Functor Cohomology

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Invariant theory

Let $\mathbb{K}$ be a field, V a dimension n vector space, and let G be a subgroup of $\mathrm{GL}(V)$. Let G act on $\mathbb{K}[V] := \mathbb{K}[x_1, \ldots, x_n]$ by linear substitution.

Theorem (Hilbert 1890)

Suppose that the representation of G on the coordinate ring $\mathbb{K}[V] = \mathbb{K}[x_1, \ldots, x_n]$ is semi-simple. Then the ring of invariants $\mathbb{K}[V]^G$ is a finitely generated $\mathbb{K}$-algebra.

Problem (Hilbert's 14th)

Let $\mathbb{F}$ be a subfield of the field of fractions $\mathbb{K}(V) := \mathbb{K}(x_1, \ldots, x_n)$ which contains $\mathbb{K}$. Is $\mathbb{K}[V] \cap \mathbb{F}$ a finitely generated $\mathbb{K}$-algebra?

Hilbert's 14th problem

In 1958, Nagata found a counter-example to Hilbert's 14th problem.
In 1960, he characterized the groups G such that, whenever G acts on a finitely generated $\mathbb{K}$-algebra, the ring of invariants A^G is a finitely generated $\mathbb{K}$-algebra as well.

Remark:
All this still makes sense when G is an affine group scheme over $\mathbb{K}$, that is a representable functor from finitely generated commutative $\mathbb{K}$-algebras to groups
or equivalently
the representing finitely generated commutative Hopf algebra $\mathbb{K}[G]$, whose *dual* is the co-commutative group algebra $\mathbb{K}G$.
Nagata's property is referred to as *geometric reductivity.*

Higher invariant theory

The same type of question has been asked for the cohomology ring $\mathrm{H}^*(G, A)$. Here we consider the cohomology defined by Hochschild, the cohomology of the cocommutative Hopf algebra $\mathbb{K}G$. This is group cohomology for a discrete group, rational cohomology for a linear algebraic group.

Problem (van der Kallen)

Whenever G acts on a finitely generated $\mathbb{K}$-algebra A, is the cohomology ring $\mathrm{H}^*(G, A)$ a finitely generated $\mathbb{K}$-algebra as well?

Higher invariant theory

Positive answers have been given when

- G is a finite group [Evens, 1961]
- $\mathbb{K}G$ is a f. d. graded connected cocommutative Hopf algebra [Wilkerson, 1981]
- $\mathbb{K}G$ is the enveloping algebra of a f. d. restricted Lie algebra [Friedlander & Parshall, 1983]
- $\mathbb{K}G$ is finite-dimensional [Friedlander & Suslin, 1997] .

In a series of recent papers, van der Kallen argues that the answer should be positive always, as soon as it is positive for the ring of invariants $A^G = \mathrm{H}^0(G, A)$, that is under Nagata's geometric reductivity.

Higher invariant theory

Proofs reduce to the case of the general linear group GL_n and use spectral sequences. Typically, Friedlander and Suslin use May's spectral sequence for the Frobenius kernels $\mathrm{GL}_{n(r)}$:

$$E_0^{2s,t} = \bigotimes_{i=1}^{r} S^s(\mathrm{gl}_n^{(i)\#}) \otimes \bigotimes_{i=1}^{r} \Lambda^t(gl_n^{(i-1)\#}) \otimes A \Longrightarrow \mathrm{H}^{2s+t}(\mathrm{GL}_{n(r)}, A)$$

where $\mathrm{gl}_n = Lie(\mathrm{GL}_n)$.
What is needed at this level is enough permanent classes to generate a subalgebra which E_2 is a finite module over.
This is an instance where functor cohomology has proven useful.

Polynomial functors (after Friedlander & Suslin, Pirashvili)

We fix a positive integer d and describe what an homogeneous polynomial functor of degree d is [Pirashvili, P & S 2003].
We start with the d-th divided power functor of a $\mathbb{K}$-vector space V:

$$\Gamma^d(V) := (V^{\otimes d})^{\mathfrak{S}_d}$$

Let $\mathcal{V}$ be the (small) category of f. d. $\mathbb{K}$-vector spaces.

