Analytic and iterative reconstructions in SPECT

J.-P. Guillement and R.G. Novikov

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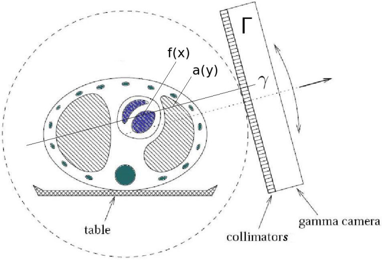
Abstract We consider analytic and iterative reconstructions in the single-photon emission computed tomography (SPECT).

- As analytic techniques we use Chang's approximate inversion formula and Novikov's exact inversion formula for the attenuated ray transform, on one hand, and Wiener-type filter for data with strong Poisson noise, on other hand.
- As iterative techniques we consider the least square and expectation maximization iterative reconstructions.
- Different comparisons are given.

Single Photon Emission Computed Tomography (SPECT)



Γ	discrete subset of the set T of all oriented straight lines in the space containing the body.
γ	point of detector set Γ
x	point of the space
$f(x)$	density of radioactive isotopes
$a(x)$	photon attenuation coeff.
$p(\gamma)$	projection data : number of photons coming from the body along oriented straight line γ to the associated detector during some fixed time.

The problem : find the isotopes distribution $f(x)$ from the projection data $p(\gamma)$ and some a priori information concerning the body (attenuation map $a(x)$).

Attenuated ray transform

In some approximation the projection data p are modeled as follows :

$$\begin{aligned}&\forall\gamma\in\Gamma,\ p(\gamma) \text{ is a realization of a Poisson variate } \mathbf{p}(\gamma)\\ &\text{with the mean } \ M\mathbf{p}(\gamma)=g(\gamma)=CP_af(\gamma),\\ &\text{all } \mathbf{p}(\gamma) \text{ are independent,}\end{aligned} \tag{1}$$

where P_af, the attenuated ray transform of f, is

$$P_af(\gamma)=\int\limits_{\gamma}\exp[-\mathcal{D}a(x,\hat{\gamma})]f(x)dx, \tag{2}$$

$\hat{\gamma}$ is the direction of γ, and $\mathcal{D}a$ the divergent beam

$$\mathcal{D}a(x,\theta)=\int\limits_0^{+\infty}a(x+t\theta)dt,\ x\in\mathbb{R}^2,\ \theta\in\mathbb{S}^1, \tag{3}$$

$C=C_1t$, t detection time.
(The SPECT problem $p\to Cf$ has been restricted to each fixed 2D plane intersecting the body)

Notation

- T the set of all oriented straight lines in $\mathbb{R}^2$, $T \approx \mathbb{R} \times \mathbb{S}^1$
- $\gamma \in T, \quad \gamma = (s, \theta) = \{x \in \mathbb{R}^2 : \; x = s\theta^\perp + t\theta, \; t \in \mathbb{R}\}$
- $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$ gives the orientation of γ
- $\theta^\perp = (-\theta_2, \theta_1)$
- $a(x) \geq 0, \; f(x) \geq 0$
- *supp* a and *supp* f are included in a disk $B_R = \{|x| \leq R\}$
- Γ is a uniform $n \times n$ sampling of

$$T_R = \{\gamma \in T : \; \gamma \cap B_R \neq \emptyset\} = \{(s, \theta) \in \mathbb{R} \times \mathbb{S}^1 : \; |s| \leq R\}$$

The standard value for n is 128

Problem 1. Find (as well as possible) Cf from p and a.

Our analytic approach to Problem 1. is based on the scheme

$$Cf \backsimeq P_a^{-1}\mathcal{W}p,$$

where

- $\mathcal{W}$ is a space-variant Wiener-type filter of [Guillement-Novikov 2008] (or some analytic method for approximate finding the noiseless data g from p)
- P_a^{-1} is a reconstruction based on some optimal combination of Novikov exact inversion formula and Chang approximate inversion formula.

  The optimal combination is constructed via a Morozov-type discrepancy principle (minimize $||P_a Cf - \mathcal{W}p||$).

Chang formula ([Chang 1978], [Novikov 2011])

$$Cf \simeq \mathcal{C}h_a g, \quad \text{where} \tag{4}$$

$$\mathcal{C}h_a g(x) = \frac{P_0^{-1}(g)(x)}{w_a(x)}$$

$$P_0^{-1}(g)(x) = \frac{1}{4\pi}\int_{\mathbb{S}^1} \theta^{\perp}\nabla_x Hg_\theta(x\theta^{\perp})d\theta, \quad \text{(FBP)}$$

$$w_a(x) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \exp[-\mathcal{D}a(x,\theta)]d\theta$$

$g = CP_a f$ (see (1) and (2)), $g_\theta(s) = g(s,\theta)$,
H, Hilbert transform, $Hu(s) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{u(t)}{s-t}dt$,
$\mathcal{D}a$ divergent beam (3).

Chang compensation formula (4) is approximate, sufficiently stable for reconstruction from discrete and noisy data p.

