3.2 Definition of Lagrangian Floer homology

In this subsection we describe the actual definition of Floer homology assuming some analytical results about the moduli spaces of homolorphic strips. Floer's insight was to understand that the crucial object in the definition of Morse homology is not the negative gradient flow, but the 1- and 2-dimensional moduli spaces of trajectories connecting two critical points. The analogue for the action functional are the 1- and 2-dimensional moduli spaces of holomorphic strips connecting intersection points. It turns out that, as Floer understood, those are well defined and can be studied using techniques from elliptic PDE's.

Before going into the definition of Floer homology we describe the geometric setting and introduce some useful terminology. For the rest of this section we fix a Liouville manifold (M, β) and denote $\omega = d\beta$. The Liouville form β on M is pulled back to a contact form α on ∂M and $\xi = \ker \alpha$ will denote the contact structure.

Definition 3.4. The *Reeb vector field* of the contact form α is the tangent vector field on ∂M which is defined by the following equations:

$$\begin{cases} \iota_R d\alpha = 0 \\ \alpha(R) = 1. \end{cases} \tag{15}$$

We consider the completion $(\hat{M}, \hat{\beta})$ defined in Section 1.5.

Definition 3.5. An almost complex structure J on $(\hat{M}, \hat{\beta})$ is *compatible with* $\hat{\beta}$ if J is compatible with ω in the sense of Definition 1.7 and, moreover, in the end $([0, +\infty) \times \partial M, d(e^\tau \alpha))$ it satisfies:

1. $J\left(\frac{\partial}{\partial \tau}\right) = R$,
2. $J(\xi) = \xi$,
3. J is compatible with $d\alpha$ on ξ, in the sense that $d\alpha|_\xi(\cdot, J\cdot)$ defines a scalar product on ξ, and
4. J is invariant by translations in the τ-direction.

We denote by $\mathcal{J}(\beta)$ the set of almost complex structures on $\hat{M}$ which are compatible with $\hat{\beta}$.

One can prove that a J-holomorphic map for an almost complex structure J which is compatible with $\hat{\beta}$ cannot have a local maximum in the end $[0, +\infty) \times \partial M$ (*maximum principle*). Then holomorphic strips with boundary on Lagrangian submanifolds in M cannot enter $[0, +\infty) \times \partial M$. In fact the only reason to introduce the completed Liouville manifold $(\hat{M}, \hat{\beta})$ is to define the almost complex structures compatible with $\hat{\beta}$ and prove the maximum principle, so that we can effectively forget about the completion and work in M.

We fix a time-dependent almost complex structure J_t on $\hat{M}$ which is compatible with $\hat{\beta}$; i.e. a map $J_\bullet \colon [0, 1] \to \mathcal{J}(\beta)$. Let L_0 and L_1 be two closed exact Lagrangian submanifolds of (M, β) which intersect transversely and let $p_+, p_- \in L_0 \cap L_1$ be intersection points.

Definition 3.6. The *moduli space* $\mathfrak{M}(p_+, p_-; J_t)$ is the set of smooth maps $u \colon \mathbb{R} \times [0, 1] \to M$ which satisfy the Cauchy-Riemann equation

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial s} = 0 \tag{16}$$

and such that

1. $u(s, 0) \in L_0$ for all $s \in \mathbb{R}$,
2. $u(s, 1) \in L_1$ for all $s \in \mathbb{R}$, and
3. $\lim_{s \to \pm\infty} u(s, t) = p_\pm$.

In the definition of Floer homology the moduli spaces of holomorphic strips $\mathfrak{M}(p_+, p_-; J_t)$ will play the same role as the moduli spaces of negative gradient flow trajectories $\mathcal{M}(p, q)$ in the definition of Morse homology. Therefore we need to define the analogue of the Morse-Smale condition and of the Morse index in the context of Floer homology.

We say that a map $u \in \mathfrak{M}(p_+, p_-; J_t)$ is a *Floer strip* from p_- to p_+ and that a map $u \in C^0(\mathbb{R} \times [0, 1], M)$ satisfying (1)–(3) in Definition 3.17, but not necessarily Equation (16), is a *Floer-like strip* from p_- to p_+. If u is smooth, it will be called a smooth Floer-like strip.