Definition

The category $\Gamma^d\mathcal{V}$ has same objects as $\mathcal{V}$ and has morphisms

$$\mathrm{Hom}_{\Gamma^d\mathcal{V}}(V, W) := \Gamma^d(\mathrm{Hom}_{\mathbb{K}}(V, W)).$$

Note that Γ^d is indeed defined over $\mathbb{Z}$, and that $\Gamma^d\mathcal{V}$ is just the $\mathbb{K}$-linearization of $\Gamma^d(ab)$, where ab is the (small) category of finite rank free abelian groups.

Polynomial functors

Definition

An homogeneous polynomial functor of degree d is a $\mathbb{K}$-linear functor $\Gamma^d\mathcal{V} \to \mathcal{V}$. We let $\mathcal{P}_d$ be the category of (natural transformations between) homogeneous polynomial functors of degree d.

That is: the structural map is an homogeneous degree d polynomial.

Examples

- the d-th tensor power functor $\otimes^d$;
- the d-th divided power functor Γ^d;
- for any f. d. representation M of $\mathfrak{S}_d$, $\mathrm{Hom}_{\mathfrak{S}_d}(M, \otimes^d)$.

Write $I := \Gamma^1 = \otimes^1$ for the identity functor.

Polynomial functors

The category $\mathcal{P}_d$ enjoys many nice properties allowing homological computations.

- Polynomial functors admit base change.

This is more easily seen if describing $\mathcal{P}_d$ as the category of additive functors from $\Gamma^d(ab)$ to $\mathcal{V}$.

- The functor $P_n := \Gamma^d(\mathrm{Hom}(\mathbb{K}^n, -))$ is a projective generator for $n \geq d$.

Note that $\mathrm{End}_{\mathcal{P}_d}(P_n) = \Gamma^d(\mathrm{End}(\mathbb{K}^n)) = \mathbb{K}[M_n]_d^{\#}$ is the Schur algebra, so that $\mathcal{P}_d$ is equivalent to the category of f. d. degree d polynomial representations of GL_n for $n \geq d$. Thus, evaluation yields

$$\mathrm{Ext}^*_{\mathcal{P}_d}(F, G) \to \mathrm{H}^*(\mathrm{GL}_n, \mathrm{Hom}(F(\mathbb{K}^n), G(\mathbb{K}^n)))$$

which is an isomorphism when $n > d$.

Cohomology computations (last century)

Most computations stem from the 1994 paper F-Lannes-Schwartz - they are just easier in $\mathcal{P}$.
Now $\mathbb{K}$ is a characteristic p finite field. For a $\mathbb{K}$-vector space V, $V^{(1)}$ is the $\mathbb{K}$-vector space obtained by extension of scalars through the Frobenius. For a functor F in $\mathcal{P}_d$, $F^{(1)}$ in $\mathcal{P}_{pd}$ is defined by: $F^{(1)}(V) := F(V^{(1)})$.

Theorem

$E_r := \mathrm{Ext}_{\mathcal{P}}(I^{(r)}, I^{(r)})$ is a divided power algebra on a generator e_0 in degree 2 truncated at height r. Thus, it is concentrated in even degrees up to $2(p^r - 1)$ and one-dimensional in those.

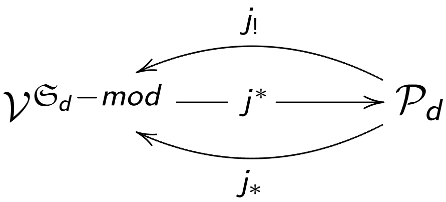
...
This has been widely extended in [FFSS, Annals Math. 1999], which provided the basic computations for Chałupnik (Annales Sc. ENS 2005).

Cohomology computations (after Chałupnik)

Chałupnik notices that functors in $\mathcal{P}_d$ can always be written

$$F = f(\otimes^d)$$

for a functor f from f. d. representations of $\mathfrak{S}_d$ to $\mathcal{V}$.
Indeed there are two natural choices, the left and the right adjoint of the functor $f \mapsto F = f(\otimes^d)$.
A representable choice $f \cong \mathrm{Hom}_{\mathfrak{S}_d}(M, -)$ is not always possible, but both adjoints take values among coherent functors (F-Pirashvili, preprint).



