Functor homology: methods

Notes of a course given to Tokyo's advanced students

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In this whole lecture, $\mathbb{k}$ denotes a commutative ground ring and, for a small category $\mathcal{C}$, $\mathcal{F}(\mathcal{C};\mathbb{k})$ denotes the (abelian) category of functors from $\mathcal{C}$ to $\mathbb{k}-\mathbf{Mod}$ (left $\mathbb{k}$-modules).

1 Change of source category: Kan extensions

1.1 Adjunction at the source

Proposition 1.1. *Let $\varphi : \mathcal{C} \to \mathcal{D}$ and $\psi : \mathcal{D} \to \mathcal{C}$ be functors between small categories. Assume that φ is left adjoint to ψ. Then $\varphi^* : \mathcal{F}(\mathcal{D};\mathbb{k}) \to \mathcal{F}(\mathcal{C};\mathbb{k})$ is right adjoint to $\psi^* : \mathcal{F}(\mathcal{C};\mathbb{k}) \to \mathcal{F}(\mathcal{D};\mathbb{k})$: one has natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{F}(\mathcal{C};\mathbb{k})}(F, \varphi^* G) \simeq \mathrm{Hom}_{\mathcal{F}(\mathcal{D};\mathbb{k})}(\psi^* F, G)$$

(for F in $\mathcal{F}(\mathcal{C};\mathbb{k})$ and G in $\mathcal{F}(\mathcal{D};\mathbb{k})$) of $\mathbb{k}$-modules, which can be derived into natural isomorphisms

$$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C};\mathbb{k})}(F, \varphi^* G) \simeq \mathrm{Ext}^*_{\mathcal{F}(\mathcal{D};\mathbb{k})}(\psi^* F, G)$$

of graded $\mathbb{k}$-modules.

In a dual way, one has natural isomorphisms

$$F \underset{\mathcal{D}}{\otimes} \psi^* G \simeq \varphi^* F \underset{\mathcal{C}}{\otimes} G$$

(for F in $\mathcal{F}(\mathcal{D}^{op};\mathbb{k})$ and G in $\mathcal{F}(\mathcal{C};\mathbb{k})$) of $\mathbb{k}$-modules, which can be derived into natural isomorphisms

$$\mathrm{Tor}^{\mathcal{D}}_*(F, \psi^* G) \simeq \mathrm{Tor}^{\mathcal{C}}_*(\varphi^* F, G)$$

of graded $\mathbb{k}$-modules.

Proof. Use the natural isomorphism $\psi^* P^{\mathcal{C}}_c \simeq P^{\mathcal{D}}_{\varphi(c)}$ (for $c \in \mathrm{Ob}\,\mathcal{C}$). □

Example 1.2. For a field k, denote by $\mathcal{V}_k$ the category of finite-dimensional k-vector spaces and by $\mathrm{I}_k : \mathcal{V}_k \to \mathbf{Ab}$ the forgetful functor. Suppose that $k \to K$ is a finite extension of degree d. Then the base change functor $T := K \underset{k}{\otimes} - : \mathcal{V}_k \to \mathcal{V}_K$ is adjoint on both sides to the forgetful functor $F : \mathcal{V}_K \to \mathcal{V}_k$. Moreover, we have $F^* \mathrm{I}_k = \mathrm{I}_K$ and $T^* \mathrm{I}_K \simeq K \underset{k}{\otimes} \mathrm{I}_k (\simeq \mathrm{I}_k^{\oplus d})$. So, Proposition 1.1 gives

$$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{V}_K)}(\mathrm{I}_K, \mathrm{I}_K) = \mathrm{Ext}^*_{\mathcal{F}(\mathcal{V}_K)}(\mathrm{I}_K, F^* \mathrm{I}_k) \simeq \mathrm{Ext}^*_{\mathcal{F}(\mathcal{V}_k)}(T^* \mathrm{I}_K, \mathrm{I}_k) \simeq \dots$$

$$K \underset{k}{\otimes} \mathrm{Ext}^*_{\mathcal{F}(\mathcal{V}_k)}(\mathrm{I}_k, \mathrm{I}_k) \simeq \mathrm{Ext}^*_{\mathcal{F}(\mathcal{V}_k)}(\mathrm{I}_k, \mathrm{I}_k)^{\oplus d}.$$

Remark 1.3. Adjunctions at the target transport also to functor categories (without changing the sides).