Equation (16) will be abbreviated $\overline{\partial}_{J_t} u = 0$ and the set of smooth Floer-like strips from p_- to p_+ will be denoted by $\mathcal{B}^\infty$. If $u \in \mathcal{B}^\infty$, then $\overline{\partial}_{J_t} u \in \Gamma(u^* TM)$. Then we can regard $\overline{\partial}_{J_t}$ as a section of a bundle $\mathcal{E}^\infty \to \mathcal{B}^\infty$ whose fibre at u is $\mathcal{E}^\infty_u = \Gamma(u^* TM)$.

We are going to define a smooth structure on $\mathfrak{M}(p_+, p_-; J_t)$ using the implicit function theorem, which requires the use of Banach spaces. For this reason we should complete $\mathcal{B}^\infty$ and $\mathcal{E}^\infty$ so that they become a Banach manifold and a Banach bundle respectively. I will discuss this completion briefly and without insisting on the technical points; the reader which is not familiar with Sobolev spaces can ignore the next few paragraphs without missing the general picture.

Given $p > 2$ we define $W^{1,p}(\mathbb{R}\times[0,1], M)$ as the set of functions $u \in C^0(\mathbb{R}\times[0,1], M)$ which are of class $W^{1,p}_{loc}$ in every chart. We denote $\mathcal{B}$ the set of Floer-like strips from p_- to p_+ of class $W^{1,p}$ and $\mathcal{E}_u$ the Banach space of sections of u^*TM of class L^p. Then we have the following proposition.

Proposition 3.7. *$\mathcal{B}$ is a banach manifold. For every $u \in \mathcal{B}$,*

$$T_u\mathcal{B} = \{\xi \in W^{1,p}(u^*TM) : \xi(s,0) \in T_{u(s,0)}L_0 \text{ and } T_{u(s,1)}L_1\}.$$

Moreover $\mathcal{E} \to \mathcal{B}$ is a bundle of Banach spaces and $\overline{\partial}_{J_t} \colon \mathcal{B} \to \mathcal{E}$ is a smooth section.

Even if we have enlarged the domain of $\overline{\partial}_{J_t}$ we have introduced no new solution: in fact elliptic regularity implies that a map $u \in W^{1,p}(\mathbb{R} \times [0,1], M)$ such that $\overline{\partial}_{J_t} u = 0$ is smooth.

Given $u \in \mathfrak{M}(p_+, p_-; J_t)$ we define the *linearized Cauchy-Riemann operator* $D_u \colon T_u\mathcal{B} \to \mathcal{E}_u$ as follows. We identify $\mathcal{B}$ with the image of the zero section of $\mathcal{E}$ so that we can regard u as an element of $\mathcal{E}$. Then there is a canonical isomorphism $T_u\mathcal{E} \cong T_u\mathcal{B} \oplus \mathcal{E}_u$ and we define D_u as the composition of the differential of $\overline{\partial}_{J_t}$ at u with the projection $T_u\mathcal{E} \to \mathcal{E}_u$. It is easy to see that $\overline{\partial}_{J_t}$ is transverse to the zero section at u if and only if D_u is surjective.

Definition 3.8. A time-dependent almost complex structure J_t is called *regular* if D_u is surjective for all Floer strip u.

Regularity for time-dependent almost complex structure is the concept which replaces the Morse-Smale condition. The following theorem is one of the main fundational results of the theory. It is the only place where working with time-dependent almost complex structures is required.[7]

Theorem 3.9. *Regular time-dependent almost complex structures are generic.*

[7] However it is still possible, in some fortunate cases, to find regular time-independent almost complex structures.

If J_t is a regular almost complex structure, then the implicit function theorem (which has a straightforward extension to infinite dimensional Banach manifolds) implies that, for all $p_+, p_- \in L_0 \cap L_1$, the moduli space $\mathfrak{M}(p_+, p_-; J_t)$ is a disjoint union of smooth manifolds. Moreover, for all $u \in \mathfrak{M}(p_+, p_-; J_t)$, we have $T_u\mathfrak{M}(p_+, p_-; J_t) = \ker D_u$.

The careful reader has surely noted that we said "disjoint union of smooth manifolds" instead of "manifold". The reason is that $\mathfrak{M}(p_+, p_-; J_t)$ in general has connected components of different dimension. This is the main difference between the implicit function theorem in finite dimension and in infinite dimension: in the finite dimensional case the dimension of the preimage of a regular value of a map $f \colon M \to N$ is $\dim M - \dim N$. However, in the infinite dimensional case, this formula makes no sense and the situation is more subtle. What makes the situation still tractable is the fact that D_u is a so-called Fredholm operator. Again the reader which is not familiar with functional analysis can skip the next few paragraphs and go directly to the Maslov index.

