EIGENVALUES AND HOLONOMY

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Abstract. We estimate the eigenvalues of connection Laplacians in terms of the non-triviality of the holonomy.

Introduction

Let $S_L = \mathbb{R}/L\mathbb{Z}$ be a circle of length L and X be the oriented unit vector field on $S = S_L$. Up to equivalence, there is exactly one Hermitian line bundle, E, over S. For a given complex number z of modulus 1, there is, again up to equivalence, exactly one Hermitian connection, ∇^E, on E with holonomy z around S.

The Laplace operator $\Delta^E = (\nabla^E)^*\nabla^E$ is essentially self-adjoint as an operator in $L^2(E)$ with domain $C^2(E)$. The spectrum of its closure is discrete and consists of the eigenvalues

$$\frac{4\pi^2}{L^2}(\rho + k)^2, \quad k \in \mathbb{Z},$$

where we write $z = \exp(2\pi i\rho)$. The corresponding eigenspaces are spanned by the functions $\exp(2\pi i(\rho + k)x/L)$. We see that, for $z \neq 1$, the spectrum does not contain 0, and that we can estimate the smallest eigenvalue in terms of L and z.

The aim of this paper is a correponding estimate for Hermitian vector bundles over closed Riemannian manifolds in higher dimensions. The results of this paper are of importance in [BBC], but seem to be also of independent interest.

Let M be a closed Riemannian manifold of dimension $n \geq 2$. Let $-(n-1)\kappa \leq 0$ be a lower bound for the Ricci curvature of M, i.e. $\mathrm{Ric}_M \geq -(n-1)\kappa$, and let D be an upper bound for the diameter of M, $\operatorname{diam} M \leq D$. Let $E \to M$ be a Hermitian vector bundle over M and ∇^E be a Hermitian connection on E. The kernel of the associated connection Laplacian $\Delta^E = (\nabla^E)^*\nabla^E$ consists of globally parallel sections of E. The estimates we obtain are in terms of quantitave measures for the non-existence of parallel sections, that is, in terms of the holonomy of E.

Assume first that ∇^E is flat and that the holonomy of ∇^E is irreducible (and nontrivial). Recall that for each point $x \in M$, the fundamental group $\pi_1(M, x)$ of M at x admits a *short basis*, that is, a generating set represented by loops of length at most $2 \operatorname{diam} M$, see [Gr]. Hence for each point $x \in M$, there is a

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constant $\alpha(x) > 0$ such that for all $v \in E_x$ there is a smooth unit speed loop $c : [0, l] \to M$ at x of length $l \le 2 \operatorname{diam} M$ with holonomy H_c satisfying

$$|H_c(v) - v| \ge \alpha(x)|v|.$$

There is also a constant $\varepsilon(x) > 0$ such that a loop at x has length $> 2 \operatorname{diam} M + \varepsilon(x)$ unless it is homotopic to a loop at x of length $\le 2 \operatorname{diam} M$. It follows that for any point $y \in M$ of distance $< \varepsilon/4$ to x, the homotopy classes of loops of length $\le 2 \operatorname{diam} M$ at y are represented by concatenated curves of the form $c_{xy}^{-1} * c * c_{xy}$, where c_{xy} denotes a fixed minimal geodesic from x to y and c is a loop at x of length $\le 2 \operatorname{diam} M$. Since ∇^E is flat, parallel translation along loops only depends on their homotopy classes. It follows that for each point y sufficiently close to x, there is a loop c of length $\le 2 \operatorname{diam} M$ at y which has the same non-trivial holonomy as the loop $c_{xy} * c * c_{xy}^{-1}$ at x. In particular, we can choose the constants $\alpha(x)$ such that they have uniform positive lower bounds locally. By the compactness of M, there is a uniform constant $\alpha > 0$ such that, for all $x \in M$ and $v \in E_x$, there is a smooth unit speed loop $c : [0, l] \to M$ at x of length $l \le 2 \operatorname{diam} M$ with holonomy H_c satisfying

$$|H_c(v) - v| \ge \alpha|v|. \tag{1}$$

Our first estimate is as follows.