An important situation where it is fruitful to use Proposition 1.1 is the following: let $\mathcal{C}$ be a small category having finite coproducts (resp. products), denoted by $\sqcup$ (resp. $\times$). The functor $\sqcup : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (resp. $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$) is left (resp. right) adjoint to the diagonal functor $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ (more generally, this adjunction extends to iterated (co)products and diagonal, between $\mathcal{C}$ and $\mathcal{C}^n$, for each $n \in \mathbb{N}$).

For example:

Corollary 1.4 (Sum/diagonal adjunction for functors)**.** *Let $\mathcal{C}$ be a small category having finite coproducts. Let A, B and F be functors in $\mathcal{F}(\mathcal{C}; \Bbbk)$. Then one has a natural isomorphism*

$$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C};\Bbbk)}(A \otimes B, F) \simeq \mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}\times\mathcal{C};\Bbbk)}(A \boxtimes B, \sqcup^* F)$$

of graded $\Bbbk$-modules.

We will see simple but important applications of this corollary later.

1.2 Kan extensions

Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. As the category of $\Bbbk$-modules has all limits and colimits, the precomposition functor $\varphi^* : \mathcal{F}(\mathcal{D}; \Bbbk) \to \mathcal{F}(\mathcal{C}; \Bbbk)$ has a left and a right adjoint, called respectively the left and the right *Kan extension* along φ (see [3], chap. X, § 3, for example) and denoted by Lan_φ and $\mathrm{Ran}_\varphi : \mathcal{F}(\mathcal{C}; \Bbbk) \to \mathcal{F}(\mathcal{D}; \Bbbk)$. We have explicit formulas:

$$\mathrm{Lan}_\varphi(F)(d) = \underset{\varphi/d}{\mathrm{colim}}\, F \circ \pi_d$$

where φ/d is the comma category associated to the functor $\mathcal{C}(-, d) \circ \varphi : \mathcal{C} \to \mathbf{Sets}^{op}$ (it has as objects the pairs (c, f) where c is an object of $\mathcal{C}$ and $f : \varphi(c) \to d$ a morphism in $\mathcal{C}$) and $\pi_d : \varphi/d \to \mathcal{C}$ is the forgetful functor ($(c, f) \mapsto c$ on objects). Dually, one has

$$\mathrm{Ran}_\varphi(F)(d) = \lim_{\varphi \backslash d} F \circ \pi'_d$$

where $\varphi \setminus d$ is the comma category associated to the functor $\mathcal{C}(d, -) \circ \varphi : \mathcal{C} \to \mathbf{Sets}$.

In general, it is not easy to give simpler expressions for Kan extensions, which are not usually exact functors. Nevertheless, in some good situations, Kan extensions are exact and given by explicit formula (without limit or colimit!). The previous paragraph gives examples of such situations (when the functor φ has an adjoint), but there are others. Let us give a useful class of examples.