Definition 3.10. Let X and Y be Banach spaces and $T \colon X \to Y$ a bounded linear operator between them. Then T is a *Fredholm operator* if the following properties hold:

- $\ker T$ is finite dimensional,
- $\operatorname{Im} T$ is closed, and
- $\operatorname{coker} T$ is finite dimensional.

If T is a Fredholm operator, the *Fredholm index* of T is

$$\operatorname{Ind}(T) = \dim \ker T - \dim \operatorname{coker} T.$$

The Fredholm index is invariant in the following sense.

Lemma 3.11. *Let $T_s \colon X \to Y$, $s \in [0, 1]$, be a family of Fredholm operators which is continuous in the operator norm. Then $\operatorname{Ind}(T_s)$ is constant in s.*

The relevance of Fredholm theory in Floer homology comes from the following proposition, which is essentially a consequence of elliptic regularity for the operator D_u.

Proposition 3.12. *If L_0 and L_1 intersect transversely at p_+ and p_- then, for all $u \in \mathfrak{M}(p_+, p_-; J_t)$, the linearized operator D_u is a Fredholm operator.*

(Although it will not be relevant in our case, it is worth noting that Proposition 3.12 does not require a regular J_t.) Proposition 3.12 implies that $T_u\mathfrak{M}(p_+, p_-; J_t)$ is finite dimensional, and its dimension is the Fredholm index of D_u. Then the Fredholm index of D_u is the quantity which plays the role in Floer homology of the difference between the Morse indeces in Morse homology. Unlike in the case of Morse theory, it turns out that $\mathrm{Ind}(D_u)$ does not depend only on the limits of u (i.e. the intersection points p_+ and p_-. However it is essentially a consequence of Lemma 3.11 that $\mathrm{Ind}(D_u)$ depends only on the homotopy class of u as a Floer-like strip. Our next task is to give a topological formula for the Fredholm index of D_u.

Let $Gr(\mathbb{C}^n)$ be the Grassmannian[8] of the Lagrangian planes in $\mathbb{C}^n$. The group $U(n)$ acts transitively on $Gr(\mathbb{C}^n)$ in the obvious way and the stabiliser of $\mathbb{R}^n$ is $O(n)$. Then $Gr(\mathbb{C}^n)$ is diffeomorphic to $U(n)/O(n)$ and an explicit diffeomorphism is induced by the map $U(n) \to Gr(\mathbb{C}^n)$, $U \mapsto U(\mathbb{R}^n)$. We define a map

$$\rho\colon Gr(\mathbb{C}^n) \to S^1$$

by $\rho(U(\mathbb{R}^n)) = \det U^2$.

Let $u\colon \mathbb{R} \times [0,1] \to M$ be a smooth Floer-like strip from p_- to p_+. We can find a unitary trivialisation of u^*TM, which is a bundle map

$$\begin{tikzcd} u^*TM \arrow[rr, "\Phi"] \arrow[dr] & & \mathbb{R}\times[0,1]\times\mathbb{C}^n \arrow[dl] \\ & \mathbb{R}\times[0,1] & \end{tikzcd}$$

such that $\Phi_{(s,t)} \circ J_t = \tilde{J}\Phi_{s,t}$, where $\Phi_{(s,t)}$ denotes the restriction of Φ to the fibre over (s,t) and $\tilde{J}$ is the almost complex structure on the fibres of $\mathbb{R}\times[0,1]\times\mathbb{C}^n \to \mathbb{R}\times[0,1]$ which is induced by the complex multiplication of $\mathbb{C}^n$. Moreover we can chose Φ such that $\Phi(T_{u(s,0)}L_0) = \mathbb{R}^n$ for all $s \in \mathbb{R}$.

We define $\lambda\colon \mathbb{R} \to Gr(\mathbb{C}^n)$ by $\lambda(s) = \Phi(T_{u(s,1)}L_1)$. The asymptotic properties of u imply that $\lim\limits_{s\to\pm\infty} \lambda(s) = \Phi(T_{p_\pm}L_1)$. Therefore we can “compactify” $\mathbb{R}$ to $[0,\pi]$ and regard λ as a map $\lambda\colon [0,\pi] \to Gr(\mathbb{C}^n)$ such that $\lambda(0) = \Phi(T_{p_-}L_1)$ and $\lambda(\pi) = \Phi(T_{p_+}L_1)$.

