A Well-Balanced Finite Volume Scheme for a Mixed Hyperbolic/Parabolic System to Model Chemotaxis

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Main motivations

Hyperbolic systems of conservation laws with source term

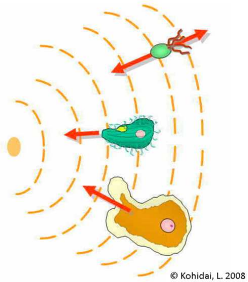
$$\partial_t w + \partial_x F(w) = \frac{1}{\nu} S(w), \quad w \in \Omega.$$

Construct Finite Volume schemes with some good properties.

- Steady states: $w \in \Omega$ such that $\partial_x F(w) = \frac{1}{\nu} S(w)$
  $\rightarrow$ exact capture of some of them.
- Preservation of asymptotic diffusive regime: $\nu \rightarrow 0$.
- Other properties of the solution (non-negativity of densities, *etc.*).

Application: chemotaxis

- Chemotaxis is the movement of an organism in response to a chemical stimulus.
- The chemical stimulus can be food, poison or a substance emitted by the organism itself to attract others.
- Example: gathering of cells to form a network of blood vessels.
- Application: development of tumors.



Hyperbolic 1D model of chemotaxis[1,2]

$$\partial_t \rho + \partial_x (\rho u) = 0, \quad (1)$$

$$\partial_t (\rho u) + \partial_x \left(\rho u^2 + p\right) = \chi \rho \partial_x \phi - \alpha \rho u, \quad (2)$$

$$\partial_t \phi - D \partial_{xx} \phi = a\rho - b\phi, \quad (3)$$

where

- $\rho(x,t) \geq 0$: particles density,
- $u(x,t) \in \mathbb{R}$: mean velocity,
- $\phi(x,t) \geq 0$: concentration of chemoattractant,
- $p(\rho) = \varepsilon \rho^\gamma$: pressure law, with $\gamma > 1$ adiabatic exponent and $\varepsilon > 0$ a constant,

[1] Natalini, Ribot, Twarogowska, CMS 2014.
[2] Twarogowska, PhD Thesis 2011.

Hyperbolic 1D model of chemotaxis[1,2]

$$\begin{aligned} \partial_t \rho + \partial_x (\rho u) &= 0, && (1)\\ \partial_t (\rho u) + \partial_x \left(\rho u^2 + p\right) &= \chi \rho \partial_x \phi - \alpha \rho u, && (2)\\ \partial_t \phi - D \partial_{xx} \phi &= a\rho - b\phi, && (3) \end{aligned}$$

where

- $\chi \geq 0$: strength of the cells response,
- $\alpha \geq 0$: strength of friction forces,
- $D > 0$: chemoattractant diffusion coefficient,
- $a > 0$: production rate,
- $b > 0$: degradation rate.

[1] Natalini, Ribot, Twarogowska, CMS 2014.
[2] Twarogowska, PhD Thesis 2011.

Objectives

- Preserve equilibrium states (at rest, with $u = 0$), that are given by

$$\begin{cases} \frac{\varepsilon}{\chi}\frac{\gamma}{\gamma-1}\rho^{\gamma-1} - \phi &= K, \\ -D\partial_{xx}\phi &= a\rho - b\phi, \end{cases} \tag{4}$$

with a constant K.

- Exact capture when $\gamma = 2$.
- W-B scheme on the hyperbolic part (1)-(2)[1,2], using a suitable extension of the hydrostatic reconstruction[3], and also on the equation (3) for ϕ.
- Deal with vacuum regions.

[1] Natalini, Ribot, Twarogowska, CMS 2014.
[2] Twarogowska, PhD Thesis 2011.
[3] Audusse, Bouchut, Bristeau, Klein, Perthame, SIAM JSC 2004.

Chosen strategy[4]

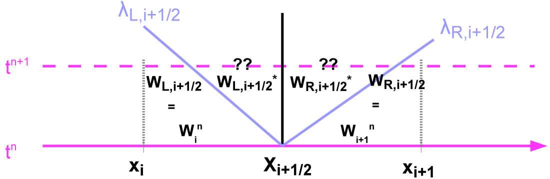
- Derive a finite volume scheme of Godunov-type.
- Construct a suitable approximate Riemann solver $\tilde{w}$:
  - that verifies the consistency condition introduced by Harten, Lax and van Leer[5],
  - by imposing the preservation of steady states of interest (with a good approximation of the source term[6] and using their explicit expression),
  - by using a cut-off procedure[7] when dealing with vanishing densities.