Novikov formula ([Novikov 2002])

$$Cf = \mathcal{N}_a g, \quad \text{where} \tag{5}$$

$$\mathcal{N}_a g(x) = \frac{1}{4\pi}\int_{\mathbb{S}^1} \theta^\perp \nabla_x K(x,\theta)d\theta,$$

$$\begin{aligned} K(x,\theta) &= \exp[-\mathcal{D}a(x,-\theta)]\,\tilde{g}(x\theta^\perp), \\ \tilde{g}(s) &= \exp\left[A_\theta(s)\right]\cos\left(B_\theta(s)\right) H(\exp\left[A_\theta\right]\cos\left(B_\theta\right) g_\theta)(s) + \\ &\quad \exp\left[A_\theta(s)\right]\sin\left(B_\theta(s)\right) H(\exp\left(A_\theta\right)\sin\left(B_\theta\right) g_\theta)(s), \end{aligned}$$

H Hilbert transform, $\mathcal{D}a$ divergent beam,
$A_\theta(s) = \frac{1}{2}P_0 a(s,\theta)$, $\quad B_\theta(s) = H\,A_\theta(s)$, $\quad g_\theta(s) = g(s,\theta)$,
$g = CP_a f$.

Formula (5) is exact (for continuous case) but not very stable for reconstruction from discrete and noisy data p.

Wiener-type filter[Guillement-Novikov 2008]

Wiener classical result Let $\hat{g}$, $\hat{\mathbf{p}}$, $\hat{p}$ on $\hat{\Gamma}$ denote the 2D discrete Fourier transforms of g, $\mathbf{p}$, p on Γ (see (1)). Let $\mathcal{W}$ denote a space-invariant linear filter on Γ that acts in the frequency domain as

$$\begin{array}{l}\hat{u}(j) \to \hat{W}(j)\hat{u}(j), \quad j \in \hat{\Gamma}, \\ \hat{W} \ \text{ is real} - \text{valued}, \quad \hat{W}(j) = \hat{W}(-j),\end{array}$$

where $\hat{W}$ is the window function of $\mathcal{W}$.
Then the mean $\mu(\mathcal{W}, g) = M\|\mathcal{W}\mathbf{p} - g\|^2_{L^2(\Gamma)}$ is minimal with respect to $\mathcal{W}$ iff

$$\begin{array}{l}\hat{W}(j) = \hat{W}^{opt}(j) \stackrel{\text{def}}{=} \frac{|\hat{g}(j)|^2}{|\hat{g}(j)|^2 + V}, \quad j \in \hat{\Gamma}, \\ V = n^{-1}\hat{g}(0) = n^{-2}\sum_{\gamma\in\Gamma} g(\gamma),\end{array} \tag{6}$$

where n is the sampling number.
But formula (6) contains unknown g.

Regularization of $\mathcal{W}^{opt}$ Let $S_1, \ldots, S_{n^*}$ a partition of $\hat{\Gamma}$ such that $-S_\alpha = S_{\beta_\alpha}$. Let a filter $\mathcal{W}$ that acts in the frequency domain as

$$\begin{array}{l} \hat{u}(j) \to \hat{W}(j)u(j), \quad j \in \hat{\Gamma}, \\ \hat{W} \ \text{ is real} - \text{valued}, \quad \hat{W}(j) = \hat{W}(-j), \\ \hat{W} \ \text{ is constant on each } S_\alpha \end{array}$$

Then $\mu(\mathcal{W}, g) = M\|\mathcal{W}\mathbf{p} - g\|^2_{L^2(\Gamma)}$ is minimal with respect to $\mathcal{W}$ iff

$$\begin{array}{l} \hat{W}(j) = \hat{W}^{r.o.}(j) \stackrel{\text{def}}{=} \frac{\Sigma_{g,\alpha(j)}}{\Sigma_{g,\alpha(j)}+V}, \quad j \in \hat{\Gamma}, \\ \Sigma_{g,\alpha} \stackrel{\text{def}}{=} \frac{1}{|S_\alpha|} \sum_{i \in S_\alpha} |\hat{g}(i)|^2, \quad \alpha = 1, \ldots, n^*, \\ V = n^{-1}\hat{g}(0) = n^{-2} \sum_{\gamma \in \Gamma} g(\gamma), \end{array}$$

- $|S_\alpha|$ number of elements in S_α
- $\alpha(j)$ denotes α such that $j \in S_\alpha$

Note : If $S_{\alpha(j)} = \{j\}\ \forall j \in \hat{\Gamma}$, then $\mathcal{W}^{r.o.}$ is reduced to $\mathcal{W}^{opt}$.

