THE TOPOLOGICAL IHX RELATION, PURE BRAIDS, AND THE TORELLI GROUP

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Abstract. We prove that the filtration on the pure braid group on g strands, induced by the lower central series of the Torelli group of a genus g surface with one boundary component, coincides with its lower central series, shifted by one.

In particular, the cubic Jacobi relations in the pure braid group are quadratic relations in the Torelli group.

1. Introduction and Statement of Results

The study of the Torelli subgroup of the mapping class group has progressed in recent years. Beginning with the work of Johnson [J1, J2, J3] in the early 80's, who computed its abelianization and gave a finite set of generators, and continuing with work of Morita [Mo1] on the Casson invariant and higher Johnson homomorphisms, the work of Garoufalidis and Levine [GL] has tied the Torelli group and Johnson homomorphisms to the rich theory of perturbative (or finite type) 3-manifold invariants [O] [Le] [LMO] [G] [Ha]. From another direction, a major leap forward in our understanding is the work of Hain [Hn], who gave a presentation of the Malcev Lie algebra of the Torelli group.

At the group level, a fundamental open problem is to determine if the Torelli group is finitely presentable or not. Indeed, to our knowledge, there does not currently exist *any* presentation of the Torelli group with explicit sets of generators and relators in genus ≥ 3.[1] We still lack an undersanding of the topological nature of relations within the Torelli group.[2]

In this paper we exhibit a relation, known as the IHX relation in perturbative theory, which exists in the Torelli group of a surface of genus $g \geq 3$. This relation seems to have appeared at least implicitly in calculations of Morita [Mo1].

To see how this relation comes about, recall from [H] that the topological IHX relation in the theory of homology spheres can be considered as arizing from the Jacobi relation in the pure braid group on 4 strands. We denote by $\Sigma_{g,r}$ a surface of genus g with r boundary components, and by $M_{g,r}$ its mapping class group. Note that $M_{0,g+1}$ is the framed pure braid group on g strands.

There is an embedding of $\Sigma_{0,g+1}$ in $\Sigma_{g,1}$ (given by considering $\Sigma_{g,0}$ to be the double of $\Sigma_{0,g+1}$ and removing a disk). Then it turns out that the cubic Jacobi relations in $M_{0,g+1}$ induce quadratic relations in the Torelli group $T_{g,1}$.

Before stating our main result, recall that $M_{g,1}$ comes equipped with a filtration

$$\cdots \subset M_{g,1}[2] \subset M_{g,1}[1] = T_{g,1} \subset M_{g,1}[0] = M_{g,1},$$

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[1] In genus 2, the Torelli group is free [M].

[2] There is the obvious commutativity of diffeomorphisms with disjoint support.

where $M_{g,1}[n]$ is the subgroup of $M_{g,1}$ acting trivially on F/F_{n+1}, where $F = \pi_1(\Sigma_{g,1})$, and for a group G, G_n denotes its lower central series.[3]

We denote by $P(g)$ the pure braid group on g strands.

Theorem 1.1. *Let $P(g) \to M_{g,1}$ denote the standard inclusion, (defined above, see also section 2). Then for all $n \geq 1$,*

$$P(g)_{n+1} = P(g) \cap (T_{g,1})_n = P(g) \cap M_{g,1}[n].$$

Let H denote F/F_2 and let $L(H) = \bigoplus_{n=1}^{\infty} L_n(H)$ denote the free Lie algebra on the $\mathbb{Z}$-module H. Define $D_n(H)$ to be the kernel of the Lie bracket map

$$[\ ,\ ]\colon H \otimes L_n(H) \to L_{n+1}(H).$$

The degree n Johnson homomorphism (see e.g., [GL]) is a map

$$J_n\colon M_{g,1}[n] \to D_{n+1}(H)$$

whose kernel is $M_{g,1}[n+1]$. One has that $(T_{g,1})_n \subset M_{g,1}[n]$, and it is an open problem to study the map

$$j'_n\colon (T_{g,1})_n/(T_{g,1})_{n+1} \to D_{n+1}(H).$$

Corollary 1.2. *j'_n is injective on the image of $P(g)_{n+1} = P(g) \cap (T_{g,1})_n$ in $(T_{g,1})_n/(T_{g,1})_{n+1}$.*

Recall that in any group, one has the Jacobi identity

$$[a^c, [b, c]][c^b, [a, b]][b^a, [c, a]] = 1,$$

where x^y denotes yxy^{-1} and $[x, y]$ denotes $xyx^{-1}y^{-1}$.

Corollary 1.3. *Every Jacobi relation in the pure braid group is a quadratic relation in the Torelli group.*

Remark. Theorem 1.1 (and hence corollary 1.3) holds also for a surface $\Sigma = \Sigma_{g,0}$ without boundary, where $M_g[n]$ is defined to be the image of $M_{g,1}[n]$ under the natural map $M_{g,1} \to M_g = M_{g,0}$. Note first that $P(g) \subset M_{0,g+1} \to M_g$ is an inclusion (see e.g., [PR]). Thus, we have

$$P(g)_{n+1} = P(g) \cap (T_{g,1})_n = P(g) \cap M_g[n].$$

To get an equivalent statement of corollary 1.2 in the closed case, we have just to replace $D_n(H)$ by a suitable quotient of $D'_n(H)$, the kernel of the Lie bracket map

$$[\ ,\ ]\colon H \otimes \mathcal{L}_n(g) \to \mathcal{L}_{n+1}(g),$$

where $\mathcal{L}_n(g) = \pi_1(\Sigma_g)_n/\pi_1(\Sigma_g)_{n+1}$ (see [Mo3]).

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[3]The lower central series of a group G is defined by $G_1 = G$, and inductively, $G_{k+1} = [G, G_k]$ is the subgroup of G_k generated by the commutators of G and G_k.

2. The Standard Embedding of $\Sigma_{0,g+1}$ in $\Sigma_{g,1}$

There are a number of models available for a surface $\Sigma_{g,1}$, but one which we find convenient for drawing pictures is the description as a 2-dimensional handlebody with a single index 0-handle and $2g$ orientation preserving index 1-handles grouped together in interlaced pairs.

Let B denote a 2-disk (considered as the 0-handle) such that its boundary ∂B is divided into two segments $\partial_+ B$, $\partial_- B$. We choose a point of $\partial_- B$ as basepoint.

