Stable homology with twisted coefficients: general framework and theorem

Notes of a course given to Tokyo's advanced students

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$\Bbbk$ denotes a ground commutative ring in this whole talk.

Introduction: what is stable homology?

Assume that

$$G_0 \to G_1 \to \dots G_n \to G_{n+1} \to \dots$$

is a sequence of group morphisms. There are three main questions when looking at the sequence induced in homology by this — note that it induced a sequence

$$H_*(G_0;\Bbbk) \to H_*(G_1;\Bbbk) \to \dots \to H_*(G_n;\Bbbk) \to H_*(G_{n+1};\Bbbk) \to \dots \quad (1)$$

of graded $\Bbbk$-modules, but also

$$H_*(G_0;M_0) \to H_*(G_1;M_1) \to \dots \to H_*(G_n;M_n) \to H_*(G_{n+1};M_{n+1}) \to \dots \quad (2)$$

if

$$M_0 \to M_1 \to \dots \to M_n \to M_{n+1} \to \dots$$

if a sequence of compatible representations of these groups (that is, M_n is a representation of G_n and $M_n \to M_{n+1}$ is G_n-equivariant with M_{n+1} endowed with the G_n-action given by restriction along the morphism $G_n \to G_{n+1}$).

1. (Homological stability) Does these sequence stabilize? That is: is it true that, for all $d \in \mathbb{N}$, the map $H_d(G_n;M_n) \to H_d(G_{n+1};M_{n+1})$ is an isomorphism for n big enough?

   One generally studies this question first for constant coefficients, and after for suitable twisted coefficients (in general, given by polynomial functors), but in fact the key point is usually the one of constant coefficients (the other one is generally a consequence of the first) — see the work of Dwyer [4] on some general linear groups and the recent work of Randal-Williams and Wahl [8] for a very general framework.

2. (Stable homology with constant coefficients) What is the colimit of the sequence (1), called stable homology of (G_n) with coefficients in $\Bbbk$?

3. (Stable homology with twisted coefficients) What is the colimit of the sequence (2), called stable homology of (G_n) with coefficients in (M_n)?

Here, we distinguished constant and twisted coefficients because the known methods are completely different; in fact, the work that we are going to present in this lecture (taken from [3]) allows to answer, in some cases, to this third question (for coefficients given by polynomial functors) *assuming that we know the answer of the second question.*

From now on, we will only deal with the third question; we will first given a general framework which covers the classical cases of symmetric, general linear, orthogonal, symplectic or unitary groups, automorphism groups of free groups (but not congruence groups or subgroups IA of automorphism groups of free groups).

1 The general framework

(The framework presented here is a variation around the one of [3], § 1, which was also a little reworked at the beginning of [2].)

Let $(\mathcal{C}, *, 0)$ be a (small) *symmetric monoidal* (see [5], for example) category and X an object of $\mathcal{C}$. We are interested in the groups

$$G_n := \mathrm{Aut}_{\mathcal{C}}(X^{*n})$$

endowed with the group morphisms

$$\mathrm{Aut}_{\mathcal{C}}(X^{*n}) \to \mathrm{Aut}_{\mathcal{C}}(X^{*(n+1)}) \qquad f \mapsto f * \mathrm{Id}_X$$

(note that we could have choosen another way of "including n factors into $n+1$", as $f \mapsto \mathrm{Id}_X * f$, but that all these choices are conjugated — so, induce the same morphisms in homology — because the monoidal structure is *symmetric*[1]).

Example 1.1.

1. Let $\mathcal{C}$ be the category of finite sets (or a small skeleton; for the moment we do not precise the arrows — there are several choices! —, which will be important later to satisfy our axioms) endowed with the disjoint union, X a set with one element: one has $G_n = \Sigma_n$ (symmetric group), with the usual inclusion $\Sigma_n \to \Sigma_{n+1}$ as arrows.
2. If R is any ring, the category of left R-modules (same remark on morphisms) with the direct sum, and $X = R$, gives $G_n = GL_n(R)$.
3. If k is a commutative ring and $\mathcal{C}$ the category of k-quadratic spaces with the orthogonal sum, and X is k^2 with the canonical hyperbolic form, the $G_n = O_{n,n}(k)$.
4. In the category of groups with the free product, $X = \mathbb{Z}$ (same remark on morphisms), we get $G_n = \mathrm{Aut}(F_n)$.