Let $\mathcal{C}$ be a small category, $\Phi : \mathcal{C} \to \mathbf{Sets}$ a functor, $\mathcal{C}_\Phi$ the corresponding category of elements and $\varphi : \mathcal{C}_\Phi \to \mathcal{C}$ the forgetful functor. Then one has canonical isomorphisms

$$\mathrm{Lan}_\varphi(F)(c) \simeq \bigoplus_{x \in \Phi(c)} F(c, x).$$

Kan extensions are also useful for tensor products of functors over a small category: if $\varphi^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ (which will be sometimes denoted simply by φ) denotes the functor having the same effect on objects and arrows as $\varphi : \mathcal{C} \to \mathcal{D}$, one has natural isomorphisms

$$(\varphi^{op})^* F \underset{\mathcal{C}}{\otimes} G \simeq F \underset{\mathcal{D}}{\otimes} \mathrm{Lan}_\varphi(G)$$

(for F in $\mathcal{F}(\mathcal{D}^{op};\Bbbk)$ and G in $\mathcal{F}(\mathcal{C};\Bbbk)$) and

$$F \underset{\mathcal{C}}{\otimes} \varphi^* G \simeq \mathrm{Lan}_{\varphi^{op}}(F) \underset{\mathcal{D}}{\otimes} G$$

(for F in $\mathcal{F}(\mathcal{C}^{op};\Bbbk)$ and G in $\mathcal{F}(\mathcal{D};\Bbbk)$).

1.3 Derived Kan extensions

In the situation of the previous paragraph, we can derive on the left (resp. right) the left (resp. right) Kan extension of φ, getting functors given on objects by:

$$\mathbf{L}_\bullet \mathrm{Lan}_\varphi(F)(d) = H_\bullet(\varphi/d; F \circ \pi_d)$$

and

$$\mathbf{R}^\bullet \mathrm{Ran}_\varphi(F)(d) = H^\bullet(\varphi \setminus d; F \circ \pi'_d).$$

The left derived Kan extension gives rise to natural spectral sequences (of composite functors)

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{F}(\mathcal{D};\Bbbk)}(\mathbf{L}_j \mathrm{Lan}_\varphi(F), G) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{F}(\mathcal{C};\Bbbk)}(F, \varphi^* G)$$

and

$$E^2_{i,j} = \mathrm{Tor}^{\mathcal{D}}_i(F, \mathbf{L}_j \mathrm{Lan}_\varphi(G)) \Rightarrow \mathrm{Tor}^{\mathcal{C}}_{i+j}(\varphi^* F, G).$$

We will use these formal facts in the second part of this lectures' serie, through the following consequence:

Proposition 1.5. *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor between small categories and F a functor in $\mathcal{F}(\mathcal{D};\Bbbk)$. Let us denote by L_i, for all integer $i \geq 0$, the functor of $\mathcal{F}(\mathcal{D}^{op};\Bbbk)$ defined by $L_i(d) = \tilde{H}_i(\varphi \setminus d; \Bbbk)$ (where $\tilde{H}$ denotes reduced homology).*

Assume that $\mathrm{Tor}^{\mathcal{D}}_*(L_i, F) = 0$ for each i. Then the canonical map

$$H_*(\mathcal{C}; \varphi^* F) \to H_*(\mathcal{D}; F)$$

is an isomorphism.

2 Künneth formula

Hypothesis 2.1. In this section, we assume that $\Bbbk$ is a field or the ring $\mathbb{Z}$ of integers.

We begin with the case of homology, which is easier than the one of cohomology.

The first result is the only one which requires no assumption on functors.

Proposition 2.2. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be small categories and F_1, F_2, G_1, G_2 functors in $\mathcal{F}(\mathcal{C}_1^{op};\Bbbk)$, $\mathcal{F}(\mathcal{C}_2^{op};\Bbbk)$, $\mathcal{F}(\mathcal{C}_1;\Bbbk)$ and $\mathcal{F}(\mathcal{C}_2;\Bbbk)$ respectively. Then one has a natural isomorphism*

$$(F_1 \boxtimes F_2) \underset{\mathcal{C}_1\times\mathcal{C}_2}{\otimes} (G_1 \boxtimes G_2) \simeq (F_1 \underset{\mathcal{C}_1}{\otimes} G_1) \otimes (F_2 \underset{\mathcal{C}_2}{\otimes} G_2).$$

Sketch of proof. Use the canonical isomorphism $P^{\mathcal{C}_1}_{c_1} \boxtimes P^{\mathcal{C}_2}_{c_2} \simeq P^{\mathcal{C}_1\times\mathcal{C}_2}_{(c_1,c_2)}$ and the fact that tensor products (over a small category, or external) commute to colimits in each variable. □

The following result is a consequence of the previous one thanks to general machinery of homological algebra (the usual Künneth formula, for tensor products of complexes of $\Bbbk$-modules).