Consider a collection $p_1, \dots, p_{4g}$ of points in $\partial_+ B$ ordered using an orientation of $\partial_+ B$. We attach orientation preserving 1-handles, H_i^α (resp. H_i^β) along neighborhoods of the 0-spheres $\{p_{4i-3}, p_{4i-1}\}$ (resp. $\{p_{4i-2}, p_{4i}\}$) in $\partial_+ B$. The handlebody $B \cup (\cup_{i=1}^g (H_i^\alpha \cup H_i^\beta))$ is a surface $\Sigma_{g,1}$ of genus g with 1 boundary component.

The core of the handle H_i^α (resp. H_i^β) is an arc A_i^α (resp. A_i^β) whose boundary is the 0-sphere $\{p_{4i-3}, p_{4i-1}\}$ (resp. $\{p_{4i-2}, p_{4i}\}$) which we orient so as to go from p_{4i-3} to p_{4i-1} (resp. p_{4i} to p_{4i-2}).

We join the endpoints of the arc A_i^α (resp. A_i^β) by an arc whose interior lies in the interior of B. This produces a simple closed curve C_i^α (resp. C_i^β) which we orient so as to induce the chosen orientation on A_i^α (resp. A_i^β). These curves can be chosen to be mutually disjoint, except that C_i^α and C_i^β meet transversely in a single point (see figure 1).

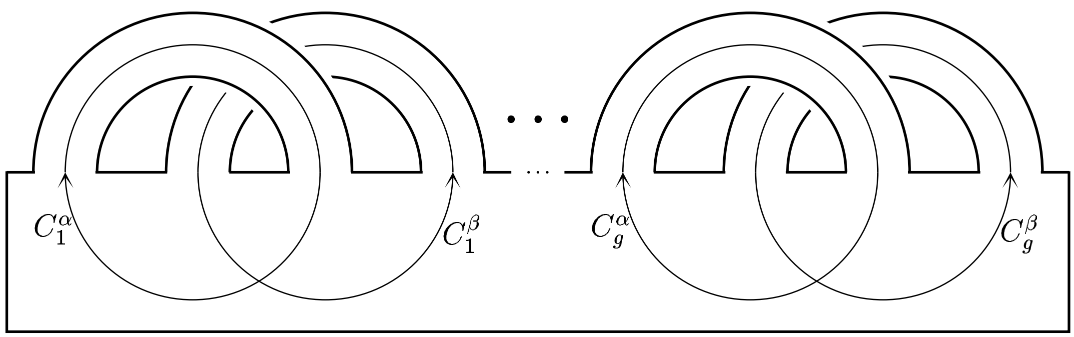


Figure 1. The curves C_i^α and C_i^β

We choose paths lying in B to connect the curves C_i^α (resp. C_i^β) to the base point. This yields well-defined elements of $\pi_1(\Sigma_{g,1})$, which will be denoted by α_i (resp. β_i) (see figure 2). The set $\{\alpha_i, \beta_i | i = 1, \dots, g\}$ is a basis of the free group $F = F(2g) = \pi_1(\Sigma_{g,1})$. With these conventions $\partial(\Sigma_{g,1})$ is represented by the element $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$.

The subsurface of $\Sigma_{g,1}$ consisting of $B \cup (\cup_{i=1}^g H_i^\alpha)$ is a surface $\Sigma_{0,g+1}$ of genus 0 with $g+1$ boundary components. If we remove a small collar neighborhood of $\partial\Sigma_{0,g+1} \setminus \partial_- B$, the resulting surface will still be denoted $\Sigma_{0,g+1}$. The closure of its complement in $\Sigma_{g,1}$ is now a connected surface, which can be made planar. It has $g+1$ boundary components, and hence is homeomorphic to $\Sigma_{0,g+1}$. $\Sigma_{g,1}$ is just the double of $\Sigma_{0,g+1}$, slit open along $\partial_- B$ to obtain a boundary component. We call this embedding of $\Sigma_{0,g+1}$ in $\Sigma_{g,1}$ the standard embedding. It was studied by Oda [Od] and Levine [L] (see also Hatcher-Thurston for an embedding of $\Sigma_{0,2g}$ in $\Sigma_{g,0}$). $\pi_1(\Sigma_{0,g+1})$ is the free group $F(g)$ on the set $\{\alpha_i | i = 1, \dots, g\}$, $\pi_1(\Sigma_{0,g+1}) = F(g) \subset F(2g) = \pi_1(\Sigma_{g,1})$.

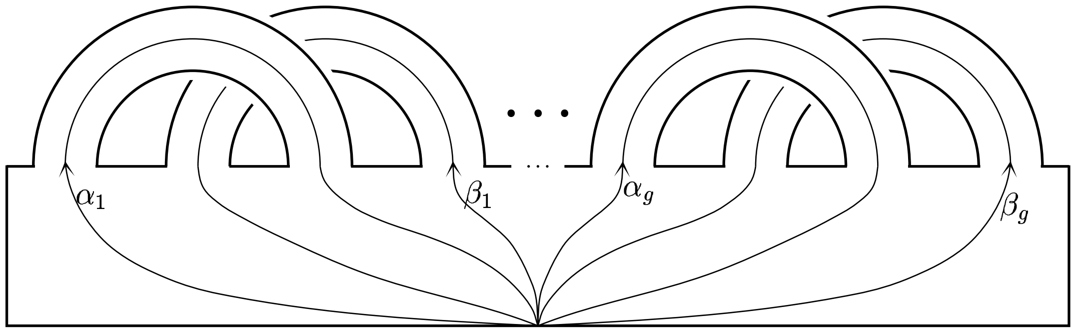


FIGURE 2. The curves α_i and β_i

We define simple closed curves $C_{i,j}$ in $\Sigma_{0,g+1}$ by taking the connected sum of C_i^α and C_j^α along an arc lying in B (see figure 3). Let τ_i denote the Dehn twist along C_i^α and let $\tau_{i,j}$ be the Dehn twist along $C_{i,j}$. The elements $\sigma_{i,j} = \tau_{i,j}\tau_i^{-1}\tau_j^{-1}$ are the generators of the pure braid group on g strands, $P(g)$, considered as a subgroup of $M_{0,g+1} = P(g) \times \mathbb{Z}^g$. As the τ_i are central (generating the $\mathbb{Z}^g$ summand of $M_{0,g+1}$), when writing commutators in the $\sigma_{i,j}$, we may dispense with the framing corrections τ_i^{-1}, τ_j^{-1} and simply write $\tau_{i,j}$.