[4] Berthon, Crestetto, Foucher, JSC 2015.
[5] Harten, Lax, van Leer, SIAM review 1983.
[6] Berthon, Chalons, Math. of Comp. 2016.
[7] Audusse, Chalons, Ung, CMS 2015.

Finite Volumes scheme for $\partial_t w + \partial_x f(w) = 0$

- 1D domain discretized by $N+1$ points: $x_i = i\Delta x$.
- Evolution in time of $w_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} w\left(x, t^n\right) \mathrm{d}x$.
- Knowing w_i^n, construct an approximate Riemann solver $\tilde{w}$ for the Riemann problem

$$\left\{\begin{array}{l} \partial_t w + \partial_x f(w) = 0, \quad t \in \left[t^n, t^{n+1}\right[, \\ w(x, t^n) = \left\{\begin{array}{ll} w_i^n, & x_i < x < x_{i+1/2}, \\ w_{i+1}^n, & x_{i+1/2} < x < x_{i+1}, \end{array}\right. \end{array}\right.$$



- Calculate $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} w\left(x, t^n + \Delta t\right) \mathrm{d}x$ thanks to $\tilde{w}$ to obtain the approximation w_i^{n+1}.

Outline

- Hyperbolic model without friction
- Hyperbolic model with friction
- Full model
- Numerical results

Hyperbolic model

We first look at the two first equations (1)-(2), that we rewrite as

$$\partial_t w + \partial_x F(w) = S(w) \tag{5}$$

with

$$w = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}, \; F(w) = \begin{pmatrix} \rho u \\ \rho u^2 + p \end{pmatrix} \text{ and } S(w) = \begin{pmatrix} 0 \\ \chi \rho \partial_x \phi - \alpha \rho u \end{pmatrix},$$

considering ϕ as a known source term.

- First study: without friction ($\alpha = 0$).
- Second study: with friction ($\alpha > 0$).

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Approximate Riemann solver

Approximate Riemann solver $\tilde{w}(\frac{x}{t}, w_L, w_R)$ defined by:

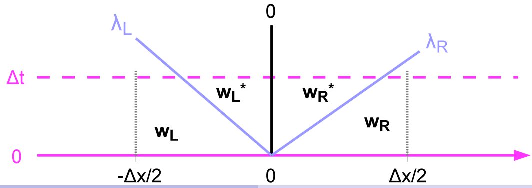
- velocities $\lambda_L < 0 < \lambda_R$:

$$\lambda_L = \min(0^-, \lambda_L^-, \lambda_R^-) \quad \text{and} \quad \lambda_R = \max(0^+, \lambda_L^+, \lambda_R^+),$$

  where $\lambda_L^\pm$ and $\lambda_R^\pm$ denote the eigenvalues of the flux Jacobian matrix:

$$\lambda^\pm = u \pm c \quad \text{where} \quad c = c(\rho) = \sqrt{P'(\rho)} = \sqrt{\varepsilon\gamma\rho^{\gamma-1}},$$

- intermediate states $w_L^\star$ and $w_R^\star$ (will be defined later),
- the CFL condition: $\frac{\Delta t}{\Delta x} \max\left(|\lambda_L|, |\lambda_R|\right) \leq \frac{1}{2}$.



HLL consistency condition

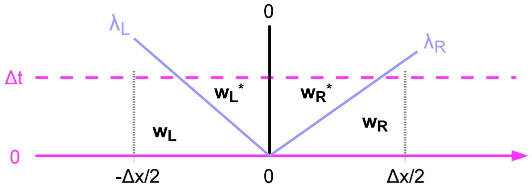
Consistency condition from Harten, Lax and van Leer:

$$\underbrace{\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}\left(\frac{x}{\Delta t}, w_L, w_R\right)dx}_{\tilde{A}} = \underbrace{\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} w_{\mathcal{R}}\left(\frac{x}{\Delta t}, w_L, w_R\right)dx}_{A_{\mathcal{R}}},$$

with $w_{\mathcal{R}}(\frac{x}{t}, w_L, w_R)$ the exact solution of the Riemann problem.