Proposition 2.3. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be small categories and F_1, F_2, G_1, G_2 functors in $\mathcal{F}(\mathcal{C}_1^{op};\Bbbk)$, $\mathcal{F}(\mathcal{C}_2^{op};\Bbbk)$, $\mathcal{F}(\mathcal{C}_1;\Bbbk)$ and $\mathcal{F}(\mathcal{C}_2;\Bbbk)$ respectively. Assume that all these functors take $\Bbbk$-flat values. Then one has, for each integer n, a (split) exact sequence*

$$0 \to \bigoplus_{i+j=n} \mathrm{Tor}^{\mathcal{C}_1}_i(F_1,G_1) \underset{\Bbbk}{\otimes} \mathrm{Tor}^{\mathcal{C}_2}_j(F_2,G_2) \to \mathrm{Tor}^{\mathcal{C}_1\times\mathcal{C}_2}_n(F_1 \boxtimes F_2, G_1 \boxtimes G_2) \to \dots$$

$$\bigoplus_{r+s=n-1} \mathrm{Tor}^{\Bbbk}_1(\mathrm{Tor}^{\mathcal{C}_1}_r(F_1,G_1), \mathrm{Tor}^{\mathcal{C}_2}_s(F_2,G_2)) \to 0,$$

so, in particular, we have a natural isomorphism

$$\mathrm{Tor}^{\mathcal{C}_1\times\mathcal{C}_2}_\bullet(F_1 \boxtimes F_2, G_1 \boxtimes G_2) \simeq \mathrm{Tor}^{\mathcal{C}_1}_\bullet(F_1,G_1) \otimes \mathrm{Tor}^{\mathcal{C}_2}_\bullet(F_2,G_2)$$

of graded $\Bbbk$-modules when $\mathrm{Tor}^{\mathcal{C}_1}_\bullet(F_1,G_1)$ (or $\mathrm{Tor}^{\mathcal{C}_2}_\bullet(F_2,G_2)$) is $\Bbbk$-flat.

Let us go to cohomology: we need here assumptions even before deriving Hom-functors.

Proposition 2.4. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be small categories and F_i, G_i functors in $\mathcal{F}(\mathcal{C}_i;\Bbbk)$ ($i = 1,2$). Assume that:*

1. *all these functors take $\Bbbk$-flat values;*
2. *F_1 and F_2 are finitely presented;*
3. *$\mathrm{Hom}_{\mathcal{F}(\mathcal{C}_1;\Bbbk)}(F_1,G_1)$ is $\Bbbk$-flat.*

Then the natural map

$$\mathrm{Hom}_{\mathcal{F}(\mathcal{C}_1;\Bbbk)}(F_1,G_1) \otimes \mathrm{Hom}_{\mathcal{F}(\mathcal{C}_2;\Bbbk)}(F_2,G_2) \to \mathrm{Hom}_{\mathcal{F}(\mathcal{C}_1\times\mathcal{C}_2;\Bbbk)}(F_1 \boxtimes F_2, G_1 \boxtimes G_2)$$

(sending $f_1 \otimes f_2$ on $f_1 \boxtimes f_2$) is an isomorphism.

Sketch of proof. Use again the canonical isomorphism $P^{\mathcal{C}_1}_{c_1} \boxtimes P^{\mathcal{C}_2}_{c_2} \simeq P^{\mathcal{C}_1\times\mathcal{C}_2}_{(c_1,c_2)}$, then take finite presentations of F_1 and F_2 and the flatness assumptions. □

Here again, the following proposition follows from general homological algebra.