THEOREM 1. *Suppose that ∇^E is flat and that the holonomy of ∇^E satisfies* (1). *Then, for each eigenvalue λ of Δ^E,*

$$\sqrt{\lambda}\, \exp\left(c_0\sqrt{\lambda + (n-1)\kappa}\, \operatorname{diam} M\right) \ge \frac{\alpha}{2 \operatorname{diam} M}$$

with a constant $c_0 = c_0(n, \sqrt{\kappa} D)$. In particular,

$$\sqrt{\lambda} \ge \min\left\{\frac{1}{c_0 \operatorname{diam} M}, \frac{\alpha}{2 \operatorname{diam} M} \exp\left(-c_0\sqrt{(n-1)\kappa}\, \operatorname{diam} M - 1\right)\right\}.$$

For each point $x \in M$ and unit vector $v \in E_x$, let $\beta(v)$ be the supremum of the ratios $|H_c(v) - v|/L(c)$, where the supremum is taken over all non-constant loops c starting at x, $L(c)$ denotes the length of c, and H_c the holonomy along c. Set

$$\beta := \inf\{\beta(v) \mid v \in E, |v| = 1\}. \tag{2}$$

Note that by the definition of the constant α in (1), we have $\beta \ge \alpha/2 \operatorname{diam} M$. In the general case, i.e. if ∇^E is not necessarily flat, we have the following estimate.

THEOREM 2. *There are positive constants $a = a(n)$ and $c_1 = c_1(n, \sqrt{\kappa} D)$ such that, for each eigenvalue λ of Δ^E,*

$$\sqrt{\lambda}\, \exp\left(c_1\sqrt{\lambda + (n-1)\kappa + n^2 r + n^2 r^2/\beta^2}\, \operatorname{diam} M\right) \ge \frac{\beta}{a},$$

where r is a uniform bound for the pointwise operator norm of R^E. In particular,

$$\sqrt{\lambda} \geq \min\left\{\frac{1}{c_1 \operatorname{diam} M}, \frac{\beta}{a}\exp\left(-c_1\sqrt{(n-1)\kappa + n^2 r + n^2 r^2/\beta^2}\operatorname{diam} M - 1\right)\right\}.$$

The constants a, c_0 and c_1 in Theorems 1 and 2 can be determined explicitly. Except for the factor $1/a$, Theorem 2 implies Theorem 1. On the other hand, the proof of Theorem 1 is more elementary than the one of Theorem 2 and exposes the main ideas more clearly. Moreover, the constant c_0 is better than the constant c_1, that is, $c_0 \leq c_1$.

Our basic analytic and geometric tools are a Sobolev inequality of Gallot [Ga], a suitable Bochner formula and Moser iteration . There are quite a few applications of Moser iteration to geometry, see for example [Li], [Ga], [LR], [PS], and [ACGR]. [1] In the proof of Theorem 1 we only need a standard version of it and refer the reader to the literature. In the proof of Theorem 2 we need a non-trivial extension of the iteration technique which does not seem to be in the literature. We therefore give a complete argument for this more general kind of iteration.

If part of the holonomy is trivial, then the corresponding space of parallel sections determines a subbundle E' of E. The above results then apply to the orthogonal complement E'' of E' in E. On the other hand, suppose $\sigma = \sum \phi_i \sigma_i$ is a section in E', where the sections σ_i are parallel. Then $\Delta^E \sigma = \sum(\Delta\phi_i)\sigma_i$, where Δ is the Laplace operator on functions of M, and hence the usual eigenvalue estimates for Δ as for example in [LY] or [Zh] apply.

Proof of Theorem 1

Let M be a closed Riemannian manifold of dimension n and volume V. Denote by $\|\cdot\|_p$ the L^p-norm with respect to the *normalized* Riemannian measure of M.

Let $-(n-1)\kappa \leq 0$ be a lower bound for the Ricci curvature of M and D be an upper bound for the diameter of M. We will use the following Sobolev inequality.

