Cohomological finite generation for p-compact groups

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Abstract

Half a century ago, Venkov, Evens et al. proved that a finite group has a finitely generated cohomology algebra. Indeed, Evens states that if a G-module M is Noetherian over a base ring k, then so is $\mathrm{H}^*(G, M)$ over $\mathrm{H}^*(G, k)$. We discuss generalizations, especially for the cohomology of a p-compact group with twisted integral coefficients, which includes the case of a compact Lie group.

Theorem (Evens 1961)

Let G be a finite group, k a ring, and M a $k[G]$-module. If M is noetherian over k, then $\mathrm{H}\ (G, M)$ is noetherian over $\mathrm{H}\ (G, k)$.

This can be seen as a version of Hilbert's 14th problem (for higher modular invariants of finite groups). Recall that a module is *Noetherian* if all of its submodules are finitely generated.

Corollary

Let G be a finite group and k a ring. If k is noetherian then $\mathrm{H}\ (G, k)$ is a finitely generated ring over k.

Lemma

Let R be a graded ring. If R is noetherian, then R^0 is noetherian, and R is finitely generated as a ring over R^0 by homogeneous elements.

The result

Theorem (K.K.S. Andersen, N. Castellana, V. F., A. Jeanneret, J. Scherer)

Let p be a prime and let X be a p-compact group.

1. *The $\mathbb{Z}_p$-algebra $H\ (BX;\mathbb{Z}_p)$ is Noetherian;*
2. *If M is a finite $\mathbb{F}_p[\pi_1 BX]$-module, $H\ (BX;M)$ is Noetherian over $H\ (BX;\mathbb{F}_p)$;*
3. *Let N be a $\mathbb{Z}_p[\pi_1 BX]$-module; If N is finitely generated over $\mathbb{Z}_p$, $H\ (BX;N)$ is Noetherian over $H\ (BX;\mathbb{Z}_p)$.*

A module is Noetherian if all its sub-modules are finitely generated.
I wish that by the end of this talk, you'll find the result unsurprising. So let's follow the yellow brick road, hoping to get back to boring home.

Hilbert's 5th

Hilbert asked how far Lie's concept of continuous groups of transformations is approachable in our investigations without the assumption of the differentiability of the functions.

For homotopists, the minimal input is an **H-space**:
a pointed topological space X endowed with a multiplication : $X \times X \to X$, the base point acting as a unit up to homotopy.

This is already quite demanding: for example, Adams showed that only the spheres of dimension 0, 1, 3, 7 are H-spaces. Among these, the sphere S^7 is not a topological group, but bears a non-associative octonian product.

Loop spaces

There is one feature that homotopists want to keep from a group: it is a loop space.

Definition

The loop space of a pointed topological space Y is the space of based loops $\Omega Y := \mathrm{map}\ (S^1, Y)$

Any topological group can be delooped: if X is a topological group, then it admits a classifying space BX, and X is homotopy equivalent to ΩBX.

Loop spaces are too many

Loop spaces are too rich for a classification: Rector proved in 1970 that our favorite Lie group S^3 has uncountably many deloopings, that is uncountably many homotopy types of Y's such that $S^3 \simeq \Omega Y$.

In 1968, Hilton-Roitberg constructed a loop space, a manifold, but not a product of Lie groups. This example is obtained by mixing different primary information. It played a role in the recognition of the arithmetic nature of homotopy, and in treating homotopy one prime at a time.

Taking one prime at a time

Doing the arithmetic one prime at a time requires a black box extracting the p-information for each prime p.

Take abelian groups first. One knows how to p-complete the integers: write them as usual by p-adic expansion, and allow infinitely many digits (on the left). This is the ring $\mathbb{Z}_p$.
For a general abelian groups A, there are two main options, depending of how you want to define the neighbourhoods of 0.
One is by $p^n A$, giving rise to the p-completion $\lim A/p^n A$.
The other by subgroup of finite p-power index, giving rise to the p-profinite completion, a limit $\lim A_i$ over the finite p-group quotients.
They agree for finitely generated abelian groups.

p-completion of spaces

For spaces, the machinery is the Bousfield-Kan p-completion, a generally frightening piece of 1970's technology. It is an analogue of abelian groups' p-completion.
An alternative is Sullivan's pro-finite completion.