Proposition 2.5. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be small categories and F_i, G_i functors in $\mathcal{F}(\mathcal{C}_i;\Bbbk)$ ($i=1,2$). Assume that:*

1. *all these functors take $\Bbbk$-flat values;*
2. *F_1 and F_2 have resolutions by finitely generated projectives;*
3. *$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}_1;\Bbbk)}(F_1,G_1)$ is $\Bbbk$-flat.*

Then we have a natural isomorphism

$$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}_1;\Bbbk)}(F_1,G_1)\otimes\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}_2;\Bbbk)}(F_2,G_2)\to\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}_1\times\mathcal{C}_2;\Bbbk)}(F_1\boxtimes F_2,G_1\boxtimes G_2)$$

of graded $\Bbbk$-modules.

Remark 2.6. The statements given in this section are not optimal (in particular, one needs less flatness assumptions than stated): we prefered to avoid too technical statements to concentrate on particularly useful special forms of Künneth formula in functor (co)homology.

3 Applications to polynomial functors

3.1 Pirashvili's lemma

The following classical result is easy but extremely useful. It has a lot of variations (with torsion groups instead of extension groups, for example).

Proposition 3.1 (Pirashvili)**.** *Let $\mathcal{C}$ be a small category with finite coproducts (resp. products) and a null object and $d\geq 0$ an integer. Let $A_0,\dots,A_d$ and F be functors in $\mathcal{F}(\mathcal{C};\Bbbk)$. Assume that the A_i are reduced (that is, $A_i(0)=0$) and F is polynomial of degree $\leq d$. Then*

$$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C};\Bbbk)}(A_0\otimes\cdots\otimes A_d,F)=0\qquad\textit{(resp. }\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C};\Bbbk)}(F,A_0\otimes\cdots\otimes A_d)=0\textit{)}.$$

Proof. One proves for example the result when $\mathcal{C}$ has finite coproducts.

Thanks to Corollary 1.4,

$$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C};\Bbbk)}(A_0\otimes\cdots\otimes A_d,F)\simeq\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}^{d+1};\Bbbk)}(A_0\boxtimes\cdots\boxtimes A_d,s^*F)$$

where $s:\mathcal{C}^{d+1}\to\mathcal{C}$ is the iterated coproduct. Now, as F is polynomial of degree $\leq d$, s^*F splits into a direct sum of functors factorizing through a projection functor $\pi:\mathcal{C}^{d+1}\to\mathcal{C}^d$ forgetting one of the factors, let us say $(a_0,\dots,a_d)\mapsto(a_1,\dots,a_d)$ for example. This functor is adjoint (on both sides) to the functor $(a_1,\dots,a_d)\mapsto(0,a_1,\dots,a_d)$, as 0 is a zero object of $\mathcal{C}$. Now, Proposition 1.1 implies

$$\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}^{d+1};\Bbbk)}(T,\pi^*U)\simeq\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}^{d};\Bbbk)}(T(0,-),U)$$

so that $\mathrm{Ext}^*_{\mathcal{F}(\mathcal{C}^{d+1};\Bbbk)}(T,\pi^*U)$ is zero if the multifunctor T is multireduced, that is zero as soon as one of its entries is zero. This gives the conclusion. □

3.2 Graded exponential functors

Graded exponential functors are particularly manageable in functor (co)homology. An example of very useful statement is the following.

Proposition 3.2. *Let $\mathcal{C}$ be a small category with finite coproducts and a null object, $E^\bullet$ a graded exponential functor in $\mathcal{F}(\mathcal{C}^{op};\Bbbk)$ and $F_1,\dots,F_n$ functors in $\mathcal{F}(\mathcal{C};\Bbbk)$.*

If $\Bbbk$ is a field, then we have a natural isomorphism

$$\mathrm{Tor}_*^{\mathcal{C}}(E^\bullet, F_1\otimes\cdots\otimes F_n)\simeq \mathrm{Tor}_*^{\mathcal{C}}(E^\bullet, F_1)\otimes\cdots\otimes \mathrm{Tor}_*^{\mathcal{C}}(E^\bullet, F_n)$$

of bigraded $\Bbbk$-modules.

