

# Functor Cohomology

Vincent Franjou

Laboratoire Jean-Leray, Université de Nantes

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## Invariant theory

Let  $\mathbb{K}$  be a field,  $V$  a dimension  $n$  vector space, and let  $G$  be a subgroup of  $\mathrm{GL}(V)$ . Let  $G$  act on  $\mathbb{K}[V] := \mathbb{K}[x_1, \dots, x_n]$  by linear substitution.

### Theorem (Hilbert 1890)

*Suppose that the representation of  $G$  on the coordinate ring  $\mathbb{K}[V] = \mathbb{K}[x_1, \dots, x_n]$  is semi-simple. Then the ring of invariants  $\mathbb{K}[V]^G$  is a finitely generated  $\mathbb{K}$ -algebra.*

### Problem (Hilbert's 14th)

*Let  $\mathbb{F}$  be a subfield of the field of fractions  $\mathbb{K}(V) := \mathbb{K}(x_1, \dots, x_n)$  which contains  $\mathbb{K}$ . Is  $\mathbb{K}[V] \cap \mathbb{F}$  a finitely generated  $\mathbb{K}$ -algebra?*

## Hilbert's 14th problem

In 1958, Nagata found a counter-example to Hilbert's 14th problem. In 1960, he characterized the groups  $G$  such that, whenever  $G$  acts on a finitely generated  $\mathbb{K}$ -algebra, the ring of invariants  $A^G$  is a finitely generated  $\mathbb{K}$ -algebra as well.

*Remark:*

All this still makes sense when  $G$  is an affine group scheme over  $\mathbb{K}$ , that is a representable functor from finitely generated commutative  $\mathbb{K}$ -algebras to groups

or equivalently

the representing finitely generated commutative Hopf algebra  $\mathbb{K}[G]$ , whose *dual* is the co-commutative group algebra  $\mathbb{K}G$ .

Nagata's property is referred to as *geometric reductivity*.

## Higher invariant theory

The same type of question has been asked for the cohomology ring  $H^*(G, A)$ . Here we consider the cohomology defined by Hochschild, the cohomology of the cocommutative Hopf algebra  $\mathbb{K}G$ . This is group cohomology for a discrete group, rational cohomology for a linear algebraic group.

### Problem (van der Kallen)

*Whenever  $G$  acts on a finitely generated  $\mathbb{K}$ -algebra  $A$ , is the cohomology ring  $H^*(G, A)$  a finitely generated  $\mathbb{K}$ -algebra as well?*

Positive answers have been given when

- $G$  is a finite group [Evens, 1961]
- $\mathbb{K}G$  is a f. d. graded connected cocommutative Hopf algebra [Wilkerson, 1981]
- $\mathbb{K}G$  is the enveloping algebra of a f. d. restricted Lie algebra [Friedlander & Parshall, 1983]
- $\mathbb{K}G$  is finite-dimensional [Friedlander & Suslin, 1997] .

In a series of recent papers, van der Kallen argues that the answer should be positive always, as soon as it is positive for the ring of invariants  $A^G = H^0(G, A)$ , that is under Nagata's geometric reductivity.

## Higher invariant theory

Proofs reduce to the case of the general linear group  $GL_n$  and use spectral sequences. Typically, Friedlander and Suslin use May's spectral sequence for the Frobenius kernels  $GL_{n(r)}$ :

$$E_0^{2s,t} = \bigotimes_{i=1}^r S^s(\mathfrak{gl}_n^{(i)\#}) \otimes \bigotimes_{i=1}^r \Lambda^t(\mathfrak{gl}_n^{(i-1)\#}) \otimes A \implies H^{2s+t}(GL_{n(r)}, A)$$

where  $\mathfrak{gl}_n = Lie(GL_n)$ .

What is needed at this level is enough permanent classes to generate a subalgebra which  $E_2$  is a finite module over.

This is an instance where functor cohomology has proven useful.

# Polynomial functors (after Friedlander & Suslin, Pirashvili)

We fix a positive integer  $d$  and describe what an homogeneous polynomial functor of degree  $d$  is [Pirashvili, P & S 2003].

We start with the  $d$ -th divided power functor of a  $\mathbb{K}$ -vector space  $V$ :

$$\Gamma^d(V) := (V^{\otimes d})^{\mathfrak{S}_d}$$

Let  $\mathcal{V}$  be the (small) category of f. d.  $\mathbb{K}$ -vector spaces.