For example, a torus is $(S^1)^n = K(\mathbb{Z}^n, 1)$, while a p-complete torus is $K((\mathbb{Z}_p)^n, 1)$.

One desired property of the p-completion is to transform a map $X \to Y$ inducing a mod p-homology isomorphism into a homotopy equivalence $X_p \to Y_p$, in a conservative way.

Fortunately, p-completion is rather innocuous in our setting. Let me explain.

p-completion of good spaces

For good spaces X, the p-completion of a space X is just a Bousfield localization : it chooses a preferred space X_p among spaces with the same mod p cohomology as X, and it comes with a natural map $X \to X_p$ inducing an isomorphism in mod p homology, which is terminal in some sense.

Good spaces include spaces whose fundamental group is a finite p-group (but a bouquet of two circles is not good).

For a simply connected space of finite type X, X_p keeps the p-torsion of its homotopy, kills other torsion, and replaces each copy of $\mathbb{Z}$ by a copy of the p-adics $\mathbb{Z}_p$. In short, in this case, the homotopy of the p-completion is the p-completion of its homotopy.

p-compact groups

Once a prime has been isolated, things are better behaved : for example, there is only one way to deloop the p-completed 3-sphere $(S^3)_p$ (Dwyer-Miller-Wilkerson 1987).

Definition (Dwyer-Wilkerson, 1994)

A p-compact group is a loop space $X \simeq \Omega BX$ with finite mod p homology such that BX is p-complete

The last condition can be replace by:

- the group of components $\pi_0 X = \pi_1(BX)$ is a finite p-group, and
- the higher homotopy groups are finite type $\mathbb{Z}_p$-modules.

Of course, a p-completed compact Lie group is a p-compact group. Note that p-completing may loose information, e. g.

$$B\mathrm{SO}(2n+1)_p \simeq B\mathrm{Sp}(n)_p \text{ for } p \text{ odd (Friedlander).}$$

A p-completed odd-dimensional sphere $(S^{2n-1})_p$ is an H-space, it is also a p-compact group when n divides $p-1$, $p \geq 3$ (Sullivan)

p-compact groups

The theory Lannes developed in the 1980's for studying homotopy fixed points, was the crucial technology breakthrough.

Eventually, a classification of all p-compact groups was completed (Andersen-Grodal-Møller-Viruel, Annals 2008).

For instance, take $p = 2$ and look for p-compact groups X_n such that $\mathrm{H}\ (BX_n, \mathbb{F}_2)$ is the Dickson algebra $\mathbb{F}_2[x_1, \ldots, x_{n+1}]^{\mathrm{GL}(n,\mathbb{F}2)}$.

$X_1 = \mathbb{Z}/2$, $X_2 = \mathrm{SO}(3)$, $X_3 = G_2$ are compact Lie groups.

The p-compact group X_4 is the only exotic example mod 2 (i.e. it is not the p-completion of a compact Lie group).

Support varieties

In the sequence X_n for $p = 2$, the index n is the 2-rank. Here it means the maximal rank of an abelian elementary p-subgroup $(\mathbb{Z}/2)^n$. It can be detected on the cohomology ring: $\mathrm{H}\ (BX_n, \mathbb{F}_2)$ is the Dickson algebra, a polynomial algebra on n generators.

As the definition suggests, a lot relies on the mod p cohomology $\mathrm{H}\ (BX, \mathbb{F}_p)$. In general, precise explicit information on the cohomology (with coefficients) can be hard to get.

A qualitative approach was initiated by Quillen (Annals 1971). it consists in taking the algebra $\mathrm{H}\ (BX)$ as the coordinate algebra of an algebraic variety (hence called the spectrum of the cohomology ring). The 2-rank of the previous examples corresponds to the usual notion of dimension in algebraic geometry.