The same statement is true for $\Bbbk=\mathbb{Z}$ if we assume moreover that all the functors E^i and F_j take torsion-free values and that the abelian groups $\mathrm{Tor}_r^{\mathcal{C}}(E^s,F_t)$ are also all torsion-free.

Sketch of proof. First use the version of Corollary 1.4 for Tor-groups (or apply directly Proposition 1.1). After apply the exponential property and Proposition 2.3. □

A variation of the same argument gives:

Proposition 3.3. *Let $\mathcal{C}$ be a small category with finite coproducts and a null object and A an additive functor in $\mathcal{F}(\mathcal{C}^{op})$ taking torsion-free values. Assume that $F_1,\dots,F_n$ are functors in $\mathcal{F}(\mathcal{C};\Bbbk)$ such that their values and the abelian groups $\mathrm{Tor}_i^{\mathcal{C}}(A^{\otimes m},F_j)$ are all torsion-free. Then we have for all integer $d\geq 0$ a natural graded isomorphism*

$$\mathrm{Tor}_*^{\mathcal{C}}(A^{\otimes d},F_1\otimes\cdots\otimes F_n)\simeq\bigoplus_{i_1+\cdots+i_n=d}\mathrm{Tor}_*^{\mathcal{C}}(A^{\otimes i_1},F_1)\otimes\cdots\otimes\mathrm{Tor}_*^{\mathcal{C}}(A^{\otimes i_n},F_n)\uparrow_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}^{\Sigma_d}$$

which is Σ_d-equivariant.

Applying this statement to the category $\mathbf{gr}$ of finitely generated free groups and to the covariant functor $\mathfrak{a}$ (abelianization) and its dual $\mathfrak{a}^\vee$ (which lies in $\mathcal{F}(\mathbf{gr}^{op})$), we get the computation of $\mathrm{Tor}_*^{\mathbf{gr}}((\mathfrak{a}^\vee)^{\otimes d},\mathfrak{a}^{\otimes n})$ (and so, of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes n},\mathfrak{a}^{\otimes d})$ by duality) from $\mathrm{Tor}_*^{\mathbf{gr}}((\mathfrak{a}^\vee)^{\otimes i},\mathfrak{a})$, which can be computed by the explicit projective resolution of $\mathfrak{a}$ given by the bar resolution on a free group (see the previous talk).

4 Use of explicit complexes

4.1 Hyper(co)homology spectral sequences

Let us remind the following classical fact of homological algebra: if

$$C^0\to C^1\to\cdots\to C^n\to\dots$$

is a cochain complex in an abelian category $\mathcal{A}$ with enough projective or injective objects, and T is an object of $\mathcal{A}$, then we have two functorial (converging) cohomological spectral sequences with first pages

$$\mathrm{I}_1^{i,j}=\mathrm{Ext}^j_{\mathcal{A}}(T,C^i)$$

and

$$\mathrm{II}_2^{i,j} = \mathrm{Ext}^i_{\mathcal{A}}(T, H^j(C^*))$$

and with the same abutment, which is called the *hypercohomology* of the complex C^* with coefficient in T. (All our cohomological spectral sequences $E_*^{\bullet,\bullet}$ are graded in such a way that the differential $d_r : E_r \to E_r$ has bidegree $(r, 1-r)$.)

This applies in particular in the functor categories $\mathcal{F}(\mathcal{C};\Bbbk)$. We have also, in these, "dual" hyperhomology spectral sequences.

4.2 Koszul, de Rham and bar complexes

In this paragraph, we give some classical complexes which are often useful in functor categories (these complexes have a lot of small variations).