To be able to construct compatible sequences of representations of these groups from functors on $\mathcal{C}$, we make the assumption that 0 is an *initial object* of $\mathcal{C}$, what implies that we have canonical arrows $A \to A*B$ ($A \simeq A*0 \xrightarrow{\mathrm{Id}_A*(0\to B)} A*B$) and $B \to A*B$ for all objects A and B of $\mathcal{C}$. So, we have a sequence

$$0 = X^{*0} \to X = X^{*1} \to X^{*2} \to \cdots \to X^{*n} \to X^{*(n+1)} \to \dots$$

[1] *braided* (or even *prebraided* — see [8]) would be enough.

of arrows of $\mathcal{C}$ (using $X^{*n} \to X^{*n} * X = X^{*(n+1)}$): for any functor F in $\mathcal{F}(\mathcal{C}; \Bbbk)$,

$$F(0) \to F(X) \to F(X^{*2}) \to \cdots \to F(X^{*n}) \to F(X^{*(n+1)}) \to \dots$$

is a compatible sequence of representations of our groups.

We are interested in the stable homology of (G_n) with coefficients into F:

$$H_*(G_\infty; F_\infty) = \operatorname*{colim}_{n \in \mathbb{N}} H_*(G_n; F(X^{*n}))$$

(group homology commutes with *filtered* colimits; G_∞ is the colimit of our sequence of groups and F_∞ the one of the previous sequence of representations).

We also met the functor homology $H_*(\mathcal{C}; F)$. It is related to the stable homology in the following way. For each $n \in \mathbb{N}$, the group G_n, seen as a category with one object, has a canonical faithful embedding into $\mathcal{C}$, which induces a natural map $H_*(G_n; F(X^{*n})) \to H_*(\mathcal{C}; F)$. Moreover, the diagram

$$\begin{array}{ccc} H_*(G_n; F(X^{*n})) & \longrightarrow & H_*(\mathcal{C}; F) \\ \downarrow & \nearrow & \\ H_*(G_{n+1}; F(X^{*(n+1)})) & & \end{array}$$

commutes thanks to the natural transformation between the functors $G_n \to \mathcal{C}$ (functors given by the canonical inclusion functor and the composition of the group morphism $G_n \to G_{n+1}$ and the canonical inclusion functor $G_{n+1} \to \mathcal{C}$), natural transformation which is given by the canonical map $X^{*n} \to X^{*(n+1)}$ (used in the previous sequence of objects of $\mathcal{C}$). So, our morphisms assemble to give a natural morphism

$$H_*(G_\infty; F_\infty) \to H_*(\mathcal{C}; F). \tag{3}$$

In order to study this morphism, we need several other assumptions on the monoidal category $(\mathcal{C}, *, 0)$.

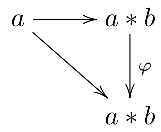
Definition 1.2. Let $(\mathcal{C}, *, 0)$ be a small symmetric monoidal category whose unit 0 is an initial object and X an object of $\mathcal{C}$. We introduce the following hypotheses.

1. (Transitivity axiom)
   - (strong form) $\mathcal{C}(x, t)$ is a transitive $\mathrm{Aut}_{\mathcal{C}}(t)$-set for all objects x and t in $\mathcal{C}$, or more generally
   - (weak form) for each $f \in \mathcal{C}(a, b)$, there is $\alpha \in \mathrm{Aut}_{\mathcal{C}}(a * b)$ such the diagram

     $$\begin{array}{ccccc} a & \xrightarrow{f} & b & \longrightarrow & a * b \\ & \searrow & & & \downarrow \alpha \\ & & & & a * b \end{array}$$

     commutes (where the arrows without label are the canonical ones).