## Definition

The category  $\Gamma^d\mathcal{V}$  has same objects as  $\mathcal{V}$  and has morphisms

$$\mathrm{Hom}_{\Gamma^d\mathcal{V}}(V, W) := \Gamma^d(\mathrm{Hom}_{\mathbb{K}}(V, W)).$$

Note that  $\Gamma^d$  is indeed defined over  $\mathbb{Z}$ , and that  $\Gamma^d\mathcal{V}$  is just the  $\mathbb{K}$ -linearization of  $\Gamma^d(ab)$ , where  $ab$  is the (small) category of finite rank free abelian groups.

# Polynomial functors

## Definition

An homogeneous polynomial functor of degree  $d$  is a  $\mathbb{K}$ -linear functor  $\Gamma^d\mathcal{V} \rightarrow \mathcal{V}$ . We let  $\mathcal{P}_d$  be the category of (natural transformations between) homogeneous polynomial functors of degree  $d$ .

That is: the structural map is an homogeneous degree  $d$  polynomial.

## Examples

- the  $d$ -th tensor power functor  $\otimes^d$ ;
- the  $d$ -th divided power functor  $\Gamma^d$ ;
- for any f. d. representation  $M$  of  $\mathfrak{S}_d$ ,  $\mathrm{Hom}_{\mathfrak{S}_d}(M, \otimes^d)$ .

Write  $I := \Gamma^1 = \otimes^1$  for the identity functor.

The category  $\mathcal{P}_d$  enjoys many nice properties allowing homological computations.

- Polynomial functors admit base change.

This is more easily seen if describing  $\mathcal{P}_d$  as the category of additive functors from  $\Gamma^d(ab)$  to  $\mathcal{V}$ .

- The functor  $P_n := \Gamma^d(\text{Hom}(\mathbb{K}^n, -))$  is a projective generator for  $n \geq d$ .

Note that  $\text{End}_{\mathcal{P}_d}(P_n) = \Gamma^d(\text{End}(\mathbb{K}^n)) = \mathbb{K}[M_n]_d^\#$  is the Schur algebra, so that  $\mathcal{P}_d$  is equivalent to the category of f. d. degree  $d$  polynomial representations of  $\text{GL}_n$  for  $n \geq d$ . Thus, evaluation yields

$$\text{Ext}_{\mathcal{P}_d}^*(F, G) \rightarrow H^*(\text{GL}_n, \text{Hom}(F(\mathbb{K}^n), G(\mathbb{K}^n)))$$

which is an isomorphism when  $n > d$ .

## Cohomology computations (last century)

Most computations stem from the 1994 paper F-Lannes-Schwartz - they are just easier in  $\mathcal{P}$ .

Now  $\mathbb{K}$  is a characteristic  $p$  finite field. For a  $\mathbb{K}$ -vector space  $V$ ,  $V^{(1)}$  is the  $\mathbb{K}$ -vector space obtained by extension of scalars through the Frobenius. For a functor  $F$  in  $\mathcal{P}_d$ ,  $F^{(1)}$  in  $\mathcal{P}_{pd}$  is defined by:  $F^{(1)}(V) := F(V^{(1)})$ .

### Theorem

$E_r := \text{Ext}_{\mathcal{P}}(I^{(r)}, I^{(r)})$  is a divided power algebra on a generator  $e_0$  in degree 2 truncated at height  $r$ . Thus, it is concentrated in even degrees up to  $2(p^r - 1)$  and one-dimensional in those.

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This has been widely extended in [FFSS, Annals Math. 1999], which provided the basic computations for Chałupnik (Annales Sc. ENS 2005).

Chałupnik notices that functors in  $\mathcal{P}_d$  can always be written

$$F = f(\otimes^d)$$

for a functor  $f$  from f. d. representations of  $\mathfrak{S}_d$  to  $\mathcal{V}$ .

Indeed there are two natural choices, the left and the right adjoint of the functor  $f \mapsto F = f(\otimes^d)$ .

