Higher invariant theory and power surjectivity

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Abstract

A classic problem in invariant theory, often referred to as Hilbert's 14th problem, asks, when a group acts on a finitely generated commutative algebra by algebra automorphisms, whether the ring of invariants is still finitely generated. The question extends to finite generation of the cohomology algebra of a group. We'll present recent progress and generalizations with a stress on the case of a general base ring.

Hilbert's 14th problem

Theorem (Hilbert 1890)

Let G be a subgroup of GL_n. Suppose that the representation of G on $\mathbb{K}[x_1, \ldots, x_n]$ is completely reducible. Then the ring of invariants is a finitely generated $\mathbb{K}$-algebra.

This led Hilbert to the formulation of his 14th problem, which asks, in particular, if finite generation holds in general.
In 1958, Nagata constructs a counter-example to Hilbert's 14th problem.
He then refines it to an algebraic group action. A new question arises:

Problem (Finite generation of invariants)

Characterize groups G such that, whenever G acts on a finitely generated commutative $\mathbb{K}$-algebra A, the ring of invariants A^G is a finitely generated $\mathbb{K}$-algebra as well.

Reductivity

Hilbert's argument works when *G-stable hyperplanes are supplemented by a G-stable line.*

This can be restated (in a form suitable for all rings $\mathbb{K}$) as lifting invariant:

Property (linear reductivity)

Let L be a cyclic $\mathbb{K}$-module with trivial G-action. Let M be a G-module, and let φ be a G-map from M onto L.
The restriction to invariants is still a surjection: $M^G \to L^G = L$.

In characteristic 0, linear reductivity is satisfied for finite groups, and for any semi-simple group (hence for GL_n).

In positive characteristic, it is rarely satisfied.

Lifting modular invariants: a misleading example

Let the additive group act on $\mathbb{Z}[\mathfrak{sl}_2] = \mathbb{Z}[X, H, Y]$
through conjugation by unipotent matrices $u(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$,
sending

$$X, \qquad H, \qquad Y$$

respectively to

$$X + aH - a^2 Y, \qquad H - 2aY, \qquad Y.$$

The mod 2 invariant H does not lift to an integral invariant,
but $H^2 + 4XY$ is an integral invariant, and it reduces to H^2.

Note that the action extends to the adjoint action of SL_2,
with $H^2 + 4XY$ still an invariant.

Reductivity and finite generation over fields

In 1964, Nagata shows that groups G having a certain property satisfy finite generation of invariants.
Indeed, the proof only uses a lift of invariants *up to a power* (cf. Springer's Springer LN). So Let me coin a powerful word:

Definition

A morphism of K-algebras: $S \to R$ is *power-surjective* if for every element in R some power lies in its image.

Property

Taking invariants under G preserves power surjectivity.

When $\mathbb{K}$ is a field, Seshadri calls the property *geometric reductivity*.
For a general ring, he restricts it to finitely generated projectives modules.
We call the better notion *power reductivity*.

Power reductivity and finite generation

Property (Power reductivity)

Taking invariants under G preserves power surjectivity.

This can be rephrased much like in the introduction of Mumford's *Geometric Invariant Theory*:

Property (Power reductivity)

Let L be a cyclic $\mathbb{K}$-module with trivial G-action. Let M be a G-module, and let φ be a G-map from M onto L.
There is a positive integer d such that the d-th symmetric power of the map φ induces a surjection: $(S^d M)^G \to S^d L$.

Proposition (Nagata's Hilbert's fourteenth)

Let G act (by algebra automorphisms) on a finitely generated commutative algebra over a Noetherian ring $\mathbb{K}$. If G is power-reductive, then the $\mathbb{K}$-algebra of invariants A^G is finitely generated as well.

Power reductivity and lifting invariants

We have already seen an example of such a lift, illustrating power reductivity of the group SL_2. More generally:

Proposition (power lifting of invariants)

Let a power-reductive group act on a K-algebra A and an ideal J. Any invariant in A/J has a power which lifts to an invariant in A.

Indeed, this property for every algebra A and invariant ideal J is *equivalent* to power-reductivity. One can even ask the weaker condition that $(A/J)^G$ is integral over the image of A^G.
Indeed, power reductivity is equivalent to finite generation over a Noetherian base ring.