- Left side:

$$\tilde{A} = \frac{1}{2}\left(w_L + w_R\right) + \frac{\Delta t}{\Delta x}\left(\lambda_L\left(w_L - w_L^{\star}\right) + \lambda_R\left(w_R^{\star} - w_R\right)\right).$$



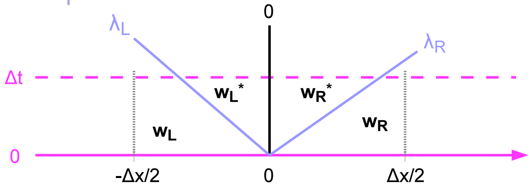
$$\partial_t w + \partial_x F(w) = S(w) \qquad (5)$$

- Right side (by integrating on $[0, \Delta t] \times \left[-\frac{\Delta x}{2}, \frac{\Delta x}{2}\right]$ the exact equation (5)) in the case $\alpha = 0$:

$$A_{\mathcal{R}} = \frac{1}{2}(w_L + w_R) - \frac{\Delta t}{\Delta x}(F(w_R) - F(w_L)) + \Delta t \begin{pmatrix} 0 \\ S_{\mathcal{R}} \end{pmatrix}.$$

- Exact source term $S_{\mathcal{R}} = \frac{1}{\Delta t \Delta x} \int_0^{\Delta t} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \chi \rho_{\mathcal{R}} \partial_x \phi \mathrm{d}x \mathrm{d}t$ approximated by $S^\star$.
- Four other unknowns: $w_L^\star = \begin{pmatrix} \rho_L^\star \\ \rho_L^\star u_L^\star \end{pmatrix}$ and $w_R^\star = \begin{pmatrix} \rho_R^\star \\ \rho_R^\star u_R^\star \end{pmatrix}$.

$\rightarrow$ We need 5 equations.



- Two first equations: $\tilde{A} = A_{\mathcal{R}} \Leftrightarrow$

$$\begin{aligned} \lambda_L\left(\rho_L - \rho_L^\star\right) + \lambda_R\left(\rho_R^\star - \rho_R\right) &= \rho_L u_L - \rho_R u_R, \\ \lambda_L\left(\rho_L u_L - \rho_L^\star u_L^\star\right) + \lambda_R\left(\rho_R^\star u_R^\star - \rho_R u_R\right) &= \rho_L u_L^2 + p_L - \rho_R u_R^2 - p_R + \Delta x S^\star. \end{aligned}$$

- Third equation: $\rho_L^\star u_L^\star = \rho_R^\star u_R^\star =: q^\star$ (continuity of momentum).

- Two last equations: study the steady states at rest.

Steady states at rest

- Steady states at rest associated to (5) given by: $\left\{ \begin{array}{l} u = 0, \\ e - \chi\phi = K, \end{array} \right.$

  where $e(\rho)$ is defined by $\partial_x e = \frac{1}{\rho}\partial_x P$ which is, since $P(\rho) = \varepsilon\rho^\gamma$:

$$e(\rho) = \varepsilon \frac{\gamma}{\gamma - 1} \rho^{\gamma-1} + e_0$$

  with e_0 an arbitrary constant.

- In the Riemann problem: $\left\{ \begin{array}{l} u_L = u_R = 0, \\ e_L - \chi\phi_L = e_R - \chi\phi_R = K. \end{array} \right.$

- In the approximated Riemann solver:

$$\left\{ \begin{array}{l} u_L^* = u_R^* = 0 \quad (\Rightarrow q^* = 0), \\ e_L^* - \chi\phi_L = e_R^* - \chi\phi_R = K. \end{array} \right.$$

Source-term approximation[6]

- We are at steady state $\Rightarrow \tilde{A} = A_{\mathcal{R}}$ becomes

$$\begin{cases} \lambda_L(\rho_L - \rho_L^\star) + \lambda_R(\rho_R^\star - \rho_R) = 0, \\ (\lambda_R - \lambda_L)q^\star + p_R - p_L = \Delta x\, S^\star. \end{cases}$$

- We want to preserve steady states $\Rightarrow$ we have to ensure $q^\star = 0$
$\Rightarrow$ we suggest to put the following consistent expression of $S^\star$:

$$S^\star = \frac{\chi}{\Delta x} \frac{p_R - p_L}{e_R - e_L} \left(\phi_R - \phi_L\right).$$

→ Fourth equation.

→ Necessary condition for Well-Balanced property.

[6] Berthon, Chalons, Math. of Comp. 2016

- For the last equation, we choose to impose

$$e_L \frac{\rho_L^\star}{\rho_L} - \chi\phi_L = e_R \frac{\rho_R^\star}{\rho_R} - \chi\phi_R,$$

which is consistent with $e_R - e_L = \chi\left(\phi_R - \phi_L\right)$ and gives a linearization of the equation $\partial_x e = \chi\partial_x\phi$.

→ Sufficient condition for Well-Balanced property.