2. (Stabilizers axiom) for all objects a and b of $\mathcal{C}$, the canonical group morphism $\mathrm{Aut}_{\mathcal{C}}(b) \to \mathrm{Aut}_{\mathcal{C}}(a * b)$ is an injection whose image is the subgroup of automorphisms φ such that the diagram

   commutes.

3. (Axiom of generation by X)

   - (Strong form) all object of $\mathcal{C}$ is isomorphic to X^{*n} for some $n \in \mathbb{N}$, or more generally
   - (Weak form) for all object t of $\mathcal{C}$, there are an object a of $\mathcal{C}$ and an integer $n \in \mathbb{N}$ such that $t * a \simeq X^{*n}$.

Remark 1.3. In [8], Randal-Williams and Wahl introduce the notion of *homogeneous categories*, with very similar axioms. A category satisfying the *strong* form of the axioms is homogeneous, but their notion of homogeneous category is more general, because they do not require the monoidal category to be symmetric (there is a weaker condition that they call *prebraided*).

Example 1.4. (This is the continuation of Example 1.1)

1. (A skeleton of) the category of finite sets, endowed with the disjoint union and X a set with one element, does *not* satisfy the axioms of Definition 1.2: the axioms of transitivity and of stabilizers fail. But the subcategory of *injections* (sometimes denoted by FI) satisfies all the axioms, in the strong forms.

2. (A skeleton of) the category of finitely generated free (or projective) left R-modules (where R is a given ring), with $X = R$ and direct sum as monoidal structure, does *not* satisfies the axioms. Even if we replace this category by its subcategory of monomorphisms, or of split monomorphisms (but with no splitting *given in the structure*), the axiom of stabilizers fails. But the category $\mathbf{S}(R)$ with the same objects and monoidal structure and with morphism split monomorphisms, *with a given splitting*, that is:

   $$\mathbf{S}(R)(M, N) = \{(u, v) \in \mathrm{Hom}_R(M, N) \times \mathrm{Hom}_R(N, M) \,|\, v \circ u = \mathrm{Id}_M\}$$

   satisfies all the axioms, in the weak version. The strong transitivity axiom is even satisfied if R is nice enough (for example, a skew field, a PID, a local ring...).

3. For any commutative ring k, (a skeleton of) the category of finitely generated free (or projective) k-modules equipped with a non-degenerate quadratic form satisfies the axioms (for the orthogonal sum and X the usual hyperbolic k-module k^2).

4. (A skeleton of) the category of finitely generated free groups with the free product and $X = \mathbb{Z}$ does *not* satisfies the axiom of stabilizers, even if we

replace morphisms by split monomorphisms (even if the splitting is given in the structure). The right category $\mathcal{G}$ (with same objects, monoidal structure and X) has the following morphisms: an arrow from G to H is a pair (u, T) where $u : G \to H$ is a group monomorphism and T a subgroup of H such that H is the internal free product of T and of the image of u. This category satisfies the axioms in the strong form.

2 First comparison of functor homology and twisted stable group homology

The dream, to use functor homology to study stable homology of groups, would be that the natural morphism (3) would be an isomorphism. Unfortunately, there is an obvious obstruction to this, for very simple coefficients: the constant ones ($F = \Bbbk$)! Indeed, as $\mathcal{C}$ has an initial object, its reduced (untwisted) homology is zero, but $\tilde{H}_*(G_\infty; \Bbbk)$ has no reason to be zero...

Nevertheless, there is a way to change a little the natural morphism (3) to "include by force" the value of stable homology with constant coefficients. Let us consider the group G_∞ as a category with a single object and denote by $\Pi : G_\infty \times \mathcal{C} \to \mathcal{C}$ the projection functor. Then we can form, for a functor F in $\mathcal{F}(\mathcal{C}; \Bbbk)$, a natural morphism

$$H_*(G_\infty; F_\infty) \to H_*(G_\infty \times \mathcal{C}; \Pi^* F) \tag{4}$$

by assembling, as in the previous section, the morphisms $H_*(G_n; F(X^{*n})) \to H_*(G_\infty \times \mathcal{C}; \Pi^* F)$ given by restriction along the functor $G_n \to G_\infty \times \mathcal{C}$ whose component $G_n \to G_\infty$ is the canonical group morphism and $G_n \to \mathcal{C}$ is the same as before.