We define $C = \{L \in Gr(C^n) : L \cap \mathbb{R}^n = \mathbf{0}\}$. We can see that the planes in C are exactely the graphs of the linear maps $A\colon \mathbb{R}^n \to i\mathbb{R}^n$ such that iA

[8]The term Grassmannian refers to a manifold which parametrises subspaces of a vector space

is represented by a symmetric matrix (in thecanonical basis). This implies that C is contractible.

We can close λ to a loop $\tilde{\lambda}\colon [0, 2\pi] \to Gr(\mathbb{C}^n)$ (with $\lambda(2\pi) = \lambda(0)$) such that $\lambda(s) \in C$ for all $s \in [\pi, 2\pi]$. Then we define the *Maslov index* $\mu(u) \in \mathbb{Z}$ by

$$\mu(u) = -\deg(\rho \circ \tilde{\lambda}).$$

Lemma 3.13. *Let $u \in \mathfrak{M}(p_+, p_-; J_t)$. Then* $\mathrm{Ind}(D_u) = \mu(u)$.

For the proof of Lemma 3.13 we do not need a regular J_t. If J_t is regular, then $\mathrm{Ind}(D_u) = \dim \ker D_u$ and we have the following corollary.

Corollary 3.14. *Suppose J_t is regular. Then, for all $u \in \mathfrak{M}(p_+, p_-; J_t)$, the dimension of the connected component of $\mathfrak{M}(p_+, p_-; J_t)$ containing u is $\mu(u)$.*

Definition 3.15. Let J_t be a regular time-dependent almost complex structure. Then we denote by $\mathfrak{M}_i(p_+, p_-; J_t)$, for $i \geq 0$, the union of the i-dimensional connected components of $\mathfrak{M}(p_+, p_-; J_t)$.

Corollary 3.14 implies that, if J_t is regular, then there is no Floer strip u with $\mu(u) < 0$ because it should belong to a negative dimensional manifold, which is impossible.

If u is a Floer strip, then $s_0 \cdot u$ defined by $s_0 \cdot u(s,t) = u(s_0 + s, t)$ is also a Floer strip. This implies that there is an action of $\mathbb{R}$ on the moduli spaces $\mathfrak{M}_i(p_+, p_-; J_t)$. Constant Floer strip at intersection points are the fixed points of this action, and the stabiliser of any non-constant Floer strip is trivial.

Definition 3.16. For $i > 0$ we define $\mathfrak{M}_i^*(p_+, p_-; J_t) = \mathfrak{M}_i(p_+, p_-; J_t)/\mathbb{R}$.

One can prove that, for a regular J_t, the quotiented moduli spaces $\mathfrak{M}_i^*(p_+, p_-; J_t)$ are smooth manifolds of dimension $i-1$. This is not automatic because quotients by the action of a non-compact group can be pathological. Now we are ready to state the structure theorem for moduli spaces of Floer strips which is the analogue to Theorem 2.12.

Theorem 3.17. *Let (M, β) be a Liouville manifold and $L_0, L_1 \subset M$ two closed exact lagrangian submanifolds which intersect transversely. Then for any regular time-dependent almost complex structure J_t compatible with $\hat{\beta}$,*

- *if u is a Floer strip with Maslov index $\mu(u) \leq 0$, then u is a constant strip $u(s,t) = p \in L_0 \cap L_1$. In particular $\mu(u) = 0$.*
- *If $p_+, p_- \in L_0 \cap L_1$, then $\mathfrak{M}_1^*(p_+, p_-; J_t)$ is a finite set.*
- *If $p_+, p_- \in L_0 \cap L_1$, then $\mathfrak{M}_2^*(p_+, p_-; J_t)$ is a 1-dimensional manifold with finitely many connected components. Every non-compact connected components of $\mathfrak{M}_2^*(p_+, p_-; J_t)$ admits a natural compactification which is homeomorphic to a closed interval. The boundary of the compactification $\overline{\mathfrak{M}}_2^*(p_+, p_-; J_t)$ of $\mathfrak{M}_2^*(p_+, p_-; J_t)$ is*

$$\partial \overline{\mathfrak{M}}_2^*(p_+, p_-; J_t) = \bigcup_{q \in L_0 \cap L_1} \mathfrak{M}_1^*(p_+, q; J_t) \times \mathfrak{M}_1^*(q, p_-; J_t).$$

Once we have Theorem 3.17 we can define the Lagrangian Floer homology in the same way we defined Morse homology.