Lemma 3 (Gallot [Ga]). *There is a positive constant $c = c(n, \sqrt{\kappa}D)$ such that, for all $p \in [1, \frac{n}{n-1}]$ and all smooth functions f on M,*

$$\|f\|_{\frac{2p}{2-p}} \leq \|f\|_2 + \frac{2c}{2-p}\operatorname{diam} M \|df\|_2.$$

Recall that the function c can be chosen to be equal to

$$c(n,d) = \left\{\frac{1}{d}\int_0^d \big(\frac{1}{2}e^{(n-1)d}\cosh t + \frac{1}{nd}\sinh t\big)^{n-1} dt\right\}^{1/n} \qquad (3)$$

with $d = \sqrt{\kappa}D$, compare [Ga].

[1] Note that not all of the arguments in the proof of the main results in [PS] are correct. In particular, Lemma 3.1, the asserted application of Moser iteration, does not hold in the generality stated there.

Let ∇ and Δ be the Levi-Civita connection and the Laplace operator on functions of M, respectively. Let $F \to M$ be a Hermitian vector bundle with a Hermitian connection ∇^F. Let Δ^F be the associated connection Laplacian.

Applying Moser iteration we obtain the following estimate, compare for example [GT, Theorem 8.15] (compare also page 215 in [GT]).

LEMMA 4. *Let $\sigma \in L^2(M,F)$ be a smooth section. Assume that (pointwise)*

$$\langle \Delta^F \sigma, \sigma \rangle \le \Lambda^2 |\sigma|^2$$

for some constant $\Lambda \ge 0$. Let $p \in (1,2) \cap [1, \frac{n}{n-1}]$. Then

$$\|\sigma\|_\infty \le c' \, \|\sigma\|_2.$$

with

$$c' = \exp(c(n, \sqrt{\kappa} D) c''(p) \Lambda \operatorname{diam} M).$$

THEOREM 5. *Suppose that $\nabla^E R^E = 0$. Then, for each eigenvalue λ of Δ^E,*

$$\sqrt{\lambda} \, \exp\big(c_0 \sqrt{\lambda + (n-1)\kappa + 2n^2 r} \, \operatorname{diam} M\big) \ge \beta$$

with $c_0 = c_0(n, \kappa\sqrt{D})$ and $r = \|R^E\|_\infty$.

Recall that $\beta \ge \alpha/2 \operatorname{diam} M$ and $r = 0$ under the assumptions of Theorem 1. Hence Theorem 5 implies Theorem 1.

Proof of Theorem 5. Let σ be a nonzero section of E with $\Delta^E \sigma = \lambda \sigma$. Let $x \in M$ and choose $\beta' < \beta$. Then there is a unit speed loop $c : [0,l] \to M$ at x, of length l, with holonomy $H_c : E_x \to E_x$ satisfying

$$|H_c(\sigma(x)) - \sigma(x)| \ge \beta' l |\sigma(x)|.$$

Let $F_1, \dots, F_k : [0,l] \to E$ be a parallel orthonormal frame along c. Express $\sigma \circ c$ as a linear combination of this frame, $\sigma \circ c = \sum \phi^i F_i$. By the assumption on the holonomy, we have

$$\begin{aligned} \beta' l |\sigma(x)| = \beta' l |\phi(0)| \le |\phi(l) - \phi(0)| &\le \int_0^l |\phi'| \, dt \\ &\le \int_0^l |(\nabla^E \sigma) \circ c| \, dt \le l \|\nabla^E \sigma\|_\infty. \end{aligned}$$

Since we use the normalized volume element for our norms, this gives

$$\beta \|\sigma\|_2 \le \beta \|\sigma\|_\infty \le \|\nabla^E \sigma\|_\infty. \tag{4}$$