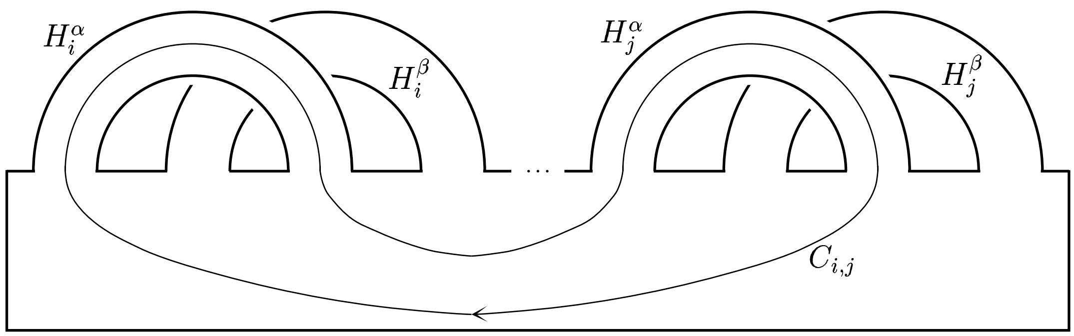


FIGURE 3. The curve $C_{i,j}$

We have that the curves $C_{i,j}$ (and thus the twists $\tau_{i,j}$) are all conjugate in the following sense: there is a diffeomorphism $\tilde{F}$ of $\Sigma_{g,1}$ which is the identity on $\partial\Sigma_{g,1}$, and which restricts to a diffeomorphism F of $\Sigma_{0,g+1}$ to itself and sends $C_{i,j}$ to $C_{1,2}$.[4] Such an F exists since cutting $\Sigma_{0,g+1}$ open along $C_{i,j}$ results in 2 surfaces, one of which is $\Sigma_{0,3}$ (and hence the other is $\Sigma_{0,g}$). F can be extended to $\Sigma_{g,1}$ simply be taking the double of F, slit open along $\partial_- B$.

3. The Main Construction

Consider a simple closed curve $D_{1,2}$ lying in $\Sigma_{2,1}$ whose homotopy class is the element $\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2$. The curve $D_{1,2}$ may be taken to intersect C_1^α and C_2^α transversely in one point. These two intersection points divide $D_{1,2}$ into two subarcs, one of which may be taken to lie entirely in B and will be called A. The other A' can be taken to miss a third arc A_0

[4] F necessarily permutes the components of $\partial\Sigma_{0,g+1}$. It thus corresponds to an element of the braid group and not the pure braid group.

connecting A to the base point in B. Note that a small neighborhood of $A_0 \cup A \cup C_1^\alpha \cup C_2^\alpha$ is a surface $\Sigma_{0,3} \subset \Sigma_{2,1}$, isotopic to the standard embedding.

The connected sum of C_1^α and C_2^α along A is the curve $C_{1,2}$. Denote by $C'_{1,2}$ the connected sum of C_1^α and C_2^α along A' (see figure 4). By pushing $A' \cup C_1^\alpha \cup C_2^\alpha$ into the complement of $A_0 \cup A \cup C_1^\alpha \cup C_2^\alpha$, i.e. into $\Sigma_{2,1} \setminus \Sigma_{0,3} \subset \Sigma_{g,1} \setminus \Sigma_{0,g+1}$, we see that the Dehn twist $\tau'_{1,2}$ along $C'_{1,2}$ commutes with elements of $M_{0,g+1}$.

Since $C_{1,2}$ and $C'_{1,2}$ are homologous, $\tau_{1,2}(\tau'_{1,2})^{-1}$ lies in $T_{g,1}$.

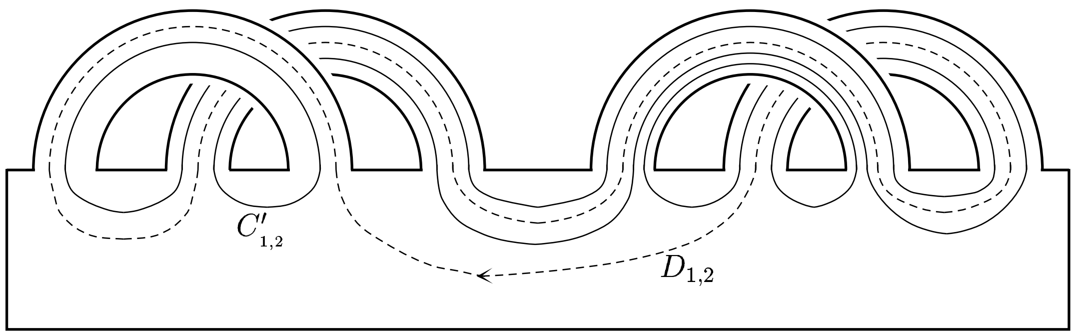


FIGURE 4. The curves $D_{1,2}$ and $C'_{1,2}$

4. Proof of the inclusion $P(g)_{n+1} \subset (T_{g,1})_n$.

Recall that we have

$$P(g) \subset M_{0,g+1} \subset M_{g,1}.$$

Set $P = P(g)$, $P_n = P(g)_n$, $T = T_{g,1}$, $T_n = (T_{g,1})_n$. Since $P \cap T_n$ is a normal subgroup of P, it suffices to see that a normal set of generators of P_{n+1} lie in T_n. For $n \geq 1$, P_{n+1} is normally generated by $[\sigma_{i,j}, f] = [\tau_{i,j}, f]$, $f \in P_n$. $\tau_{i,j}^f$ is the Dehn twist along the image of $c_{i,j}$ by the diffeomorphism f (or rather a representative of its class).

Since $f(c_{i,j})$ is homologous to $c_{i,j}$, $[\tau_{i,j}, f] = \tau_{i,j}(\tau_{i,j}^f)^{-1}$ lies in T. This proves that $P_2 \subset T$. Suppose inductively we have that P_n lies in T_{n-1}.

We first suppose $\{i,j\} = \{1,2\}$. Since $\tau'_{1,2}$ commutes with $M_{0,g+1}$, we have $[\tau_{1,2}, f] = [\tau_{1,2}(\tau'_{1,2})^{-1}, f]$. But as $\tau_{1,2}(\tau'_{1,2})^{-1} \in T$ and $f \in P_n \subset T_{n-1}$, we have $[\tau_{1,2}, f] \in T_n$.

Now let $\{i,j\} \neq \{1,2\}$. Let $(\tilde{F}, F)$ be a diffeomorphism (c.f. section 2) of the pair $(\Sigma_{g,1}, \Sigma_{0,g+1})$ sending $c_{i,j}$ to $c_{1,2}$. Then $[\tau_{i,j}, f]^F = [\tau_{1,2}, f^F] \in T_n$, since f^F lies in P_n, because conjugation by F sends P_n to itself.[5] It follows that $[\tau_{i,j}, f] \in T_n$, since T_n is normal in $M_{g,1}$.