- Solving the system of five equations ⇒ expressions for $\rho_L^\star$, $\rho_R^\star$, $q^\star$ and $S^\star$.

- Theorem: this approximate Riemann solver preserves steady states at rest on ρ and ρu.

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Case with friction: $\alpha > 0$

- What changes? The second equation:

$$\partial_t (\rho u) + \partial_x \left(\rho u^2 + p\right) = \chi \rho \partial_x \phi - \alpha \rho u.$$

- Integrating on $[0, \Delta t] \times \left[-\frac{\Delta x}{2}, \frac{\Delta x}{2}\right]$ gives

$$\mathcal{F}(\Delta t) = \frac{1}{2}(\rho_L u_L + \rho_R u_R) - \frac{\Delta t}{\Delta x}\left(\rho_R u_R^2 + p_R - \rho_L u_L^2 - p_L\right) + \Delta t S_{\mathcal{R}} - \alpha \int_0^{\Delta t} \mathcal{F}(t)\,\mathrm{d}t,$$

where

$$\mathcal{F}(t) = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} (\rho u)_{\mathcal{R}}(x,t)\,\mathrm{d}x.$$

$\rightarrow$ Equation in $\mathcal{F}(\Delta t)$:

$$\mathcal{F}'(\Delta t) = -\frac{1}{\Delta x}\left(\rho_R u_R^2 + p_R - \rho_L u_L^2 - p_L\right) + S_{\mathcal{R}} - \alpha \mathcal{F}(\Delta t).$$

- Solution:

$$\mathcal{F}(\Delta t) = K e^{-\alpha \Delta t} + \frac{1}{\alpha}\left[-\frac{1}{\Delta x}\left(\rho_R u_R^2 + p_R - \rho_L u_L^2 - p_L\right) + S_{\mathcal{R}}\right],$$

  with

$$K = \frac{1}{2}\left(\rho_L u_L + \rho_R u_R\right) - \frac{1}{\alpha}\left[-\frac{1}{\Delta x}\left(\rho_R u_R^2 + p_R - \rho_L u_L^2 - p_L\right) + S_{\mathcal{R}}\right].$$

- HLL consistency condition: $\tilde{A}_2 = \mathcal{F}(\Delta t)$.
- Approximation of $S_{\mathcal{R}}$ in order to preserve equilibrium ($u = 0$): $S^\star$ defined previously suits.
- $\rho_L^\star$ and $\rho_R^\star$ not changed.
- $q^\star$ is the only quantity that has to be modified when taking $\alpha > 0$.

Correction for the positivity

Min-Max procedure[7] in order to ensure $\rho_L^\star \geq 0$ and $\rho_R^\star \geq 0$, where

$$\rho_L^\star = \rho_L + \frac{\lambda_R \mathcal{R}}{\delta_R \lambda_L - \delta_L \lambda_R} - \delta_R \frac{\rho_L u_L - \rho_R u_R}{\delta_R \lambda_L - \delta_L \lambda_R},$$

$$\rho_R^\star = \rho_R + \frac{\lambda_L \mathcal{R}}{\delta_R \lambda_L - \delta_L \lambda_R} - \delta_L \frac{\rho_L u_L - \rho_R u_R}{\delta_R \lambda_L - \delta_L \lambda_R},$$

with $\delta_R := \frac{e_R}{\rho_R}$, $\delta_L := \frac{e_L}{\rho_L}$ and $\mathcal{R} := \chi\left(\phi_R - \phi_L\right) - e_R$.

- For simplicity reasons: $\lambda_R = -\lambda_L$.
- Consistency gives now:

$$\tilde{A}_1 = A_{\mathcal{R},1} \;\Leftrightarrow\; \rho_L^\star + \rho_R^\star = \rho_L + \rho_R - \frac{\rho_R u_R - \rho_L u_L}{\lambda_R} =: 2\rho_{HLL}.$$

- We take: $$\rho_R^\star = \min\left(\max\left(0, \rho_R^\star\right), 2\rho_{HLL}\right), \quad \rho_L^\star = \min\left(\max\left(0, \rho_L^\star\right), 2\rho_{HLL}\right).$$

[7] Audusse, Chalons, Ung, CMS 2015.

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Full model

We now look at the equation (3) on ϕ, that we rewrite as:

$$\partial_t \phi - D\partial_x \psi = a\rho - b\phi,$$

where ρ is assumed to be known (since computed previously) and $\psi := \partial_x \phi$.