On the other hand, $\nabla^E \sigma$ is a one-form with values in E, that is, a section of the bundle $F = \Lambda^1(T^*M) \otimes E$. This bundle inherits a connection, ∇^F, from the Levi–Civita connection ∇ of M and the connection ∇^E of E. In terms of a local orthonormal frame $X_1, \dots, X_n$ of M and a further local vector field Z, the corresponding Bochner formula is

$$(5)\quad (\Delta^F \nabla^E \sigma)(Z) = \nabla^E_Z(\Delta^E\sigma) - \nabla^E_{\operatorname{Ric} Z}\sigma - 2\sum R^E(X_i, Z)\nabla^E_{X_i}\sigma - \sum(\nabla^E_{X_i} R^E)(X_i, Z)\sigma,$$

see e.g. Lemma 3.3.1 of [LR]. In particular, since $\Delta^E\sigma = \lambda\sigma$ and $\nabla^E R^E = 0$,

$$(6)\qquad \langle \Delta^F(\nabla^E\sigma), \nabla^E\sigma\rangle \le \big(\lambda + (n-1)\kappa + 2n^2 r\big)|\nabla^E\sigma|^2,$$

where we are somewhat generous in the estimate of the curvature term. Now $\|\nabla^E\sigma\|_2 = \sqrt{\lambda}\|\sigma\|_2$ since $\Delta^E\sigma = \lambda\sigma$. Applying Lemma 4 with $p = (n+1)/n$, we obtain the asserted inequality. □

Proof of Theorem 2

We cannot apply the previous argument directly to prove Theorem 2. The reason is that, in general, the Bochner formula (5) only gives the estimate

$$(7)\quad \langle \nabla^E\sigma, \Delta^F\nabla^E\sigma\rangle \le \big(\lambda + (n-1)\kappa + n^2 r\big)|\nabla^E\sigma|^2 - \sum_{i,j}\big\langle (\nabla^E_{X_i}R^E)(X_i, X_j)\sigma + R^E(X_i, X_j)\nabla^E_{X_i}\sigma, \nabla^E_{X_j}\sigma\big\rangle.$$

Note that we distribute the terms arising from $2\sum R^E(X_i, Z)\nabla^E_{X_i}\sigma$ in (5) to both terms on the right hand side in (7). Now (7) involves σ on the right hand side, hence the standard Moser iteration procedure does not work.

We let $f_\varepsilon := \sqrt{|\nabla^E\sigma|^2 + \varepsilon^2}$. By the Kato inequality and (7), we have the pointwise estimate

$$f_\varepsilon\Delta f_\varepsilon \le \operatorname{Re}\langle\nabla^E\sigma, \Delta^F\nabla^E\sigma\rangle \le \big(\lambda + (n-1)\kappa + n^2 r\big) f_\varepsilon^2 - \sum_{i,j}\big\langle (\nabla^E_{X_i}R^E)(X_i, X_j)\sigma + R^E(X_i, X_j)\nabla^E_{X_i}\sigma, \nabla^E_{X_j}\sigma\big\rangle.$$

Let $k \ge 1$. Then

$$\begin{aligned}
\|df_\varepsilon^k\|_2^2 &= k^2\langle f_\varepsilon^{k-1}df_\varepsilon, f_\varepsilon^{k-1}df_\varepsilon\rangle_2 \\
&= \frac{k^2}{2k-1}\langle df_\varepsilon, df_\varepsilon^{2k-1}\rangle_2 = \frac{k^2}{2k-1}\langle \Delta f_\varepsilon, f_\varepsilon^{2k-1}\rangle_2 \\
&= \frac{k^2}{2k-1}\big(\lambda + (n-1)\kappa + n^2 r\big)\|f_\varepsilon\|_{2k}^{2k} \\
&\quad - \frac{k^2}{2k-1}\int_M \sum_{i,j}\big\langle \nabla^E_{X_i}(R^E(X_i, X_j)\sigma), \nabla^E_{X_j}\sigma\big\rangle f_\varepsilon^{2k-2},
\end{aligned}$$

where it is understood that we choose, for each point $x \in M$, an orthonormal frame $X_1, \dots, X_n$ with $(\nabla_{X_i} X_j)(x) = 0$. As in [LR], the divergence theorem gives