5. Proof of Theorem 1.1

We have seen that for all $n \geq 1$,

$$P(g)_{n+1} \subset (T_{g,1})_n.$$

Since $(T_{g,1})_n \subset M_{g,1}[n]$, we have that

$$P(g)_{n+1} \subset (T_{g,1})_n \cap P(g) \subset M_{g,1}[n] \cap P(g).$$

[5] N.b., $f \mapsto f^F$ is <u>not</u> an inner automorphism of $P(g)$.

To prove equality, it is therefore enough to show that

$$M[n] \cap P \subset P_{n+1}$$

where $P = P(g)$ and $M[n] = M_{g,1}[n]$.

This was shown by Oda [Od] (see also Levine [L]). For the convenience of the reader, we give the details of the argument, as in [L].

For $n = 1$, the map $\frac{P}{P_2} \to \frac{M[0]}{M[1]} = Sp(2g)$ sends the elements $\sigma_{i,j}$ to the matrix $\begin{pmatrix} I & E_{i,j} + E_{j,i} \\ O & I \end{pmatrix}$ in the basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$, (or rather their homology classes). Since the $\sigma_{i,j}$ are a basis of $\frac{P}{P_2}$ and the $E_{i,j} + E_{j,i}$ are linearly independent, it follows that $\frac{P}{P_2}$ injects into $Sp(2g)$. Thus, we get $M[1] \cap P \subset P_2$.

Let $D_n(H) = \ker([\ ,\ ]\colon H \otimes L_n(H) \to L_{n+1}(H))$, where $H = H_1(\Sigma_{g,1})$ and where $L(H) = \bigoplus_{n=1}^{\infty} L_n(H)$ denotes the free Lie algebra on the $\mathbb{Z}$-module H. The degree n Johnson homomorphism (see e.g. [GL]) is a map $J_n : M[n] \to D_{n+1}(H)$ which has kernel $M[n+1]$. On the other hand, setting $H' = H_1(\Sigma_{0,g+1})$, the degree n Milnor invariants give a map $\mu_n : P_n \to D_n(H')$ whose kernel is P_{n+1} (see e.g. [HL, HM]). Thus the inclusion $M[n] \cap P \subset P_{n+1}$ follows from the following by induction.

Claim: Denote by $p_n : D_n(H) \to D_n(H')$ the morphism induced by the map $H \to H'$ obtained by forgetting the β_i's. Then, the diagram

$$\begin{array}{ccc} P_n & \hookrightarrow & M[n-1] \\ \downarrow \mu_n & & \downarrow J_{n-1} \\ D_n(H') & \xleftarrow{p_n} & D_n(H) \end{array}$$

commutes up to sign.

We recall briefly the definitions of the maps μ_n, J_n. Recall that if F is a free group and $H = \frac{F}{F_2}$, then $\frac{F_n}{F_{n+1}} \simeq L_n(H)$. Recall that a string link σ induces an automorphism A_σ of $\frac{F(g)}{F(g)_{n+2}}$. If we let ℓ_i denote the longitude in $\pi = \pi_1(D^2 \times I \setminus \sigma)$, then $A_\sigma(x_i) = x_i^{\lambda_i}$ where the x_i's are the free generators of π and λ_i denotes the image of ℓ_i in $\frac{\pi}{\pi_{n+1}} \approx \frac{F(g)}{F(g)_{n+1}}$. Thus if the λ_i lie in $\frac{F(g)_n}{F(g)_{n+1}} = L_n(H')$, we define $\mu_n(\sigma) = \sum_i x_i \otimes \lambda_i \in H' \otimes L_n(H')$. Actually, $\mu_n(\sigma)$ lies in $D_n(H')$.

Similarly if $f \in M_{g,1}[n]$, then the induced automorphism f_* of $\frac{F(2g)}{F(2g)_{n+2}}$ takes α_i to $\alpha_i \eta_i^\alpha$ and β_i to $\beta_i \eta_i^\beta$, where $\eta_i^\alpha, \eta_i^\beta \in \frac{F(2g)_{n+1}}{F(2g)_{n+2}} = L_{n+1}(H)$, and we set $J_n(f) = \sum_i \alpha_i \otimes \eta_i^\beta - \beta_i \otimes \eta_i^\alpha$. Actually, $J_n(f)$ lies in $D_{n+1}(H)$.

For $\sigma \in P$, let us denote by f_σ the corresponding element in $M_{g,1}$. If $C(f_\sigma)$ is the mapping cylinder of f_σ, then $C(f_\sigma)$ is an handlebody of genus $2g$ (see figure 5). The boundary of $C(f_\sigma)$ can be decomposed in two pieces: $\partial(C(f_\sigma)) = \Sigma_{g,1}^+ \cup \Sigma_{g,1}^-$. We denote by $i^\pm$ the inclusions of $\Sigma_{g,1}^\pm$ in $C(f_\sigma)$ and we identify $\Sigma_{g,1}^+$ with $\Sigma_{g,1}$, and $F(2g)$ with $\pi_1(C(f_\sigma))$ via i_*^+. Then, by definition, we have

$$\eta_i^\alpha = i_*^+(\alpha_i)^{-1} i_*^-(\alpha_i) \quad \text{and} \quad \eta_i^\beta = i_*^+(\beta_i)^{-1} i_*^-(\beta_i).$$

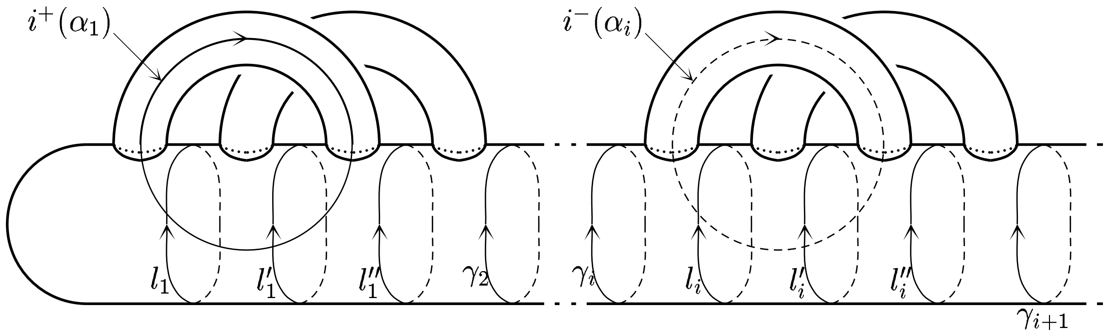


FIGURE 5. The mapping cylinder of f_σ and the curves l_i, l_i', l_i'', γ_i

Now, if l_i, l_i', l_i'' and γ_i are the curves described in figure 5, then one has (after connecting these curves to the basepoint),

$$i^-(\alpha_i) = l_i^{-1} i^+(\alpha_i) l_i' \quad \text{and} \quad i^-(\beta_i) = (l_i'')^{-1} i^+(\beta_i) l_i'$$

and we get

$$\eta_i^\alpha = [\alpha_i^{-1}, l_i^{-1}] l_i^{-1} l_i' \quad \text{and} \quad \eta_i^\beta = [\beta_i^{-1}, (l_i'')^{-1}](l_i'')^{-1} l_i'.$$

