields (5.3).

Set $W(t) = (I + A)^{-\alpha} e^{itA}$. We have just seen that for every $t \in \mathbb{R}$, the operator $W(t)$ is bounded on $L^p(\Omega)$, $1 \leq p \leq +\infty$. Let us now prove that the mapping $t \to W(t)$ is strongly continuous from $\mathbb{R}$ to $\mathcal{L}(L^p)$ for $p \in [1, +\infty[$. One can write, for $t > 0$, $s \in \mathbb{R}$, and $f \in L^p$,

$$\|e^{-(t+is)A}(I + A)^{-\alpha} f - e^{-isA}(I + A)^{-\alpha} f\|_p \leq \|W(s)\|_{p\to p} \|f - e^{-tA} f\|_p.$$

According to (5.3), the first factor is bounded by $C(1 + |s|)^\alpha$, and the second goes to zero with t, since $-A$ is a generator on L^p, for $1 \leq p < +\infty$ (see the remark at the end of Section 4.2). Therefore, $e^{-(t+is)A}(I + A)^{-\alpha} f$ tends to $e^{-isA}(I + A)^{-\alpha} f = W(s) f$ in L^p, uniformly in $s \in [-T, T]$, as $t \to 0^+$. Again by the previously mentioned remark, since $z \to e^{-zA}$ is holomorphic on L^p on the right half-plane, $s \to e^{-(t+is)A}(I + A)^{-\alpha} f$ is continuous from $\mathbb{R}$ to L^p, for every $t > 0$ and $f \in L^p$. Hence, $W(s)$ is strongly continuous in L^p.

We leave to the reader the task to check that iA is indeed the generator of the regularised group $W(t)$. □

5.3. *α-integration of the Schrödinger group*

Formally, one obtains $\frac{t^\alpha}{\Gamma(\alpha)} I_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} e^{-isA}\, ds$ by integrating α times the group e^{-itA}. As a matter of fact, a theory of integrated semigroups has been developped by W. Arendt in order to provide a framework for Cauchy problems which are not governed by C^0 semigroups (see for instance [4], [36], [31]).

Let B an unbounded linear operator with domain $\mathcal{D}(B)$ in a Banach space E, and let $\alpha > 0$; B is said to generate an (exponentially bounded) α-integrated semigroup on E, if there exists $w \in \mathbb{R}$ and a strongly continuous mapping $S : [0, +\infty[\to \mathcal{L}(E)$ such that

$$\sup_{t\in\mathbb{R}} e^{-wt} \|S(t)\| < +\infty,$$

every $\lambda > w$ lies in the resolvent set of B and

$$(\lambda I - B)^{-1} = \lambda^\alpha \int_0^{+\infty} e^{-\lambda t} S(t)\, dt.$$

Similarly, B is said to generate an α-integrated group on E: $S : \mathbb{R} \to \mathcal{L}(E)$, if B and $-B$ generate α-integrated semigroups $S(t)$ and $S(-t)$, $t \geq 0$, respectively.

Following [23], we say that the α-integrated group S is globally tempered if

$$\|S(t)\| \leq C|t|^\alpha, \ \forall t \in \mathbb{R}. \tag{5.4}$$

The estimate (5.1) can be used to show the following.

THEOREM 5.3. *For $1 \leq p < +\infty$ and $\alpha > \nu|\frac{1}{2} - \frac{1}{p}|$, iA generates a globally tempered α-integrated group S on $L^p(\Omega)$.*

Proof. Set $S(t) = \frac{t^\alpha}{\Gamma(\alpha)} I_\alpha(t)$, $t > 0$, which is by Theorem 5.1 a family of bounded operators on $L^p(\Omega)$ satisfying (5.4), for $1 \le p \le +\infty$. For $1 \le p < +\infty$, S is strongly continuous on $L^p(\Omega)$. Indeed, $(I+A)^{-k}L^p$ is dense in L^p, therefore it is enough to check that the mapping

$$t \to \int_0^t (t-s)^{\alpha-1}(I+A)^{-k}e^{-isA}\,ds$$

is strongly continuous. Since the latter operator may be written as

$$\int_0^R \chi_{[0,t]}(u)u^{\alpha-1}(I+A)^{-k}e^{-i(t-u)A}\,du,$$

for R large enough, this follows immediately from the continuity of $t \to W(t)$ which has been proved in Section 5.2.

Finally the formula

$$(\lambda I - iA)^{-1} = \lambda^\alpha \int_0^{+\infty} e^{-\lambda t}S(t)\,dt$$

is true by spectral theory on L^2 and extends to L^p by density. The same argument applies to $S(-t)$. □

For instance this result has the following consequence. Here $\mathcal{D}_p(B)$ denotes the domain of the operator B acting in $L^p(\Omega)$.