This is also a way of getting qualitative results on representations, by adding coefficients in the cohomology.

But as a starting point we need finite generation.

Cohomological finite generation

Not surprisingly, the first theorem in the theory of p-compact groups is:

Theorem (Dwyer-Wilkerson, Annals 1994)

The mod p cohomology $\mathrm{H}^{*}(BX, \mathbb{Z}/p)$ *of a p-compact group X is a finitely generated algebra.*

What about CFG over other rings?
Because the classifying space BX of a p-compact group is p-complete by definition, and $\mathbb{Z}_p$ is not a finitely generated abelian group, only p-complete rings make sense.

We concentrate the rest of the talk on getting CFG over the p-complete ring of p-adic integers $\mathbb{Z}_p$.

Torsion control

To relate CFG mod p and CFG over the integers, the first step is algebraic:

Theorem

*Let Y be a connected space with finite fundamental group.
Then, the graded $\mathbb{Z}_p$-algebra $\mathrm{H}\ (Y;\mathbb{Z}_p)$ is Noetherian if, and only if, the graded $\mathbb{F}_p$-algebra $\mathrm{H}\ (Y;\mathbb{F}_p)$ is Noetherian and the torsion in $\mathrm{H}\ (Y;\mathbb{Z}_p)$ is (uniformly) bounded.*

The condition is necessary, because the torsion is finitely generated as an ideal in the Noetherian algebra $\mathrm{H}\ (Y;\mathbb{Z}_p)$. We'll get back to the converse's proof later (note the condition on π_1).

A p-compact group's cohomology has bounded torsion

We first explain why this applies to our p-compact groups.

A finite group's cohomology has bounded torsion

Let us go back to the case of finite groups. One important feature is the **transfer**: if G is a subgroup of G, summing over classes defines a transfer map in the "other" way: $\mathrm{H}\ (BG\ ) \to \mathrm{H}\ (BG)$. Composed with the "normal" way, the restriction, it is multiplication by the index of G in G on $\mathrm{H}\ (BG)$. This proves that:

- $\mathrm{H}\ (BG)$ is annihilated by the order of G;
- $\mathrm{H}\ (BG, \mathbb{Z}/p)$ embeds in the cohomology $\mathrm{H}\ (BS, \mathbb{Z}/p)$ of a p-Sylow subgroup of G.

A p-compact group's cohomology has bounded torsion

For p-compact groups, there is a weak version of a transfer map. It allows to reduce to the case of a *toral* p-compact group S, an extension of a "torus" $T = \mathbb{Z}_{p^\infty}^n$ by a finite p-group P.

Again, multiplication by the order of P on $\mathrm{H}\ (BS_p\ , \mathbb{Z}_p)$ factors through $\mathrm{H}\ (BT_p\ , \mathbb{Z}_p)$. This has no torsion.

This concludes the geometric part of the argument.

Lifting to the integers

We end the talk with the proof of:

Theorem

Let Y be a connected space with finite fundamental group.
Then, the graded $\mathbb{Z}_p$-algebra $\mathrm{H}^{*}(Y;\mathbb{Z}_p)$ is Noetherian if the graded $\mathbb{F}_p$-algebra $\mathrm{H}^{*}(Y;\mathbb{F}_p)$ is Noetherian and the torsion in $\mathrm{H}^{*}(Y;\mathbb{Z}_p)$ is (uniformly) bounded.

Commutative algebra tells us that to prove $\mathrm{H}^{*}(Y;\mathbb{Z}_p)$ Noetherian, it is enough to prove that

- its mod p reduction $H^{*}(Y;\mathbb{Z}_p)\otimes\mathbb{F}_p$ is Noetherian;
- each $\mathrm{H}^n(Y;\mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p$.

Where the finite fundamental group assumption comes in

Commutative algebra tells us that to prove $\mathrm{H}\ (Y;\mathbb{Z}_p)$ Noetherian, it is enough to prove that

- its mod p reduction $H\ (Y;\mathbb{Z}_p)\otimes\mathbb{F}_p$ is Noetherian;
- each $\mathrm{H}^n(Y;\mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p$.