Theorem 2.1 ([3]). *Under the assumptions of Definition 1.2, the natural morphism (4) of graded $\Bbbk$-modules is an isomorphism for all functor F of $\mathcal{F}(\mathcal{C}; \Bbbk)$.*

Sketch of proof. We are going to show the result only when the *strong* assumptions of Definition 1.2 are satisfied (the general case is not very different, but technically a little harder).

By standard facts of homological algebra, as the morphism (4) is a morphism of δ-functors commuting to filtered colimits, is it enough to show that is an isomorphism when F is a projective functor $P^{\mathcal{C}}_{X^{*i}}$ for an $i \in \mathbb{N}$. For $n \geq i$, the transitivity and stabilizers axioms give an isomorphism

$$\mathcal{C}(X^{*i}, X^{*n}) \simeq \mathrm{Aut}_{\mathcal{C}}(X^{*n})/\mathrm{Aut}_{\mathcal{C}}(X^{*(n-i)}) = G_n/G_{n-i}$$

of G_n-sets. Thus

$$H_*(G_n; P^{\mathcal{C}}_{X^{*i}}(X^{*n})) \simeq H_*(G_{n-i}; \Bbbk)$$

thanks to Shapiro Lemma. One checks easily that, after taking colimit on m, it allows to identify the map (4) as

$$\operatorname*{colim}_{n\in\mathbb{N}} H_*(G_{n-i}; \Bbbk) \to \operatorname*{colim}_{n\in\mathbb{N}} H_*(G_n; \Bbbk) = H_*(G_\infty; \Bbbk) \simeq H_*(G_\infty \times \mathcal{C}; P^{\mathcal{C}}_{X^{*i}})$$

induced by the group monomorphisms $G_{n-i} \to G_n$, what gives the conclusion. □

Relations between $H_*(G_\infty; F_\infty)$, $H_*(\mathcal{C}; F)$ and $H_*(G_\infty; \Bbbk)$ The (classical) Künneth formula gives the following corollaries of Theorem 2.1.

Corollary 2.2. *If $\Bbbk$ is a field, then, under the assumptions of Definition 1.2, we have a natural isomorphism*

$$H_*(G_\infty; F_\infty) \simeq H_*(G_\infty; \Bbbk) \underset{\Bbbk}{\otimes} H_*(\mathcal{C}; F)$$

of graded $\Bbbk$-vector spaces.

Corollary 2.3. *If $\Bbbk$ is the ring of integers $\mathbb{Z}$, then, under the assumptions of Definition 1.2, we have a natural short exact sequence of abelian groups*

$$0 \to \bigoplus_{i+j=n} H_i(G_\infty; \mathbb{Z}) \otimes H_j(\mathcal{C}; \Bbbk) \to H_n(G_\infty; F_\infty) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i(G_\infty; \mathbb{Z}), H_j(\mathcal{C}; F)) \to 0$$

for all $n \in \mathbb{N}$.

Case of symmetric groups ([1], revisited) Remind that Γ is the skeleton of the category of finite pointed sets whose objets are $[i] := \{0, \ldots, i\}$ with 0 as base-point (for $i \in \mathbb{N}$). Si F is an object of $\mathcal{F}(\Gamma; \Bbbk)$, we define the cross-effect

$$cr_i(F) := \mathrm{Ker}\,(F([i]) \to F([i-1])^{\oplus i})$$

where the arrows are induced by the morphisms $r_j : [i] \to [i-1]$ (for $j = 1, \ldots, i$) sending t on t for $t < j$, j on 0 and t on $t-1$ for $j < t \leq i$. The $\Bbbk$-module $cr_i(F)$ is endowed with a natural Σ_i-action (see [7] for details about these cross-effects).

Theorem 2.4 (Betley)**.** *Let F be a functor in $\mathcal{F}(\Gamma; \Bbbk)$. There is a natural isomorphism*

$$H_*(\Sigma_\infty; F_\infty) \simeq \bigoplus_{i \in \mathbb{N}} H_*(\Sigma_\infty \times \Sigma_i; cr_i(F))$$

where Σ_∞ acts trivially on $cr_i(F)$.