But one has $(l_i'')^{-1} l_i' = i^-(\alpha_i)\gamma_i^{-1} l_i i^-(\alpha_i)^{-1}$, thus we have

$$\eta_i^\beta = [\beta_i^{-1}, (l_i'')^{-1}][i^-(\alpha_i), (\gamma_i)^{-1} l_i](\gamma_i)^{-1} l_i.$$

On the other hand, since $f_\sigma \in M_{0,g+1} \subset M_{g,1}$, the handles H_i^β remain fixed via f_σ and it follows that we have $l_i = l_i'$, and $l_i'' = \gamma_{i+1}$. Thus, we get

$$\eta_i^\alpha = [\alpha_i^{-1}, l_i^{-1}] \quad \text{and} \quad \eta_i^\beta = [\beta_i^{-1}, (\gamma_{i+1})^{-1}][i^-(\alpha_i), (\gamma_i)^{-1} l_i](\gamma_i)^{-1} l_i.$$

Now, suppose that $\sigma \in P_n$. Then, $f_\sigma \in M[n-1]$ and $\eta_i^\alpha, \eta_i^\beta \in \frac{F(2g)_n}{F(2g)_{n+1}}$. Thus, via the projection $H \to H'$, we get

$$\widetilde{\eta_i^\alpha} = [x_i^{-1}, (\tilde{l}_i)^{-1}] \in L_n(H') \quad \text{and} \quad \widetilde{\eta_i^\beta} = [\widetilde{i^-(\alpha_i)}, (\tilde{\gamma}_i)^{-1}\tilde{l}_i](\tilde{\gamma}_i)^{-1}\tilde{l}_i \in L_n(H')$$

where $\tilde{z}$ denotes the projection of z.

On the other hand, one has $A_\sigma(x_i) = (\tilde{l}_i)^{-1} x_i \tilde{l}_i$ and we have $\tilde{l}_i \in F(g)_n$ since $\sigma \in P_n$. Furthermore, if $\delta_i = l_i''(l_i')^{-1} l_i$, we have the following inductive relations :

$$\delta_i = [\alpha_i^{-1}, \gamma_i l_i^{-1}]\gamma_i \quad \text{and} \quad \gamma_{i+1} = [\beta_i l_i'(l_i'')^{-1}, \delta_i (l_i'')^{-1}]\delta_i$$

which implies, since $l_i = l_i'$ and $l_i'' = \gamma_{i+1}$,

$$\tilde{\delta}_i = \widetilde{\gamma_{i+1}} = [x_i^{-1}, \tilde{\gamma}_i(\tilde{l}_i)^{-1}]\tilde{\gamma}_i.$$

So, we have $\tilde{\gamma}_i \in F(g)_{n+1}$ by induction and we get

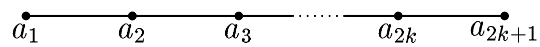
$$\widetilde{\eta_i^\alpha} = 0 \quad \text{and} \quad \widetilde{\eta_i^\beta} = \tilde{l}_i \quad \text{in} \quad L_n(H').$$

This proves that

$$p_n J_{n-1}(f_\sigma) = p_n(\sum \alpha_i \otimes \eta_i^\beta - \beta_i \otimes \eta_i^\alpha) = \sum \alpha_i \otimes \tilde{l}_i = -\mu_n(\sigma).$$

6. A relation in $(T_{g,1})_2$.

We want to give explicitly the relation in $(T_{g,1})_2$ obtained from the Jacobi relation via the map $P(g)_3 \to (T_{g,1})_2$. We will use Johnson's notation [J1]. Let $(a_1, \dots, a_{2k+1})$ be an odd chain, that is to say an ordered collection of oriented simple closed curves in a surface Σ whose intersection graph is the one below (defined by using a vertex for each curve, an edge between two vertices if the corresponding curves intersect transversely in a single point and no edge if they are disjoint), and such that the algebraic intersection $a_i \cdot a_{i+1}$ is $+1$.



This chain defines an element of $T_{g,1}$ as follows. A neighborhood of $a_1 \cup \ldots \cup a_{2k+1}$ in Σ is homeomorphic to $\Sigma_{k,2}$. Denote by γ (resp. γ') the boundary component which is on the left (resp. on the right) of $a_1, a_3, \dots, a_{2k+1}$. Then, γ and γ' are homologous and $\tau_\gamma \tau_{\gamma'}^{-1}$ is in the Torelli group of Σ. This element will be written $[a_1; \dots ; a_{2k+1}]$ and will be called the k-chain map induced by $(a_1, \dots, a_{2k+1})$.

Now, consider the curves C_1^α, $D_{1,2}$, C_2^α, C_3^α introduced in section 2 and 3, and denote by $D_{2,3}$ a curve in $\Sigma_{g,1}$ whose homotopy class is $\alpha_2^{-1}\beta_2^{-1}\alpha_3\beta_3$. Then, $(C_1^\alpha, D_{1,2}, C_2^\alpha, D_{2,3}, C_3^\alpha)$ forms a 5-chain and we have

Lemma 6.1.

$[\tau_{1,2}, \tau_{2,3}] = [C_1^\alpha; D_{1,2}+C_2^\alpha+D_{2,3}; C_3^\alpha]^{-1}[C_1^\alpha; D_{1,2}; C_2^\alpha]^{-1}[C_2^\alpha; D_{2,3}; C_3^\alpha]^{-1}[C_1^\alpha; D_{1,2}; C_2^\alpha; D_{2,3}; C_3^\alpha]$
where $D_{1,2} + C_2^\alpha + D_{2,3}$ *denotes the curve* $\tau_{D_{2,3}}^{-1}\tau_2^{-1}(D_{1,2})$.

Remark. It will be more convenient here to use the model of $\Sigma_{g,1}$ given in figure 6.