$$
\begin{aligned}
-\int_M \sum_{i,j} \big\langle \nabla^E_{X_i}(R^E(X_i,X_j)\sigma), \nabla^E_{X_j}\sigma \big\rangle f_\varepsilon^{2k-2} \\
&= \int_M \sum_{i,j} \big\langle R^E(X_i,X_j)\sigma, \nabla^E_{X_i}\nabla^E_{X_j}\sigma \big\rangle f_\varepsilon^{2k-2} \\
&+ 2(k-1)\int_M f_\varepsilon^{2k-3} \sum_{i,j} df_\varepsilon(X_i)\big\langle R^E(X_i,X_j)\sigma, \nabla^E_{X_j}\sigma \big\rangle.
\end{aligned}
$$

Now $R(X_i, X_j) = -R(X_j, X_i)$; therefore, with the above choice of frames,

$$
\sum_{i,j} \big\langle R^E(X_i,X_j)\sigma, \nabla^E_{X_i}\nabla^E_{X_j}\sigma \big\rangle = \frac{1}{2}\sum_{i,j} \big| R^E(X_i,X_j)\sigma \big|^2.
$$

Hence

$$
\begin{aligned}
-\int_M \sum_{i,j} \big\langle \nabla^E_{X_i}(R^E(X_i,X_j)\sigma), \nabla^E_{X_j}\sigma \big\rangle f_\varepsilon^{2k-2} \\
&\le \frac{n^2r^2}{2}\int_M |\sigma|^2 f_\varepsilon^{2k-2} + 2(k-1)nr\int_M |\sigma| f_\varepsilon^{2k-2}|df_\varepsilon| \\
&\le \frac{n^2r^2}{2}\|\sigma\|_\infty^2 \int_M f_\varepsilon^{2k-2} + 2\frac{k-1}{k} nr\|\sigma\|_\infty \int_M f_\varepsilon^{k-1}|df_\varepsilon^k| \\
&\le \frac{n^2r^2}{2}\|\sigma\|_\infty^2 \int_M f_\varepsilon^{2k-2} + 2nr\|\sigma\|_\infty \int_M f_\varepsilon^{k-1}|df_\varepsilon^k|.
\end{aligned}
$$

But

$$
\frac{2k(k-1)}{2k-1} nr\|\sigma\|_\infty \int_M f_\varepsilon^{k-1}|df_\varepsilon^k| \le \\
\frac{1}{2}\int_M |df_\varepsilon^k|^2 + \left(\frac{k(k-1)}{2k-1}\right)^2 2n^2r^2\|\sigma\|_\infty^2 \int_M f_\varepsilon^{2k-2}
$$

and $\|\sigma\|_\infty \le \|\nabla^E\sigma\|_\infty/\beta \le \|f_\varepsilon\|_\infty/\beta$, hence

$$
\begin{aligned}
\|df_\varepsilon^k\|_2^2 &\le \frac{2k^2}{2k-1}\left(\lambda + (n-1)\kappa + n^2 r + \left(\frac{1}{2} + \frac{2(k-1)^2}{2k-1}\right)\frac{n^2r^2}{\beta^2}\right)\|f_\varepsilon\|_\infty^2 \|f_\varepsilon^{k-1}\|_2^2 \\
&\le 2k^2\left(\lambda + (n-1)\kappa + n^2 r + \left(\frac{1}{2} + \frac{2(k-1)^2}{(2k-1)^2}\right)\frac{n^2r^2}{\beta^2}\right)\|f_\varepsilon\|_\infty^2 \|f_\varepsilon^{k-1}\|_2^2.
\end{aligned}
$$

Set $L^2 := 2\big(\lambda + (n-1)\kappa + n^2 r + n^2r^2/\beta^2\big)$. Since $k \ge 1$,

$$
\|df_\varepsilon^k\|_2^2 \le L^2k^2\|f_\varepsilon\|_\infty^2 \|f_\varepsilon\|_{2k-2}^{2k-2} \le L^2k^2\|f_\varepsilon\|_\infty^2 \|f_\varepsilon\|_{2k}^{2k-2}.
$$