Power reductivity is preserved by base change, and has descent (for faithfully flat $\mathrm{K} \to R$).

Power reductivity: example

Consider again the additive group of 2×2 upper triangular matrices with diagonal 1: this is just an additive group.
Let it act on M with basis $\{x, y\}$ by linear substitutions:
$u(a)$ sends x, y respectively to x, $ax + y$.

Sending x to 0 defines $M \to L$;
since $(S^*M)^U = \mathrm{K}[x]$ is sent to $\mathrm{K} \subset S^*(L) = \mathrm{K}[y]$, power reductivity fails.

Note, though here S^*M^U still is finitely generated, that Nagata's counter-example is for a suitable subgroup of a product of additive groups.

Power reductivity: example

Consider the group SL_2 acting on 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by conjugation.
Let L be the line of homotheties in $M := \mathfrak{gl}_2(\mathbb{Z})$.
The restriction: $M^\# \to L^\#$ extends to

$$\mathbb{Z}[M] = \mathbb{Z}[a, b, c, d] \to \mathbb{Z}[\lambda] = \mathbb{Z}[L].$$

Take SL_2-invariants:

$$\mathbb{Z}[a, b, c, d]^{\mathrm{SL}_2} = \mathbb{Z}[t, D] \to \mathbb{Z}[\lambda].$$

The trace $t = a + c$ is sent to 2λ, so λ does not lift to an invariant.
The determinant $D = ad - bc$ is sent to λ^2,
illustrating power reductivity of SL_2.

Mumford's conjecture

It would thus be nice to know if your favorite group is power-reductive. Indeed, D. Mumford conjectured in the introduction of his book *Geometric Invariant Theory* that it is so:

Theorem (Habousch 1975)

Every Chevalley group is power-reductive over a field.

Chevalley group = a subgroup of GL_n governed by an admissible lattice
= connected split reductive algebraic group, defined over $\mathbb{Z}$.

Mumford conjecture over an arbitrary base

Theorem (F.-van der Kallen)

Every Chevalley group is power-reductive for every base $\mathbb{K}$.

Proof.

Go local. Lift the Steinberg modules so we can follow our favorite modular proof [CPSvdK, Inventiones 1977].
It uses cohomology and Kempf's vanishing theorem.
Conclude with the Nakayama lemma and the universal coefficient theorem. □

Higher invariant theory — the finite generation problem in cohomology

The same question has been asked for the cohomology ring $\mathrm{H}^*(G, A)$.

This is the cohomology defined by Hochschild, *aka* group cohomology for a finite group, rational cohomology for an algebraic group.
The algebra structure is given by the cup-product.

Problem (van der Kallen)

Let K be a Noetherian ring and let G be an affine algebraic group scheme over K.
Let A be a finitely generated commutative K-algebra on which G acts through algebra automorphisms.
Is the cohomology ring $\mathrm{H}^*(G, A)$ a finitely generated K-algebra?

Cohomological finite generation or CFG is this property for every A.

Cohomological finite generation for fields

Cohomological finite generation has been proved in its generality when

- G is a finite group, for all rings K [Evens, 1961]
  if M is Noetherian over a ring K, then so is $\mathrm{H}^*(G, M)$ over $\mathrm{H}^*(G, \mathrm{K})$.
  this is a equivalent to CFG (take the symmetric algebra S^*M for A).
- $\mathbb{K}[G]$ is finite, $\mathbb{K}$ a field [Friedlander & Suslin, 1997]
  This is the infinitesimal case. Indeed, $\mathrm{H}^*(G, A)$ is Noetherian over an explicit algebra.
- G is power-reductive, $\mathbb{K}$ a field [Touzé & van der Kallen, 2010].
  This was van der Kallen's conjecture.
  The upshot is that, over a field, CFG is equivalent to FG. But easier:
  $\mathrm{H}^1(\mathbb{G}_a, \mathbb{F}_p)$ is not finite dimentional.

Functor cohomology for universal classes

Proofs of the last two results reduce to the general linear group GL_n and use spectral sequences *à la* Evens.
What is needed is enough permanent classes to generate a subalgebra which the first page E_1 is a noetherian module over.
Then, the spectral sequence will stop, and finite generation follows.

Friedlander and Suslin use functor techniques to produce universal classes.