Proof. Let us denote by $C_{1,2,3}$ and $C'_{1,2,3}$ the two boundary curves of a neighborhood in $\Sigma_{g,1}$ of $C_1^\alpha \cup D_{1,2} \cup C_2^\alpha \cup D_{2,3} \cup C_3^\alpha$ (see figure 6). Then, one has

$$[C_1^\alpha; D_{1,2}; C_2^\alpha; D_{2,3}; C_3^\alpha] = \tau_{1,2,3}(\tau'_{1,2,3})^{-1}.$$

where $\tau_{1,2,3}$ (resp. $\tau'_{1,2,3}$) is the Dehn twist along $C_{1,2,3}$ (resp. $C'_{1,2,3}$).

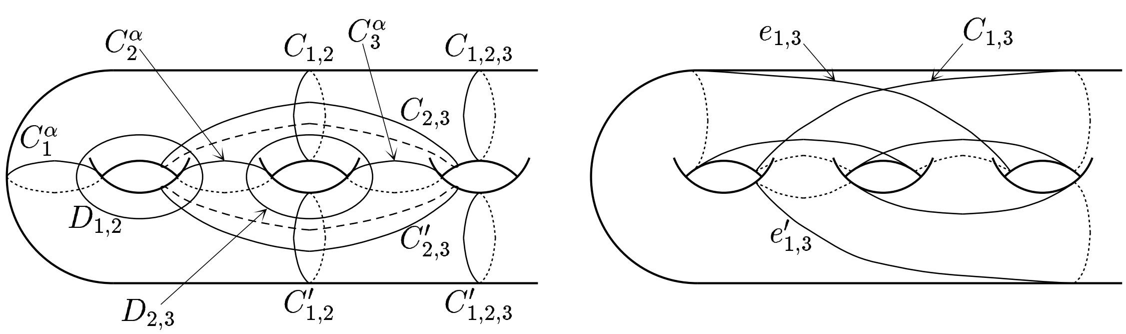


Figure 6.

The four curves C_1^α, C_2^α, C_3^α and $C_{1,2,3}$ bound a subsurface of $\Sigma_{0,g+1}$ which is homeomorphic to $\Sigma_{0,4}$. Thus, they yield the following lantern relation (see [J1]):

$$\tau_1\tau_2\tau_3\tau_{1,2,3} = \tau_{1,3}\tau_{2,3}\tau_{1,2} = \tau_{1,2}\tau_{1,3}(\tau_{1,2})^{-1}\tau_{1,2}\tau_{2,3} \qquad (\mathrm{L}_1).$$

From this, we get $[\tau_{1,2},\tau_{2,3}] = \tau_{1,2}(\tau_{1,3})^{-1}(\tau_{1,2})^{-1}\tau_{1,3}$ $(\star)$.

On the other hand, C_1^α, C_2^α, C_3^α and $C'_{1,2,3}$ also bound a subsurface homeomorphic to $\Sigma_{0,4}$ and so, yield the lantern relation:

$$\tau_1\tau_2\tau_3\tau'_{1,2,3} = \tau'_{1,2}\tau'_{2,3}\tau_{e'_{1,3}} \qquad (\mathrm{L}_2).$$

Remark. τ_1, τ_2, τ_3, $\tau_{1,2}$, $\tau_{1,3}$, $\tau_{2,3}$ and $\tau'_{1,2}$ are precisely the twists considered in the preceding sections, whereas $\tau_{1,2}\tau_{1,3}(\tau_{1,2})^{-1}$, $\tau_{e'_{1,3}}$ and $\tau'_{2,3}$ are the twists along the curves $e_{1,3}$, $e'_{1,3}$ and $C'_{2,3}$ described in figure 6.

Multiplying (L_1) by the inverse of (L_2), we get

$$\tau_{1,2,3}(\tau'_{1,2,3})^{-1} = \tau_{1,3}(\tau_{e'_{1,3}})^{-1}\tau_{2,3}(\tau'_{2,3})^{-1}\tau_{1,2}(\tau'_{1,2})^{-1}$$

which yields by $(\star)$

$$[\tau_{1,2},\tau_{2,3}] = (\tau_{e_{1,3}})^{-1}\tau_{e'_{1,3}}(\tau_{e'_{1,3}})^{-1}\tau_{1,3} = (\tau_{e_{1,3}})^{-1}\tau_{e'_{1,3}}(\tau_{1,2})^{-1}\tau'_{1,2}(\tau_{2,3})^{-1}\tau'_{2,3}\tau_{1,2,3}(\tau'_{1,2,3})^{-1}.$$

Now, noticing that

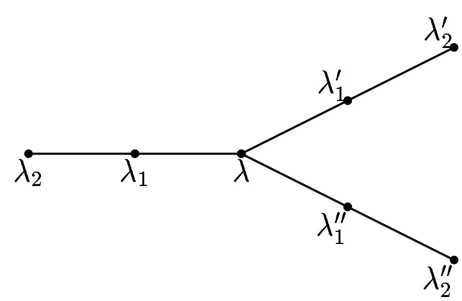
$$\tau_{1,2}(\tau'_{1,2})^{-1} = [C_1^\alpha; D_{1,2}; C_2^\alpha], \qquad \tau_{2,3}(\tau'_{2,3})^{-1} = [C_2^\alpha; D_{2,3}; C_3^\alpha],$$

$$\text{and} \quad \tau_{e_{1,3}}(\tau_{e'_{1,3}})^{-1} = [C_1^\alpha; D_{1,2} + C_2^\alpha + D_{2,3}; C_3^\alpha],$$

we get the required relation.

□

Now, suppose that we are given a collection of seven curves $\lambda, \lambda_1, \lambda_2, \lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2$ whose intersection graph is



and consider the following elements of $\mathcal{T}_{g,1}$:

$$Y_1 = [\lambda_2; \lambda_1; \lambda; \lambda'_1; \lambda'_2], \quad Y_2 = [\lambda'_2; \lambda'_1; \lambda; \lambda''_1; \lambda''_2], \quad Y_3 = [\lambda''_2; \lambda''_1; \lambda; \lambda_1; \lambda_2],$$

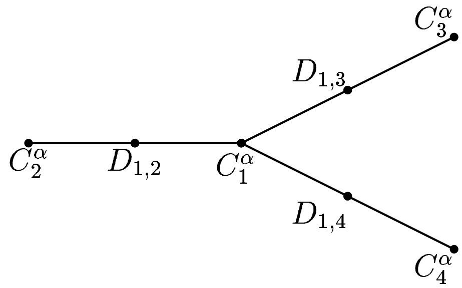
$$A = [\lambda; \lambda_1; \lambda_2], \quad A' = [\lambda; \lambda'_1; \lambda'_2], \quad A'' = [\lambda; \lambda''_1; \lambda''_2],$$

$$B_1 = [\lambda_2; \lambda_1 + \lambda + \lambda'_1; \lambda'_2], \quad B_2 = [\lambda'_2; \lambda'_1 + \lambda + \lambda''_1; \lambda''_2], \quad B_3 = [\lambda''_2; \lambda''_1 + \lambda + \lambda_1; \lambda_2].$$