HLL consistency condition

Consistency condition from Harten, Lax and van Leer:

$$\underbrace{\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\tilde{\phi}\left(\frac{x}{\Delta t},\phi_L,\phi_R\right)\mathrm{d}x}_{\tilde{A}} = \underbrace{\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\phi_{\mathcal{R}}\left(\frac{x}{\Delta t},\phi_L,\phi_R\right)\mathrm{d}x}_{A_{\mathcal{R}}},$$

with $\phi_{\mathcal{R}}$ the exact Riemann solver and $\tilde{\phi}$ the approximated one.

- Left side:

$$\tilde{A} = \frac{1}{2}\left(\phi_L + \phi_R\right) + \frac{\Delta t}{\Delta x}\left(\lambda_L\left(\phi_L - \phi_L^\star\right) + \lambda_R\left(\phi_R^\star - \phi_R\right)\right).$$

$$\partial_t \phi - D\partial_x \psi = a\rho - b\phi \qquad (3)$$

- Right side (by integrating on $[0, \Delta t] \times \left[-\frac{\Delta x}{2}, \frac{\Delta x}{2}\right]$ the exact equation (3)):

$$A_{\mathcal{R}} = \frac{1}{2}\left(\phi_L + \phi_R\right) + \frac{D\Delta t}{\Delta x}\left(\psi_R - \psi_L\right) + \frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2} a\rho_{\mathcal{R}} - b\phi_{\mathcal{R}} \mathrm{d}x\mathrm{d}t.$$

- First part of the model:

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \rho_{\mathcal{R}}\left(x, t\right)\mathrm{d}x = \frac{1}{2}\left(\rho_L + \rho_R\right) - \frac{t}{\Delta x}\left(\rho_R u_R - \rho_L u_L\right).$$

- We get:

$$\mathcal{G}\left(\Delta t\right) := A_{\mathcal{R}} = \frac{1}{2}\left(\phi_L + \phi_R\right) + \frac{D\Delta t}{\Delta x}\left(\psi_R - \psi_L\right) + a\left(\frac{\Delta t}{2}\left(\rho_L + \rho_R\right) - \frac{\Delta t^2}{2\Delta x}\left(\rho_R u_R - \rho_L u_L\right)\right) - b\int_0^{\Delta t}\mathcal{G}\left(t\right)\mathrm{d}t.$$

→ Equation on $\mathcal{G}(\Delta t)$:

$$\mathcal{G}'(\Delta t)+b\mathcal{G}(\Delta t)=\frac{D}{\Delta x}(\psi_R-\psi_L)+a\left(\frac{1}{2}(\rho_L+\rho_R)-\frac{\Delta t}{\Delta x}(\rho_R u_R-\rho_L u_L)\right).$$

- Solution:

$$\mathcal{G}(\Delta t)=\frac{1}{2}(\phi_L+\phi_R)e^{-b\Delta t}+\beta_1\Delta t+\beta_2\left(1-e^{-b\Delta t}\right)$$

with $\beta_1=-\frac{a}{b\Delta x}(\rho_R u_R-\rho_L u_L)$ and

$$\beta_2=\frac{D}{b\Delta x}(\psi_R-\psi_L)+\frac{a}{2b}(\rho_L+\rho_R)+\frac{a}{b^2\Delta x}(\rho_R u_R-\rho_L u_L).$$

- Consistency condition: $\tilde{A}=A_{\mathcal{R}}=\mathcal{G}(\Delta t)$.

\+ Preserve the variations of ϕ: $\phi_L-\phi_L^\star=\phi_R-\phi_R^\star$.

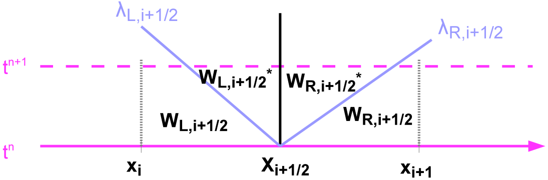
→ Expressions for $\phi_L^\star$ and $\phi_R^\star$.

- And what about ψ_L and ψ_R? Answer later...