Cohomology computations (after Chałupnik)

Chałupnik then proves formulae expressing $\mathrm{Ext}^*_{\mathcal{P}_{dp^r}}(F^{(r)}, G^{(r)})$ in terms of f, g and the graded $\mathfrak{S}_d^{op} \times \mathfrak{S}_d$-representation

$$T_r := \mathrm{Ext}^*_{\mathcal{P}_{dp^r}}(\otimes^{d(r)}, \otimes^{d(r)}) = E_r^{\otimes^d} \otimes \mathbb{K}\mathfrak{S}_d.$$

In many cases, for instance for a projective F, it is given by:

$$\mathrm{Ext}^*_{\mathcal{P}_{dp^r}}(F^{(r)}, G^{(r)}) \cong g \otimes f^{\#}(T_r).$$

As a consequence:

$$\mathrm{Ext}^*_{\mathcal{P}_{dp^r}}(\Gamma^{d(r)}, G^{(r)}) = G(\mathrm{Ext}^*_{\mathcal{P}_{p^r}}(I^{(r)}, I^{(r)})).$$

Cohomology computations (after Chałupnik)

Resolving a general F by projectives, we get from the resulting spectral sequence:

Theorem (F-Pirashvili)

There is a natural graded $\mathbb{K}$-linear functor $e(F,G)$ defined on $\mathfrak{S}_d^{op} \times \mathfrak{S}_d$-modules such that $e(F,G)(T_r)$ is the second page of a spectral sequence converging to $\mathrm{Ext}_{\mathcal{P}}(F^{(r)}, G^{(r)})$, for all polynomial functors F, G of deg d.

Note that collapsing gives a functor-proof of twist injectivity, hence, combined with [Kuhn, 1995, p.286-287], a functor-proof of Ext-injectivity through precomposition

$$\mathrm{Ext}^*_{\mathcal{P}_d}(F,G) \to \mathrm{Ext}^*_{\mathcal{P}_{da}}(F \circ A, G \circ A).$$

Bifunctor cohomology

In van der Kallen's finite generation problem, desirable classes live in

$$\mathrm{H}(\mathrm{GL}_n, \Gamma^d \mathrm{gl}_n).$$

This can be expressed in terms of functor cohomology if one allows two-variable functors. Define: $\mathrm{gl}(V,W) := \mathrm{Hom}(V,W)$.

Proposition (F-Friedlander 2007)

Let $\mathcal{P}_{(1,1)}$ be the category of strict polynomial functors defined on $\mathcal{V}^{op} \times \mathcal{V}$ and let B in $\mathcal{P}_{(1,1)}$ be of bidegree (d,d). For $n \geq d$, there is an isomorphism:

$$\mathrm{Ext}_{\mathcal{P}_{(1,1)}}(\Gamma^d \mathrm{gl}, B) \cong \mathrm{H}(\mathrm{GL}_n, B(\mathbb{K}^n, \mathbb{K}^n)).$$

We denote by $\mathrm{H}(\mathrm{GL}, B)$ this stable cohomology.
The one variable case is recovered by:

$$\mathrm{Ext}_{\mathcal{P}}(F,G) \cong \mathrm{Ext}_{\mathcal{P}_{(1,1)}}(\Gamma^d \mathrm{gl}, \mathrm{gl} \circ (F,G)).$$

Touzé's cohomology computations

Theorem (A. Touzé 2007)

The Poincaré series of $\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, S^{d(r)}\mathrm{gl})$ *is equal to the Poincaré series of the coinvariants of* $T_r = \mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, \otimes^{d(r)}\mathrm{gl})$ *under the action of the symmetric group* $\mathfrak{S}_d$.

Touzé's methods don't give yet an answer for $\mathrm{H}(\mathrm{GL}, \Gamma^{d(r)}\mathrm{gl})$. However, he obtains several relevant results, as the top cohomological degree, or:

Theorem (A. Touzé 2007)

The Euler-Poincaré characteristic of $\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, F\mathrm{gl})$ *and of* $\mathrm{H}^*_{\mathcal{P}}(\mathrm{GL}, F^\sharp\mathrm{gl})$ *are equal.*

Here $F^\#$ is the Kuhn dual of F, for example: $S^{d\#} = \Gamma^d$.