If V is a (left) $\Bbbk$-module and i and integer, let us define $S^i(V)$ (resp. $\Lambda^i(V)$) be the i-th symmetric (resp. exterior) power or V (by convention, it is zero if $i < 0$). So, $S^\bullet$ and $\Lambda^\bullet$ are quotients of the graded endofunctor $T^\bullet$ (tensor algebra) of $\Bbbk$-modules. Moreover, $S^\bullet$ and $\Lambda^\bullet$ are graded exponential functors [1]. For all integers i and j, one can define natural transformations

$$\kappa : S^i \otimes \Lambda^j \to S^{i+1} \otimes \Lambda^{j-1}$$

and

$$d : S^i \otimes \Lambda^j \to S^{i-1} \otimes \Lambda^{j+1}$$

as the composites

$$\kappa : S^i \otimes \Lambda^j \to S^i \otimes (\Lambda^1 \otimes \Lambda^{j-1}) \simeq (S^i \otimes S^1) \otimes \Lambda^{j-1} \to S^{i+1} \otimes \Lambda^{j-1}$$

using the coproduct on $\Lambda^\bullet$ and the product on $S^\bullet$ and

$$d : S^i \otimes \Lambda^j \to (S^{i-1} \otimes S^1) \otimes \Lambda^j \simeq S^{i-1} \otimes (\Lambda^1 \otimes \Lambda^j) \to S^{i-1} \otimes \Lambda^{j+1}$$

using the coproduct on $S^\bullet$ and the product on $\Lambda^\bullet$.

It is an easy exercice to check that $d \circ d$ and $\kappa \circ \kappa$ are zero. So, d and κ define on $S^* \otimes \Lambda^\bullet$ to structures of complexes, which are called respectively de Rham and Koszul complexes.

The following result is classical.

Proposition 4.1. *The Koszul complex has zero cohomology, except in total degree* 0*, on projective* $\Bbbk$*-modules.*

If $\Bbbk$ *is a field of characteristic* 0*, then the de Rham complex has zero cohomology, except in total degree* 0*.*

Sketch of proof. As all functors in these complexes commute to filtered colimits, it is enough to prove the result on finitely generated $\Bbbk$-modules. It is even enough to restrict to finitely generated free $\Bbbk$-modules. The key point is that Koszul and de Rham differentials are compatible, in a suitable sense, with the *bigraded exponential structure* on $S^*\otimes\Lambda^\bullet$. So, the (usual) Künneth formula (which applies easily because, on projective modules, our functors take projective values) shows that is indeed enough to prove that our complexes are acyclic when evaluated on $\Bbbk$, what is a very simple computation. □

[1] The source category ($\Bbbk$-modules) is not equivalent to a small one, but it does not matter.

If $\Bbbk$ is a field of positive characteristic, the de Rham complex is no longer acyclic, but its cohomology can be computed (see [2], § 3).

The reduced (or normalized) bar complex, truncated and shifted, on a group V gives a functorial complex

$$\cdots \to \bar{\Bbbk}[V]^{\otimes n} \to \bar{\Bbbk}[V]^{\otimes (n-1)} \to \cdots \to \bar{\Bbbk}[V],$$

where $\bar{\Bbbk}[V]$ denotes the augmentation ideal of the $\Bbbk$-algebra of the group V, whose differential is given by a usual (simplicial) alternated sum and whose homology is naturally isomorphic (in non-negative degree) to the group homology $H_{*+1}(V;\Bbbk)$. The previous lecture gave applications of this when V is a free group. It is also very useful when V is an *abelian* group. In the following paragraph, we will deal with $\Bbbk = \mathbb{Q}$: we have then a natural isomorphism $H_*(V;\mathbb{Q}) \simeq \Lambda^*(V \otimes \mathbb{Q})$. (Note that $V \mapsto H_*(V;\mathbb{Z})$ is not well understood as a graded endofunctor of abelian groups.)