Finite Volumes scheme

- 1D domain $[0, L]$ discretized by $N+1$ points: $x_i = i\Delta x$, $i = 0, \dots, N$, $\Delta x = \frac{L}{N}$.
- Evolution in time of $w_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} w(x, t^n)\,\mathrm{d}x$ and $\phi_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \phi(x, t^n)\,\mathrm{d}x$, where $t^n = n\Delta t$ for a time step Δt.
- Scheme coming from the previously defined Riemann solvers:

$$\begin{cases} w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(\lambda_{i-\frac{1}{2},R}\left(w_i^n - w^\star_{i-\frac{1}{2},R}\right) - \lambda_{i+\frac{1}{2},L}\left(w_i^n - w^\star_{i+\frac{1}{2},L}\right)\right), \\ \phi_i^{n+1} = \phi_i^n - \frac{\Delta t}{\Delta x}\left(\lambda_{i-\frac{1}{2},R}\left(\phi_i^n - \phi^\star_{i-\frac{1}{2},R}\right) - \lambda_{i+\frac{1}{2},L}\left(\phi_i^n - \phi^\star_{i+\frac{1}{2},L}\right)\right). \end{cases}$$



Definition of ψ_i^n

- ψ_i^n such that good approximation of $(\partial_x \phi)_i^n$, for example of the form
$$\psi_i^n = \frac{1}{2\Delta x}\left(\phi_{i+1}^n - \phi_{i-1}^n\right) \times \mathit{1}(\Delta x)$$
where $\mathit{1}(\Delta x)$ has to be consistent with 1 when Δx tends to 0.
- $\mathit{1}(\Delta x)$: correction term such that $\phi_i^{n+1} = \phi_i^n$ at equilibrium in the particular case $\gamma = 2$.

Equilibrium for $\gamma = 2$

- Equilibrium given by:

$$\begin{cases} u = 0, \\ \phi = \frac{2\varepsilon}{\chi}\rho + K, \\ D\partial_{xx}\phi - b\phi = -a\rho, \end{cases} \Rightarrow \begin{cases} u = 0, \\ \partial_{xx}\rho - \frac{\chi}{2\varepsilon D}\left(\frac{2\varepsilon b}{\chi} - a\right)\rho = Kb\frac{\chi}{2\varepsilon D}. \end{cases}$$

- Solutions[1,2]:

$$\text{if } \rho = 0: \quad \phi(x) = A\cosh\left(x\sqrt{b/D}\right) + B\sinh\left(x\sqrt{b/D}\right),$$

$$\text{if } \rho > 0,\ C < 0: \quad \phi(x) = A\cos\left(x\sqrt{|C|}\right) + B\sin\left(x\sqrt{|C|}\right) - \phi_p, \quad \rho(x) = \frac{\chi}{2\varepsilon}\left(\phi(x) - K\right),$$

$$\text{if } \rho > 0,\ C > 0: \quad \phi(x) = A\cosh\left(x\sqrt{C}\right) + B\sinh\left(x\sqrt{C}\right) - \phi_p, \quad \rho(x) = \frac{\chi}{2\varepsilon}\left(\phi(x) - K\right),$$

where A and B are some constants, $C = \frac{1}{D}\left(b - \frac{a\chi}{2\varepsilon}\right)$ and $\phi_p = \frac{Ka\chi}{2\varepsilon b - a\chi}$.

[1] Natalini, Ribot, Twarogowska, CMS 2014.
[2] Twarogowska, PhD Thesis 2011.

- Injecting these solutions in the Finite Volumes scheme and imposing $\phi_i^{n+1} = \phi_i^n$ gives the appropriate expression of $\mathcal{I}(\Delta x)$:

$$\begin{aligned}
&\text{if } \rho = 0: && \mathcal{I}(\Delta x) = \frac{\Delta x^2}{2} \frac{b/D}{\cosh\left(\sqrt{b/D}\,\Delta x\right) - 1},\\
&\text{if } \rho > 0,\ C < 0: && \mathcal{I}(\Delta x) = \frac{\Delta x^2}{2} \frac{C}{\cos\left(\sqrt{|C|}\,\Delta x\right) - 1},\\
&\text{if } \rho > 0,\ C > 0: && \mathcal{I}(\Delta x) = \frac{\Delta x^2}{2} \frac{C}{\cosh\left(\sqrt{C}\,\Delta x\right) - 1}.
\end{aligned}$$

- Theorem: with these expressions of $\mathcal{I}(\Delta x)$ and ψ_i^n:

$$\psi_i^n = \frac{1}{2\Delta x} \left(\phi_{i+1}^n - \phi_{i-1}^n\right) \times \mathcal{I}(\Delta x)\,,$$

steady states are exactly preserved for $\gamma = 2$ on ϕ and well approximated for $\gamma > 1$ on ρ and ρu.
Moreover, the scheme is non-negative preserving.