Advertising

They introduce a category $\mathcal{P}$ of polynomial functors between vector spaces (their structure map is a polynomial) which enjoys many nice properties allowing cohomological computations.
It still computes rational cohomology, indeed it is equivalent to the category of modules over the Schur algebra. For instance:

$$\mathrm{H}^*(\mathrm{GL}_n, \mathfrak{gl}_n^{(r)}) \cong \mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)}), \; n \geq p^r.$$

Here, $I^{(r)}$ associates to a $\mathbb{K}$-vector space V, the $\mathbb{K}$-vector space $V^{(r)}$ obtained by extension of scalars through iterated Frobenius.

Functor cohomology for universal classes

Theorem

$\mathrm{Ext}_{\mathcal{P}}(I^{(r)}, I^{(r)})$ *is a divided power algebra on a generator* e_1 *in degree* 2*, truncated at height* r*.*

The class e_1 is the class of the extension:

$$0 \to Id^{(1)} \to S^p \to \Gamma^p \to Id^{(1)} \to 0.$$

It corresponds to the class c_1 in $\mathrm{H}^2(\mathrm{GL}, \mathfrak{gl}^{(1)})$ represented by the Witt vectors extension of algebraic groups:

$$0 \to \mathfrak{gl}^{(1)} \to \mathrm{GL}(W_2) \to \mathrm{GL} \to 1.$$

Bifunctor cohomology

In van der Kallen's finite generation problem, more classes are needed in

$$\mathrm{H}^*(\mathrm{GL}_n, \Gamma^d \mathfrak{gl}_n^{(r)}).$$

This can be expressed in terms of functor cohomology if one allows two-variable functors, such as: $gl(V, W) := \mathrm{Hom}_{\mathbb{K}}(V, W)$.

Proposition (F-Friedlander 2007)

Let $\mathcal{P}_{(1,1)}$ be the category of strict polynomial bifunctors and let B in $\mathcal{P}_{(1,1)}$ be of bidegree (d, d). For $n \geq d$, there is an isomorphism:

$$\mathrm{Ext}_{\mathcal{P}_{(1,1)}}(\Gamma^d gl, B) \cong \mathrm{H}(\mathrm{GL}_n, B(\mathbb{K}^n, \mathbb{K}^n)).$$

We denote by $\mathrm{H}(\mathrm{GL}, B)$ this stable cohomology.

Touzé's universal classes

To construct the desirable classes in $\mathrm{H}^*(\mathrm{GL}, \Gamma^d gl^{(1)})$, Touzé uses Troesch's resolutions of twists. One difficulty is their lack of naturality, and it forces Touzé to perform a tour de force.
Touzé now has a new proof. He uses adjoints of precomposition in the derived category (this is due to Chałupnik) to take care of naturality.
So from Touzé treatment of Troesch's complexes, only a formality result for maps between twists is needed.
Just ask him when he comes and visit.
This new understanding of precomposition (or plethism) seems full of potential.

van der Kallen's conjecture over an arbitrary base

Other steps of the proof seem to generalize to an arbitrary base.
This includes a clever argument using resolution of the diagonal [van der Kallen 2011] (based on the field case with Srinivas).
This proves CFG in finite characteristic, and more.

Theorem (van der Kallen 2011; F-vdK for $\mathbb{Z}$)

Let G be a Chevalley group over a Noetherian ring K.
Let A be a finitely generated commutative algebra on which G acts through algebra automorphisms.
Let M be a Noetherian A-module with compatible G-action.

1. *For each integer n, $\mathrm{H}^n(G, M)$ is a noetherian A^G-module.*
2. *If $\mathrm{H}^n(G, A)$ is a finitely generated K-algebra, then $\mathrm{H}^n(G, M)$ is a Noetherian $\mathrm{H}^n(G, A)$-module*
3. *The K-algebra $\mathrm{H}^*(G, A)$ is finitely generated if, and only if, it has bounded torsion.*

van der Kallen's conjecture over an arbitrary base

Thus one asks:

Problem

Let K be a Noetherian ring and let G be a Chevalley group over K.
Let A be a finitely generated K-algebra on which G acts rationally through algebra automorphisms.
Is the cohomology ring $\mathrm{H}^*(G, A)$ a finitely generated K-algebra?