For the second point, we have indeed:

Proposition

Let Y be a connected space with finite fundamental group.
The group $\mathrm{H}^n(Y;\mathbb{F}_p)$ is finite for every n if and only if the $\mathbb{Z}_p$-module $\mathrm{H}^n(Y;\mathbb{Z}_p)$ is finitely generated for every n.

This was known for simply-connected spaces. In the general case, we consider the universal cover $\tilde{Y}$, with group $\pi_1(Y) =: G$. To apply the simply-connected case, we need to show that $\mathrm{H}^n(\tilde{Y};\mathbb{F}_p)$ is finite. We could only prove that $\mathrm{H}^n(\tilde{Y};\mathbb{F}_p)^G$ is finite. We then use finiteness of G to conclude.

Torsion and integral lift

Commutative algebra tells us that to prove $\mathrm{H}\ (Y;\mathbb{Z}_p)$ Noetherian, it is enough to prove that

- its mod p reduction $H\ (Y;\mathbb{Z}_p)\otimes\mathbb{F}_p$ is Noetherian ;
- each $\mathrm{H}^n(Y;\mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p$.

For the first point, more commutative algebra tells us that it is enough to prove that the Noetherian algebra $\mathrm{H}\ (Y;\mathbb{F}_p)$ is finitely generated over $\mathrm{H}\ (Y;\mathbb{Z}_p)\otimes\mathbb{F}_p$ – the structure is given by the reduction mod p: $\mathrm{H}\ (Y;\mathbb{Z}_p)\otimes\mathbb{F}_p \xrightarrow{\rho} \mathrm{H}\ (Y;\mathbb{F}_p)$.

The universal coefficient exact sequence

$$0 \to \mathrm{H}\ (Y;\mathbb{Z}_p)\otimes\mathbb{F}_p \xrightarrow{\rho} \mathrm{H}\ (Y;\mathbb{F}_p) \to \mathrm{Tor}(\mathrm{H}\ ^{+1}(Y;\mathbb{Z}_p);\mathbb{Z}/p) \to 0$$

shows that torsion is an obstruction to an integral lift of mod p classes. What can be said when there is torsion?

Torsion and power lift

The following is directly inspired by [Benson-Habegger, 1987] for the cohomology of a finite group.

Proposition

Let Y be a connected space and let d be an integer.
Suppose that the p-torsion in $\mathrm{H}\ (Y;\mathbb{Z}_p)$ is bounded by p^d. Then, for any class u in $\mathrm{H}\ (Y;\mathbb{F}_p)$, the class u^{p^d} belongs to the image of $\rho : H\ (Y;\mathbb{Z}_p)\otimes\mathbb{F}_p \to H\ (Y;\mathbb{F}_p)$.

We say that ρ is *power-surjective*. Power surjective is stronger than integral, it implies Noetherian.

Torsion and power lift

Proof.

The connecting morphism in the CLES for the SES of coefficients:

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^{k+1} \to \mathbb{Z}/p^k \to 0$$

is called the Bockstein. Use that it is a derivation – over the ring $\mathrm{H}\ (Y; \mathbb{F}_p)$ – to deduce that any p-th power u^p lifts from $\mathrm{H}\ (Y; \mathbb{Z}/p^k)$ to $\mathrm{H}\ (Y; \mathbb{Z}/p^{k+1})$. By induction, any u^{p^d} lifts to $\mathrm{H}\ (Y; \mathbb{Z}/p^{d+1})$.

In the commutative diagram of UCS

$$0 \to \mathrm{H}\ (\mathbb{Z}_p) \otimes \mathbb{Z}/p^{d+1} \to \mathrm{H}\ (\mathbb{Z}/p^{d+1}) \to \mathrm{Tor}(\mathrm{H}\ ^{+1}(\mathbb{Z}_p); \mathbb{Z}/p^{d+1})$$

$$\downarrow \qquad\qquad \downarrow$$