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Testcase 1: perturbation of an equilibrium solution

- Exact equilibrium solution[1,2]:

$$\phi(x) = \begin{cases} \frac{2\varepsilon bK}{\tau\chi D}\frac{\cos\left(\sqrt{\tau}x\right)}{\cos\left(\sqrt{\tau}\overline{x}\right)} - \frac{aK}{\tau D}, & \text{for } x \in [0, \overline{x}], \\ -\frac{2\varepsilon K}{\chi}\frac{\cosh\left(\sqrt{\frac{b}{D}}(x-L)\right)}{\cosh\left(\sqrt{\frac{b}{D}}(\overline{x}-L)\right)}, & \text{for } x \in]\overline{x}, L], \end{cases}$$

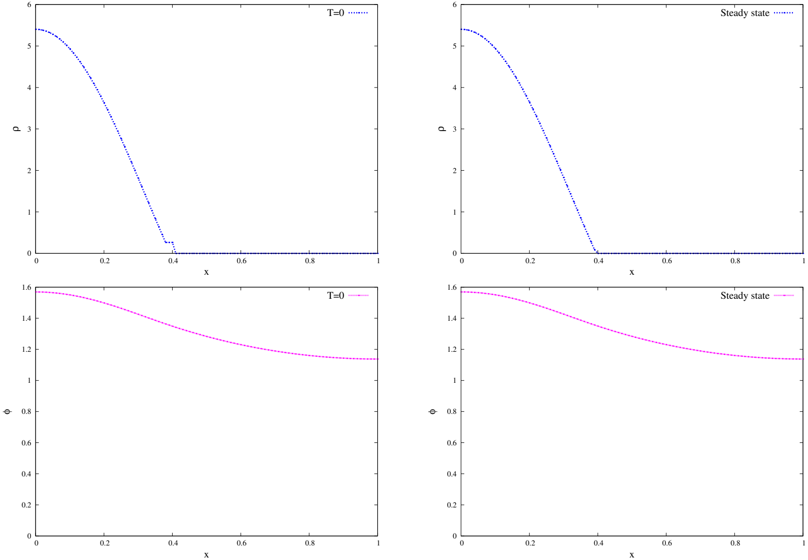
$$\rho(x) = \begin{cases} \frac{\chi}{2\varepsilon}\phi(x) + \frac{D}{b}\frac{M\tau^{3/2}}{\tan\left(\sqrt{\tau}\overline{x}\right) - \sqrt{\tau}\overline{x}}, & \text{for } x \in [0, \overline{x}], \\ 0, & \text{for } x \in]\overline{x}, L], \end{cases}$$

$$u(x) = 0,$$

where $\tau = \frac{1}{D}\left(\frac{a\chi}{2\varepsilon} - b\right)$ and $\overline{x}$ s.t. $\sqrt{\frac{b}{\tau D}}\tan\left(\sqrt{\tau}\overline{x}\right) = \tanh\left(\sqrt{\frac{b}{D}}\left(\overline{x} - L\right)\right)$.

[1] Natalini, Ribot, Twarogowska, CMS 2014.
[2] Twarogowska, PhD Thesis 2011.

ρ and ϕ with $a = b = D = \varepsilon = 1$, $\gamma = 2$, $\chi = 50$, $L = 1$, $\Delta x = 0.01$.