Theorem 6.2. *One has the following relation in $(\mathcal{T}_{g,1})_2 \subset \mathcal{T}_{g,1}$:*

$$[Y_3B_3^{-1}A'', Y_2B_2^{-1}A'(A'')^{-1}][Y_2B_2^{-1}A', Y_1B_1^{-1}A(A')^{-1}][Y_1B_1^{-1}A, Y_3B_3^{-1}A''(A)^{-1}] = 1.$$

Proof. With the notation of section 2, consider curves $D_{1,3}$ and $D_{1,4}$ in $\Sigma_{4,1}$ whose homotopy classes are respectively $\beta_1^{-1}\alpha_2\alpha_3\beta_3$ and $\beta_1^{-1}\alpha_2\alpha_3\beta_3\alpha_3^{-1}\beta_3^{-1}\alpha_4\beta_4$. Then, the intersection graph of $C_1^\alpha, D_{1,2}, C_2^\alpha, D_{1,3}, C_3^\alpha, D_{1,4}, C_4^\alpha$ is



and there exists a homeomorphism of $\Sigma_{g,1}$ which send respectively $\lambda, \lambda_1, \lambda_2, \lambda_1', \lambda_2', \lambda_1''$ and λ_2'' to $C_1^\alpha, D_{1,2}, C_2^\alpha, D_{1,3}, C_3^\alpha, D_{1,4}$ and C_4^α. Thus, it suffices to prove the relation in this case.

Consider first the 5-chain $(C_2^\alpha, D_{1,2}, C_1^\alpha, D_{1,3}, C_3^\alpha)$. We claim that

$$B_1^{-1}A(A')^{-1}Y_1 = [\tau_{1,2}, \tau_{1,3}].$$

Indeed, there is a diffeomorphism h of $\Sigma_{g,1}$ (for exemple, $h = (\tau_1\tau_2\tau_{D_{1,2}})^2$) which sends respectively $C_2^\alpha, D_{1,2}, C_1^\alpha, D_{1,3}, C_3^\alpha, C_{1,2}$ and $C_{1,3}$ to $C_1^\alpha, D_{1,2}, C_2^\alpha, D_{2,3}, C_3^\alpha, C_{1,2}$ and $C_{2,3}$. Thus, up to a conjugation, the required relation is the one of lemma 6.1. In the same way, we have the following two relations:

$$B_2^{-1}A'(A'')^{-1}Y_2 = [\tau_{1,3}, \tau_{1,4}] \quad \text{and} \quad B_3^{-1}A''A^{-1}Y_3 = [\tau_{1,4}, \tau_{1,2}].$$

From these three relations, we deduce

$$(\tau_{1,2})^{\tau_{1,4}}(\tau_{1,2}')^{-1} = [\tau_{1,4}, \tau_{1,2}]\, A = Y_3B_3^{-1}A'',$$

$$(\tau_{1,4})^{\tau_{1,3}}(\tau_{1,4}')^{-1} = [\tau_{1,3}, \tau_{1,4}]\, A'' = Y_2B_2^{-1}A',$$

$$(\tau_{1,3})^{\tau_{1,2}}(\tau_{1,3}')^{-1} = [\tau_{1,2}, \tau_{1,3}]\, A' = Y_1B_1^{-1}A,$$

where $A = \tau_{1,2}(\tau_{1,2}')$, $A' = \tau_{1,3}(\tau_{1,3}')$ and $A'' = \tau_{1,4}(\tau_{1,4}')$.

Now, the Jacobi identity in P_3 gives

$$[(\tau_{1,2})^{\tau_{1,4}}, [\tau_{1,3}, \tau_{1,4}]][(\tau_{1,4})^{\tau_{1,3}}, [\tau_{1,2}, \tau_{1,3}]][(\tau_{1,3})^{\tau_{1,2}}, [\tau_{1,4}, \tau_{1,2}]] = 1.$$

Since $\tau_{1,2}', \tau_{1,3}'$ and $\tau_{1,4}'$ commute with $\tau_{1,2}, \tau_{1,3}$ and $\tau_{1,4}$, we get in $(T_{g,1})_2$ the following relation:

$$[Y_3B_3^{-1}A'', Y_2B_2^{-1}A'(A'')^{-1}][Y_2B_2^{-1}A', Y_1B_1^{-1}A(A')^{-1}][Y_1B_1^{-1}A, Y_3B_3^{-1}A''(A)^{-1}] = 1.$$

□

7. The Topological IHX Relation

Let G be a group. We set $\mathcal{L}(G) = \bigoplus_{n\geq 0} \mathcal{L}_n(G)$, where $\mathcal{L}_n(G) = \frac{G_n}{G_{n+1}}$. It is a Lie algebra with bracket defined by $[\bar{a}, \bar{b}] = \overline{[a, b]}$, where $a \in G_n$, $b \in G_m$ represent $\bar{a} \in \frac{G_n}{G_{n+1}}, \bar{b} \in \frac{G_m}{G_{m+1}}$. The Jacobi group relation $[a^c, [b, c]][c^b, [a, b]][b^a, [c, a]] = 1$ implies the Jacobi relation in $\mathcal{L}(G)$. We will write $\mathcal{L}_n^{\mathbb{Q}}(G)$ for $\mathcal{L}(G) \otimes \mathbb{Q}$. If $G^{ab} = \frac{G}{G_2}$, then one has a surjective mapping $L(G^{ab}) \to \mathcal{L}(G)$ given by the isomorphism $L_1(G^{ab}) = G^{ab} = \frac{G}{G_2} = \mathcal{L}_1(G)$, where $L(G^{ab})$ is the free

Lie algebra on G^{ab}. We write $L(G^{ab}_{\mathbb{Q}})$ for the free $\mathbb{Q}$ Lie algebra on $G^{ab}_{\mathbb{Q}} = G^{ab} \otimes \mathbb{Q}$. Then, $L(G^{ab}_{\mathbb{Q}}) \to \mathcal{L}^{\mathbb{Q}}(G)$ is also surjective. The kernel $R(G)$ of this map is the ideal of relations.

If $T = T_{g,1}$, $g \geq 4$, we deduce a relation in $R_2(T)$, which we call the IHX relation, as it arizes from the Jacobi relation in $P = P(g)$.