Using Lemma 3 with $p = (n+2)/(n+1)$, we get

$$\begin{aligned}\|f_\varepsilon\|_{2kq}^k = \|f_\varepsilon^k\|_{2q} &\le \|f_\varepsilon\|_{2k}^k + CLk\|f_\varepsilon\|_\infty \|f_\varepsilon\|_{2k}^{k-1} \\ &\le (1+CLk)\|f_\varepsilon\|_\infty \|f_\varepsilon\|_{2k}^{k-1},\end{aligned}$$

where $q := p/(2-p) = (n+2)/n$ and $C := (2n+2)c(n, \sqrt{\kappa}D)\operatorname{diam} M/n$. Letting $\varepsilon \to 0$, we obtain

$$\|\nabla^E\sigma\|_{2kq} \le (1+CLk)^{1/k}\, \|\nabla^E\sigma\|_\infty^{1/k} \|\nabla^E\sigma\|_{2k}^{1-1/k}.$$

We iterate this inequality with $k = q^j$, $j \in \mathbb{N}$. Setting $p_i := 1 - 1/q^i$, we get

$$\begin{aligned}\|\nabla^E\sigma\|_{2q^{j+1}} &\le \left(1+CLq^j\right)^{1/q^j} \|\nabla^E\sigma\|_\infty^{1-p_j} \|\nabla^E\sigma\|_{2q^j}^{p_j} \\ &\le \prod_{i=1}^{j}\left(1+CLq^i\right)^{p_{i+1}\cdot\ldots\cdot p_j/q^i} \|\nabla^E\sigma\|_\infty^{1-p_1\cdot\ldots\cdot p_j} \|\nabla^E\sigma\|_{2q}^{p_1\cdot\ldots\cdot p_j} \\ &\le \prod_{i=1}^{j}\left(1+CLq^i\right)^{1/q^i} \|\nabla^E\sigma\|_\infty^{1-p_1\cdot\ldots\cdot p_j} \|\nabla^E\sigma\|_{2q}^{p_1\cdot\ldots\cdot p_j},\end{aligned}$$

where we use, for the latter inequality, that $0 < p_i < 1$ and that $x^p \le x$ if $x \ge 1$ and $0 < p < 1$. The limit

$$\varepsilon = \varepsilon(n) := \prod_{i=1}^{\infty} p_i$$

exists and satisfies $0 < \varepsilon < 1$. Moreover, using the inequality

$$1 + CLq^i \le (1+CL)q^i$$

we obtain

$$\prod_{i=1}^{\infty}\left(1+CLq^i\right)^{1/q^i} \le (1+CL)^{\sum_{i=1}^\infty 1/q^i} \cdot q^{\sum_{i=1}^\infty i/q^i} \le a_1(n)e^{b(n)CL}$$

with $a_1(n) = q^{\sum_{i=1}^\infty i/q^i}$ and $b(n) = \sum_{i=1}^\infty 1/q^i$. We conclude that

$$\|\nabla^E\sigma\|_\infty \le a_2(n)\exp\big(b(n)CL/\varepsilon(n)\big)\|\nabla^E\sigma\|_{2q}$$

with $a_2(n) = a_1(n)^{1/\varepsilon(n)}$. We also have

$$\begin{aligned}\|\nabla^E\sigma\|_{2q} &\le \|\nabla^E\sigma\|_2^{1/q} \cdot \|\nabla^E\sigma\|_\infty^{(q-1)/q} \\ &\le \|\nabla^E\sigma\|_2^{n/(n+2)} \cdot \|\nabla^E\sigma\|_\infty^{2/(n+2)},\end{aligned}$$

where we recall that $q = (n+2)/n$. Hence finally

$$\|\nabla^E\sigma\|_\infty \le a(n)\exp\big((n+2)b(n)CL/(n\varepsilon(n))\big)\|\nabla^E\sigma\|_2$$

with $a(n) = a_2(n)^{(n+2)/n}$. The rest of the argument is as before.

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