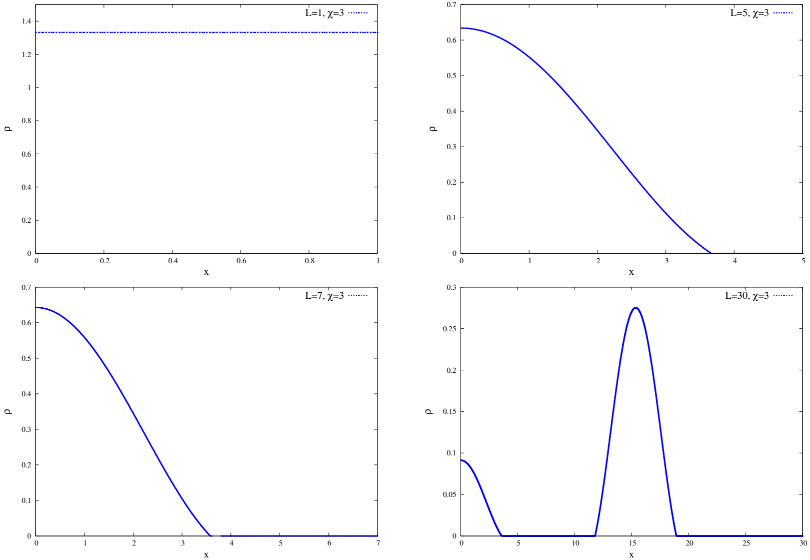


Testcase 2: influence of parameters on the steady state

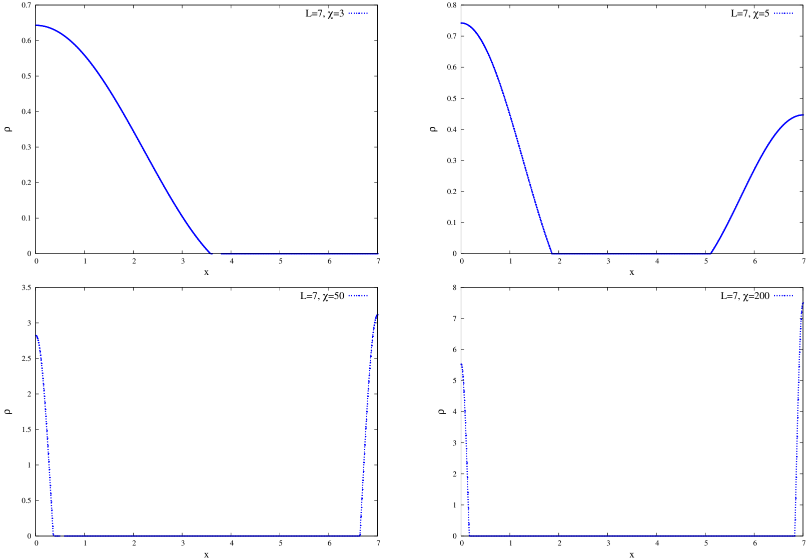
- Initial conditions: $\rho(x,0) = 1 + \sin\left(4\pi|x - \frac{L}{4}|\right),$
  $u(x,0) = 0,$
  $\phi(x,0) = 0.$
- Study
  - Density ρ as a function of x at steady state.
  - Influence of L and χ with $a = b = D = \varepsilon = 1$.
  - Influence of γ with $a = 20,\ b = 10,\ D = 0.1,\ \varepsilon = 1$.
- Validation? No analytical solution. But results similar to those of Twarogowska[1,2].

[1] Natalini, Ribot, Twarogowska, CMS 2014.
[2] Twarogowska, PhD Thesis 2011.

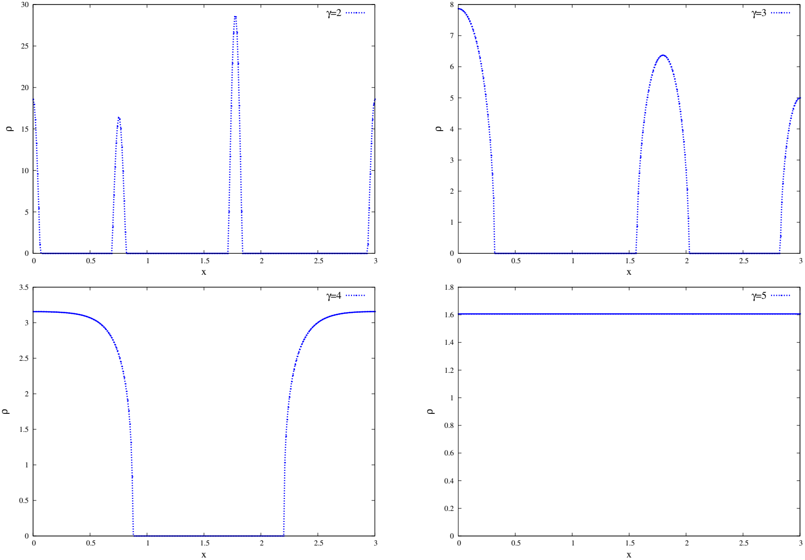
Influence of L at $\gamma = 2$, $\chi = 3$, $\Delta x = 0.01$.



Influence of χ at $\gamma = 2$, $L = 7$, $\Delta x = 0.01$.



Influence of γ at $\chi = 10$, $L = 3$, $\Delta x = 0.01$.



Conclusions...

- HLL consistent Riemann solver.
- Equilibrium states exactly preserved when $\gamma = 2$ and well approached when $\gamma > 2$.
- Non-negativity of ρ and ϕ.
- No problem with vacuum $\rho = 0$.
- AP scheme in the case $\alpha > 0$.

... and perspectives

- Understand the behaviour of the asymptotic-parabolic model.
- Extension of the model to the 2D.
- Application of this technique to other systems.

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Thank you for your attention!