First recall that the degree 1 Johnson homomorphism is a map $J_1 \colon T \to \wedge^3(H)$, $H = H_1(\Sigma_{g,1})$. Rationally this gives an isomorphism $\mathcal{L}^{\mathbb{Q}}_1(T) = T^{ab}_{\mathbb{Q}} \to \wedge^3(H^{\mathbb{Q}})$. Thus, $R_2(T)$ lies in $L_2(\wedge^3(H^{\mathbb{Q}}))$.

There is an obvious relation in $R_2(T)$ (see [Hn]) coming from the commutativity of diffeomorphisms having disjoint support, namely $[a_1 \wedge a_2 \wedge b_2, b_3 \wedge a_3 \wedge a_4]$, where $(a_i, b_i)_{1 \leq i \leq g}$ is a symplectic basis of H. Substituting $a_1 + b_3$ for b_3 also gives a relation. The difference gives the relation $[a_1 \wedge a_2 \wedge b_2, a_1 \wedge a_3 \wedge a_4]$.

The Jacobi relation in $\frac{P_3}{P_4}$ determines a relation in $\frac{T_2}{T_3}$ and hence in $\mathcal{L}^{\mathbb{Q}}_2(T)$. We can compute examples as follows. For $a = \sigma_{1,2}$, we have $[a, [b, c]] = [\tau_{1,2}(\tau'_{1,2})^{-1}, [b, c]]$. We claim that $J_1(\tau_{1,2}(\tau'_{1,2})^{-1}) = \alpha_1 \wedge \alpha_2 \wedge \beta_2 - \alpha_1 \wedge \alpha_2 \wedge \beta_1$. To see this, note that the curves $C^{\alpha}_1, D_{1,2}, C^{\alpha}_2$ of section 3 define a 3-chain and that $J_1([x, y, z]) = -x \wedge y \wedge z$ for a 3-chain $[x, y, z]$ (see [J2]). Thus,

$$J_1(\tau_{1,2}(\tau'_{1,2})^{-1}) = -\alpha_1 \wedge (-\beta_1 + \alpha_2 + \beta_2) \wedge \alpha_2 = \alpha_1 \wedge \alpha_2 \wedge \beta_2 - \alpha_1 \wedge \alpha_2 \wedge \beta_1.$$

If we take $b = \sigma_{1,3}$, $c = \sigma_{1,4}$, then $J_1([b, c]) = \alpha_1 \wedge \alpha_3 \wedge \alpha_4$. Thus, $[\sigma_{1,2}, [\sigma_{1,3}, \sigma_{1,4}]]$ defines $[\alpha_1 \wedge \alpha_2 \wedge \beta_2 - \alpha_1 \wedge \alpha_2 \wedge \beta_1, \alpha_1 \wedge \alpha_3 \wedge \alpha_4]$ in $\mathcal{L}^{\mathbb{Q}}_2(T)$. Now note that there is a diffeomorphism f taking $M_{4,1}$ to itself which on homology takes α_1, β_1 to themselves and cyclically permutes α_i, β_i for $i = 2, 3, 4$. Thus f cyclically permutes the curves $C_{1,2}$, $C_{1,3}$, $C_{1,4}$. This proves that the Jacobi relation yields the element

$$[\alpha_1 \wedge \alpha_2 \wedge \beta_2 - \alpha_1 \wedge \alpha_2 \wedge \beta_1, \alpha_1 \wedge \alpha_3 \wedge \alpha_4] + [\alpha_1 \wedge \alpha_3 \wedge \beta_3 - \alpha_1 \wedge \alpha_3 \wedge \beta_1, \alpha_1 \wedge \alpha_4 \wedge \alpha_2] + [\alpha_1 \wedge \alpha_4 \wedge \beta_4 - \alpha_1 \wedge \alpha_4 \wedge \beta_1, \alpha_1 \wedge \alpha_2 \wedge \alpha_3]$$

in $R_2(T)$. Since terms like $[\alpha_1 \wedge \alpha_2 \wedge \beta_2, \alpha_1 \wedge \alpha_3 \wedge \alpha_4]$ are in $R_2(T)$, this reduces to

$$[\alpha_1 \wedge \alpha_2 \wedge \beta_1, \alpha_1 \wedge \alpha_3 \wedge \alpha_4] + [\alpha_1 \wedge \alpha_3 \wedge \beta_1, \alpha_1 \wedge \alpha_4 \wedge \alpha_2] + [\alpha_1 \wedge \alpha_4 \wedge \beta_1, \alpha_1 \wedge \alpha_2 \wedge \alpha_3].$$

In genus $g \geq 5$, we may substitute $\alpha_1 + \alpha_5$ for α_1 and the difference yields

$$[\alpha_5 \wedge \alpha_2 \wedge \beta_1, \alpha_1 \wedge \alpha_3 \wedge \alpha_4] + [\alpha_5 \wedge \alpha_3 \wedge \beta_1, \alpha_1 \wedge \alpha_4 \wedge \alpha_2] + [\alpha_5 \wedge \alpha_4 \wedge \beta_1, \alpha_1 \wedge \alpha_2 \wedge \alpha_3]$$

plus terms like $[\alpha_1 \wedge \alpha_2 \wedge \beta_1, \alpha_5 \wedge \alpha_3 \wedge \alpha_4]$ and $[\alpha_5 \wedge \alpha_2 \wedge \beta_1, \alpha_5 \wedge \alpha_3 \wedge \alpha_4]$ which lie in $R_2(T)$. Permuting the indices, we have shown the following.

Theorem 7.1. *If $g \geq 5$, the relation*

$$[\alpha_1 \wedge \alpha_4 \wedge \beta_5, \alpha_2 \wedge \alpha_3 \wedge \alpha_5] + [\alpha_2 \wedge \alpha_4 \wedge \beta_5, \alpha_3 \wedge \alpha_1 \wedge \alpha_5] + [\alpha_3 \wedge \alpha_4 \wedge \beta_5, \alpha_1 \wedge \alpha_2 \wedge \alpha_5] = 0$$

holds in $\mathcal{L}^{\mathbb{Q}}_2(T_{g,1})$.

Remark. This relation appears at least implicitly in work of Morita (see [Mo1, Mo2]), when he computed the image of the second Johnson homorphism.

Remark. In [HS] (following Hain [Hn]) a presentation of the Malcev Lie algebra of $T_{g,1}$ is given. The relation of theorem 7.1 is related to another quadratic relation R_2 (see [HS], §3) determined by the boundary of $\Sigma_{g,1}$.

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