Definition 3.18. Let (M, β) be a Liouville manifold, $L_0, L_1 \subset M$ two closed exact lagrangian submanifolds which intersect transversely, and fix a regular time-dependent almost complex structure J_t compatible with $\hat{\beta}$. We define the *Floer chain complex* $CF(L_0, L_1, J_t)$ by

$$CF(L_0, L_1, J_t) = \bigoplus_{p \in L_0 \cap L_1} (\mathbb{Z}/2\mathbb{Z})p$$

with boundary

$$\partial p = \sum_{q \in L_0, L_1} \# \mathfrak{M}_1^*(p, q, J_t) q.$$

The following theorem is the algebraic translation of the structure of the compactification of the moduli spaces of *Floer strips.* Its proof is completely analogue to the proof of the corresponding theorem for Morse homology (Theorem 2.14) and therefore will be left to the reader.

Theorem 3.19. *In the situation of Definition 3.18, $\partial^2 = 0$.*

The homology

$$HF(L_0, L_1, J_t) = \frac{\ker \partial}{\operatorname{Im} \partial}$$

is called the *Lagrangian Floer homology* of L_0 and L_1.

3.3 Properties and applications

Theorem 3.20. *The Floer homology $HF(L_0, L_1, J_t)$ is independent of the choice of the regular time-dependent almost complex structure J_t. Moreover a Hamiltonian isotopy φ_t, with $\varphi_0 = Id$ and $\varphi_1 = \varphi$ defines isomorphisms*

$$HF(L_0, L_1, J_t) \xrightarrow{\cong} HF(L_0, \varphi(L_1), J_t),$$

$$HF(L_0, L_1, J_t) \xrightarrow{\cong} HF(\varphi(L_0), L_1, J_t),$$

which depend only on the homotopy class of φ_t in the group of Hamiltonian diffeomorphisms of M relative to the endpoints.

In view of the invariance of Lagrangian Floer homology, the time-dependent almost complex structure J_t will from now on dropped from notation. We will therefore write $HF(L_0, L_1)$.

Theorem 3.21. *Let (M, β) be a Liouville manifold and $L \subset M$ a closed exact lagrangian submanifold. If φ is a Hamiltonian diffeomorphism of M such that $\varphi(L)$ is transverse to L, then*

$$HF(L, \varphi(L)) \cong H_*(L; \mathbb{Z}/2\mathbb{Z}).$$

We give two applications of Theorem 3.21.

Theorem 3.22. *Let L be a compact n-dimensional smooth manifold and consider its cotangent bundle T^*L with its canonical symplectic form $-d\lambda$. Denote L_0 the zero section of T^*L. If φ is a Hamiltonian symplectomorphism of $(T^*L, -d\lambda)$ such that L_0 and $\varphi(L_0)$ intersect transversely, then*

$$\#(\varphi(L_0) \cap L_0) \geq \sum_{i=0}^{n} \dim H_i(L; \mathbb{Z}/2\mathbb{Z}). \tag{17}$$

It is important to note that $\#(\varphi(L_0) \cap L_0)$ is the naive number of intersection points (i.e. counted without signs). This thorem was originally proved, with different methods, by Laudenbach and Sikorav.

Proof. By theorem 3.21, $HF(L, \varphi(L)) \cong H_*(L; \mathbb{Z}/2\mathbb{Z})$. Moreover

$$\dim HF(L, \varphi(L)) \leq \dim CF(L, \varphi(L), J_t)$$

because the dimension of a chain complex is always at least the dimension of its homology. Since $\dim CF(L, \varphi(L), J_t) = \#(\varphi(L_0) \cap L_0)$ we have Inequality (17). $\square$

Another application of Floer homology is the following theorem of Gromov.

Theorem 3.23. *There is no closed exact Lagrangian submanifold in* (R^{2n}, ω_0).

Proof. Suppose there is such an exact Lagrangian L. Consider the function $f\colon \mathbb{R}^{2n} \to \mathbb{R}$ such that $f(x_1, y_1, \ldots, x_n, y_n) = x_1$. The Hamiltonian flow φ_t of f consists of translations in the coordinate y_1 so, for T large enough, $L \cap \varphi_T(L) = \emptyset$. This implies that $HF(L, \varphi_T(L)) = 0$. However we know by Theorem 3.21 that $HF(L, \varphi_T(L)) \cong H_*(L; \mathbb{Z}/2\mathbb{Z})$ and, for a closed manifold, $H_*(L; \mathbb{Z}/2\mathbb{Z}) \neq 0$. $\square$