Sketch of proof. Let Ω (resp. $\Theta(= FI)$) the categories having as objects the finite sets $\underline{n} := \{1, \ldots, n\}$ for $n \in \mathbb{N}$ and as morphisms the surjective (resp. injective) functions. We have a canonical faithful functor $\alpha : \Theta \to \Gamma$ which adds an extra base-point on objects ($\underline{n} \mapsto [n]$) and has the obvious effect on arrows.

As Θ satisfies the assumptions of Definition 1.2, Theorem 2.1 implies that it is enough to exhibit a natural isomorphism

$$H_*(\Theta; \alpha^* F) \simeq \bigoplus_{i \in \mathbb{N}} H_*(\Sigma_i; cr_i(F)).$$

The left Kan extension $\mathrm{Lan}_{\alpha^{op}}$ is an *exact* functor given by

$$\mathrm{Lan}_{\alpha^{op}}(F)(E) = \bigoplus_{A \in \mathcal{Q}(E)} F(A \setminus \{*_A\})$$

where $\mathcal{Q}(E)$ denotes the set of quotients of the pointed set E and $*_A$ the base-point of A.

So, we get a natural isomorphism

$$H_*(\Theta; \alpha^* F) \simeq \mathrm{Tor}_*^\Gamma(\mathrm{Lan}_{\alpha^{op}}(\Bbbk), F) \simeq \mathrm{Tor}_*^\Gamma(\Bbbk[\mathcal{Q}], F).$$

We can then use Pirashvili's theorem "à la Dold-Kan" (see [6] or [7]) to get

$$H_*(\Theta; \alpha^* F) \simeq \mathrm{Tor}_*^{\Omega}(\mathrm{cr}\,\Bbbk[\mathcal{Q}], \mathrm{cr}\, F)$$

where cr is an explicit equivalence $\mathcal{F}(\Gamma; \Bbbk) \to \mathcal{F}(\Omega; \Bbbk)$ (or $\mathcal{F}(\Gamma^{op}; \Bbbk) \to \mathcal{F}(\Omega^{op}; \Bbbk)$), which satisfies $\mathrm{cr}\, F(\underline{i}) \simeq cr_i(F)$ as Σ_i-module and $\mathrm{cr}\,\Bbbk[\mathcal{Q}] \simeq \Bbbk[Q]$ where $Q(E)$, for a finite set E, is the set of its quotients. As we have an isomorphism

$$Q \simeq \bigsqcup_{i \in \mathbb{N}} \Omega(-, \underline{i})/\Sigma_i$$

of functors $\Omega^{op} \to \mathbf{Set}$ and Σ_i acts freely on the sets $\Omega(\underline{n}, \underline{i})$, we get

$$\mathrm{Tor}_*^{\Omega}(\Bbbk[Q], T) \simeq \bigoplus_{i \in \mathbb{N}} H_*(\Sigma_i; T(\underline{i}))$$

what gives the conclusion. □

3 Strategy for other comparison results

The previous case of homology of symmetric groups with coefficients into Γ-modules is exceptionnally easy because:

- we get a general result for twisted stable homology, without any polynomial assumption (indeed, this assumption is almost free: *all* Γ-module is analytic, that is, colimit of polynomial functors);
- the proof is not hard from the general Theorem 2.1: one has only to play with Kan extensions which are exact and explicit.

In other interesting examples, the situation is harder: Theorem 2.1 gives an interpretation of twisted stable group homology in terms of functor homology groups whose computation is out of direct reach (in particular because the tools in functor homology studied in previous talks do not apply to categories satisfying the assumptions of Definition 1.2).

The strategy is then to compare the result directly given by Theorem 2.1 with functor homology given by more managable categories (for which some computations are available) through spectral sequences associated to derived Kan extensions. This requires a polynomial hypothesis: we can not get good comparison results without this. In each particular situation, we need different "concrete" arguments to do this, but Scorichenko's work [9] (which applies directly for general linear group) is a general source of inspiration.

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