COROLLARY 5.4. *Let $p \in [1, +\infty[$ and let $k \in \mathbb{N}^*$ be such that $k > \nu|\frac{1}{2} - \frac{1}{p}|$, then for any $f \in \mathcal{D}_p(A^{k+1})$, there is a unique solution $u \in C^0(\mathbb{R}, \mathcal{D}_p(A^k)) \cap C^1(\mathbb{R}, L^p)$ to the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} = iAu \\ u(0) = f \in \mathcal{D}_p(A^{k+1}). \end{cases}$$

6. Some L^p functional calculus

We show in this section a result on mapping properties of functions of the operator A. Our result is similar to the one of A. Jensen and S. Nakamura [30], in the case where A is a Schrödinger type operator $-\Delta + V$ in the Euclidean space. We begin, as in [30] and [14] to introduce the following class of functions. Let $\beta \in \mathbb{R}$, we say that a smooth function f on $\mathbb{R}_+$ is in $\mathcal{S}(\beta)$ if for any $k \in \mathbb{N}$, there is a constant C_k with

$$\left|\frac{d^k f}{d\lambda^k}(\lambda)\right| \le C_k(1+\lambda)^{\beta-k}, \ \forall \lambda \in \mathbb{R}_+.$$

We can now state the following.

THEOREM 6.1. *Let $p \in [1, +\infty[$. If $f \in \mathcal{S}(-\beta)$ for some $\beta > 0$, then $f(A)$ extends to a bounded operator on $L^p(\Omega)$. Moreover if $f \in \mathcal{S}(-\beta)$ for a $\beta > \nu|\frac{1}{2} - \frac{1}{p}|$ then $f(A)e^{itA}$ extends to a bounded operator on $L^p(\Omega)$ and one has*

$$\|f(A)e^{itA}\|_{p\to p} \leq C_\varepsilon(1+|t|)^{\nu|\frac{1}{2}-\frac{1}{p}|+\varepsilon}, \forall t \in \mathbb{R}$$

for every $\varepsilon > 0$.

This theorem follows directly from the works of A. Jensen and S. Nakamura [30] or Davies [14] and the following classical bound on the resolvent.

PROPOSITION 6.2. *If $p \in [1, +\infty[$, then for any $\beta > \nu|\frac{1}{2} - \frac{1}{p}|$ we have*

$$\|(A - zI)^{-1}\|_{p\to p} \leq C\frac{|z|^\beta}{|\mathrm{Im} z|^{\beta+1}}, \ \forall z \in \mathbb{C}\backslash\mathbb{R}.$$

Proof. Set $z = re^{-i\theta}$ and assume that $\theta \in]0, \pi]$. Since the resolvent is the Laplace transform of the semigroup, we can write

$$\begin{aligned}(A - zI)^{-1} &= e^{-i\frac{\pi-\theta}{2}}(Ae^{-i\frac{\pi-\theta}{2}} + re^{i\frac{\pi-\theta}{2}}I)^{-1} \\ &= e^{-i\frac{\pi-\theta}{2}}\int_0^{+\infty} e^{-rte^{i\frac{\pi-\theta}{2}}} e^{-te^{-i\frac{\pi-\theta}{2}}A}dt.\end{aligned}$$

Now we apply (4.3) to obtain

$$\|(A - zI)^{-1}\|_{p\to p} \leq \frac{C}{(\sin\frac{\theta}{2})^\beta}\int_0^{+\infty} e^{-r\sin\frac{\theta}{2}t}dt,$$

which gives the desired estimate. If $\theta \in [-\pi, 0[$, we change π to $-\pi$ in the above argument and argue similarly.

The theorem now follows from the argument of [30] or [14] which we recall here. If $f \in \mathcal{S}(\alpha)$, A. Jensen and S. Nakamura construct an almost analytic extension of f, that is to say an $\tilde{f} \in C^\infty(\mathbb{C}, \mathbb{C})$ such that $\tilde{f} = f$ on $\mathbb{R}_+$, and for every $k \in \mathbb{N}$ there is a constant C_k such that

$$\left|\frac{\partial\tilde{f}}{\partial\bar{z}}\right| \leq C_k(1+|z|)^{\alpha-1-k}|\mathrm{Im} z|^k, \ \forall z \in \mathbb{C}.$$

Then we use the Helffer-Sjöstrand formula:

$$f(A) = \frac{1}{2i\pi}\int_{\mathbb{C}} \frac{\partial\tilde{f}}{\partial\bar{z}}(z)(A - zI)^{-1}dzd\bar{z},$$

which is valid on $L^2 \cap L^p$ ([14]). On $L^p(\Omega)$ we have the bound

$$\|f(A)\|_{p\to p} \leq C\int_{\mathbb{C}}(1+|z|)^{\alpha-1-k}|\mathrm{Im} z|^k\frac{|z|^\beta}{|\mathrm{Im} z|^{\beta+1}}dzd\bar{z},$$

where $\beta > \nu|\frac{1}{2} - \frac{1}{p}|$. Now this integral converges if $\alpha < 0$ and $k > \beta$; this finishes the proof of the first part of the theorem. In order to prove the second part, note first that if $f \in \mathcal{S}(-\beta)$, then $(1+\lambda)^\alpha f \in \mathcal{S}(-\beta+\alpha)$. Now, for $\beta > \alpha > \nu|\frac{1}{2} - \frac{1}{p}|$, we write

$$f(A)e^{itA} = f(A)(I+A)^\alpha (I+A)^{-\alpha} e^{itA},$$

and apply the first part of the theorem and Theorem 5.2 to finish the proof. □

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