A representable choice  $f \cong \text{Hom}_{\mathfrak{S}_d}(M, -)$  is not always possible, but both adjoints take values among coherent functors (F-Pirashvili, preprint).

$$\begin{array}{ccc} & j_! & \\ \mathcal{V}^{\mathfrak{S}_d\text{-mod}} & \xrightarrow{j^*} & \mathcal{P}_d \\ & j_* & \end{array}$$

Chałupnik then proves formulae expressing  $\text{Ext}_{\mathcal{P}_{dpr}}^*(F^{(r)}, G^{(r)})$  in terms of  $f, g$  and the graded  $\mathfrak{S}_d^{op} \times \mathfrak{S}_d$ -representation

$$T_r := \text{Ext}_{\mathcal{P}_{dpr}}^*(\otimes^{d(r)}, \otimes^{d(r)}) = E_r^{\otimes d} \otimes \mathbb{K}\mathfrak{S}_d.$$

In many cases, for instance for a projective  $F$ , it is given by:

$$\text{Ext}_{\mathcal{P}_{dpr}}^*(F^{(r)}, G^{(r)}) \cong g \otimes f^\#(T_r).$$

As a consequence:

$$\text{Ext}_{\mathcal{P}_{dpr}}^*(\Gamma^{d(r)}, G^{(r)}) = G(\text{Ext}_{\mathcal{P}_{pr}}^*(I^{(r)}, I^{(r)})).$$

Resolving a general  $F$  by projectives, we get from the resulting spectral sequence:

## Theorem (F-Pirashvili)

*There is a natural graded  $\mathbb{K}$ -linear functor  $e(F, G)$  defined on  $\mathfrak{S}_d^{op} \times \mathfrak{S}_d$ -modules such that  $e(F, G)(T_r)$  is the second page of a spectral sequence converging to  $\text{Ext}_{\mathcal{P}}(F^{(r)}, G^{(r)})$ , for all polynomial functors  $F, G$  of deg  $d$ .*

Note that collapsing gives a functor-proof of twist injectivity, hence, combined with [Kuhn, 1995, p.286-287], a functor-proof of Ext-injectivity through precomposition

$$\text{Ext}_{\mathcal{P}_d}^*(F, G) \rightarrow \text{Ext}_{\mathcal{P}_{da}}^*(F \circ A, G \circ A).$$

## Bifunctor cohomology

In van der Kallen's finite generation problem, desirable classes live in

$$H(\text{GL}_n, \Gamma^d \text{gl}_n).$$

This can be expressed in terms of functor cohomology if one allows two-variable functors. Define:  $\text{gl}(V, W) := \text{Hom}(V, W)$ .

## Proposition (F-Friedlander 2007)

*Let  $\mathcal{P}_{(1,1)}$  be the category of strict polynomial functors defined on  $\mathcal{V}^{op} \times \mathcal{V}$  and let  $B$  in  $\mathcal{P}_{(1,1)}$  be of bidegree  $(d, d)$ . For  $n \geq d$ , there is an isomorphism:*

$$\text{Ext}_{\mathcal{P}_{(1,1)}}(\Gamma^d \text{gl}, B) \cong H(\text{GL}_n, B(\mathbb{K}^n, \mathbb{K}^n)).$$

We denote by  $H(\text{GL}, B)$  this stable cohomology.

The one variable case is recovered by:

$$\text{Ext}_{\mathcal{P}}(F, G) \cong \text{Ext}_{\mathcal{P}_{(1,1)}}(\Gamma^d \text{gl}, \text{gl} \circ (F, G)).$$

## Theorem (A. Touzé 2007)

*The Poincaré series of  $H_{\mathcal{P}}^*(GL, S^{d(r)}gl)$  is equal to the Poincaré series of the coinvariants of  $T_r = H_{\mathcal{P}}^*(GL, \otimes^{d(r)}gl)$  under the action of the symmetric group  $\mathfrak{S}_d$ .*

Touzé's methods don't give yet an answer for  $H(GL, \Gamma^{d(r)}gl)$ . However, he obtains several relevant results, as the top cohomological degree, or:

## Theorem (A. Touzé 2007)

*The Euler-Poincaré characteristic of  $H_{\mathcal{P}}^*(GL, Fgl)$  and of  $H_{\mathcal{P}}^*(GL, F^{\#}gl)$  are equal.*

Here  $F^{\#}$  is the Kuhn dual of  $F$ , for example:  $S^{d\#} = \Gamma^d$ .