4.3 An example of application

Proposition 4.2. *Let $\mathcal{A}$ be a small additive category, a an object of $\mathcal{A}$ and A an additive functor in $\mathcal{F}(\mathcal{A};\mathbb{Q})$. Then we have $\mathrm{Ext}^i_{\mathcal{F}(\mathcal{A};\mathbb{Q})}(\mathcal{A}(a,-) \otimes \mathbb{Q}, A) = 0$ for $i > 0$.*

Proof. Let us precompose the previous version of the bar complex (for $\Bbbk = \mathbb{Q}$) with $\mathcal{A}(a,-)$: we get a complex

$$\cdots \to (\bar{P}^{\mathcal{A}}_a)^{\otimes n} \to (\bar{P}^{\mathcal{A}}_a)^{\otimes (n-1)} \to \cdots \to \bar{P}^{\mathcal{A}}_a$$

of functors of $\mathcal{F}(\mathcal{A};\mathbb{Q})$ whose homology in degree n is $\Lambda^{n+1}(\mathcal{A}(a,-)\otimes\mathbb{Q})$. Applying $\mathrm{Ext}^*(-,A)$ to this complex, we get two hypercohomology spectral sequences with the same abutment and with first pages:

- $\mathrm{I}_1^{i,j} = \mathrm{Ext}^j((\bar{P}^{\mathcal{A}}_a)^{\otimes (i+1)}, A)$, which is zero except for $i = j = 0$, because $(\bar{P}^{\mathcal{A}}_a)^{\otimes (i+1)}$ is projective and $\mathrm{Hom}((\bar{P}^{\mathcal{A}}_a)^{\otimes (i+1)}, F) \simeq cr_{i+1}(F)(a,\dots,a)$;
- $\mathrm{II}_2^{i,j} = \mathrm{Ext}^i(\Lambda^{j+1}(\mathcal{A}(a,-) \otimes \mathbb{Q}), A)$. This term is zero for $j > 0$, because Λ^{j+1} is rationally a direct summand of T^{j+1}, and $\mathrm{Ext}^i((\mathcal{A}(a,-) \otimes \mathbb{Q})^{\otimes (j+1)}, A)$ vanishes thanks to Proposition 3.1.

So, the second spectral sequence collapses at the second page. As it has the same abutment as the first spectral sequence, which 0 except in degree 0, we deduce that $\mathrm{II}_2^{i,0} = 0$ for $i > 0$, what is the wished conclusion. □

Remark 4.3. 1. The situation is completely different if $\mathbb{Q}$ is replaced by a finite field $\mathbb{F}$. In this case, Franjou, Lannes and Schwartz proved in [2] that, in the category $\mathcal{F}(\mathcal{V}_{\mathbb{F}};\mathbb{F})$, the dimension of the $\mathbb{F}$-vector space $\mathrm{Ext}^n_{\mathcal{F}(\mathcal{V}_{\mathbb{F}};\mathbb{F})}(\mathrm{Id},\mathrm{Id})$ is 0 when n is odd and 1 when n is even. The ingredients of the proof include hypercohomology spectral sequences associated to de Rham and Koszul complexes, Pirashvili's cancellation result (Proposition 3.1) but also another specific lemma, due to Kuhn; moreover, the proof is quite more long and tricky (nevertheless, when $\mathbb{F}$ has characteristic 2, one can reach quicker the result — see § 2 of [2]). Note that the result of [2] was obtained earlier by Breen in the very long paper [1], with a completely different point of view (there is no functor category in this work!).

2. By using Proposition 4.2, one can see that, for all integer d, the inclusion of the abelian subcategory $\mathcal{F}_d(\mathcal{A};\mathbb{Q})$ of polynomial functors of degree $\leq d$ into $\mathcal{F}(\mathcal{A};\mathbb{Q})$ induces isomorphism between extension groups. For $d = 1$, it is an easy exercise from the proposition; the general case requires more work but is also reachable with the tools introduced in this lecture.

References

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