Approximation result The window

$$\hat{A}(j) = \begin{cases} \frac{\Sigma_{p,\alpha(j)} - V_p}{\Sigma_{p,\alpha(j)}} & \text{if } \Sigma_{p,\alpha(j)} - V_p > 0 \\ 0 & \text{if } \Sigma_{p,\alpha(j)} - V_p \leq 0 \end{cases}$$

$$\Sigma_{p,\alpha} = \frac{1}{|S_\alpha|} \sum_{i \in S_\alpha} |\hat{p}(i)|^2,$$

$$V_p = n^{-1}\hat{p}(0) = n^{-2} \sum_{\gamma \in \Gamma} p(\gamma),$$

is a very efficient approximation to $\mathcal{W}^{r.o.}$, under the condition that

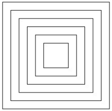
$$|S_{\alpha(j)}| \text{ is great enough in comparison with } |j| \qquad (7)$$

Filters

$\mathcal{A}^{simp}$: $S_{\alpha(j)} = \{j\}$. (7) is not fulfilled.

$\mathcal{A}^{1d}$: $S_{\alpha(j)} = \{z = (z_1, z_2) \in \hat{\Gamma},\ z_1 = j_1\}$. (7) is fulfilled.

$\mathcal{A}^{sym}$: S_α = squares centered at 0 in $\hat{\Gamma}$. (7) is fulfilled.



$\mathcal{A}^{sym}_{l_1,l_2}$: space-variant version of $\mathcal{A}^{sym}$, uses space-invariant considerations in $l_1 \times l_2$ neighborhood of each detector $\gamma \in \Gamma$. In our numerical examples $l_1 = l_2 = 8$.

Optimized analytic reconstruction (OAR)

$$Cf \backsimeq Cf_\alpha = \mathcal{N}_{a_\alpha}(\mathcal{W}p)_\alpha + \mathcal{C}h_a(\mathcal{W}p - (\mathcal{W}p)_\alpha) \qquad (8)$$

where

- $\mathcal{N}_a$ and $\mathcal{C}h_a$ are the inversion operators of formulas (5) and (4),
- $\mathcal{W}$ is a space-variant Wiener-type filter of [Guillement-Novikov 2008],
- $(\mathcal{W}p)_\alpha$ and a_α are the low-frequency parts of $\mathcal{W}p$ and a, obtained via some standard $2D$ space-invariant filtering dependent on α,
- α is an optimization parameter choosed to minimize the discrepancy $||P_a Cf_\alpha - \mathcal{W}p||_{L^2(\Gamma)}$.

The ansatz Cf_α of (8) is motivated by the facts that $\mathcal{N}_a p$ of the exact formula is sufficiently stable on sufficiently low frequency part of p and a, whereas $\mathcal{C}h_a p$ of the Chang approximate formula is sufficiently stable on reasonably high frequency part of p and a.

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Gradient

Find f wich minimizes $\|P_a f - p\|_{L^2}^2$, with p the projection data. In discrete model, this corresponds to
Find $X = \{f(x_i)\}$ that minimizes $\Phi = 1/2\|AX - Y\|_2^2$

- A the matrix corresponding to P_a
- Y the vector formed by the projections $p(\gamma)\}$.

It is a quadratic least square problem which can be treated by the gradient method, the conjugate gradient... In addition, to avoid adjustment of the noise, one can add a term for regularization like $\alpha \Lambda f.f, (\Lambda = -Laplacian)$ or filter the data.
The gradient iterations are

$$X' = X - \rho d_0, \quad d_0 = \nabla\Phi = A^*(AX - Y),\ \rho = \frac{\|d_0\|^2}{A^* A d_0 . d_0}.$$

AX and $A^* Z$ are computed as

- $AX(s,\theta) = \int e^{-\mathcal{D}_a(x(t),\theta)} X(s\theta^\perp + t\theta)dt$
- $A^* Z(x) = \int e^{-\mathcal{D}_a(x,\theta)} Z(x\theta^\perp, \theta)d\theta$

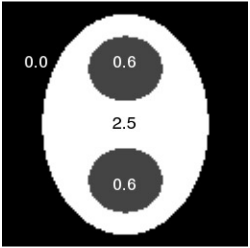
Expectation Maximization (EM)

EM is a parameter adjustment method by maximum likelihood principle. Introduced by [Shepp-Vardi 1982] in emission tomography PET, the algorithm is designed to compute the maximum of the probability $L(p|f)$ so that the emission map f generates the projections p. For that, one consider a discrete model for which $f(x)$ and $p(\gamma)$ are Poisson distributions. EM iterations for the maximum are simple and give good results in SPECT. OSEM (Ordered Subsets EM) [Hudson-Larkin 1994] accelerates the EM iterations by limiting back-projection computation on "ordered" subsets of projections.

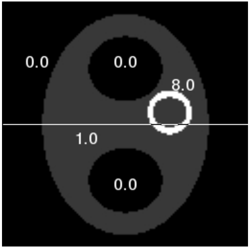
EM iterations Start with $f_0 \equiv 1$. Iteratively

- compute $p_n = P_a f_n$ and compare to supplied projections p by
- $q_n = \frac{p}{p_n}$
- update f estimation by
  $f_{n+1}(x) = f_n(x) \int e^{-\mathcal{D}_a(x,\theta)} q_n(x\theta^\perp, \theta) d\theta \frac{1}{\int e^{-\mathcal{D}_a(x,\theta)} d\theta}$

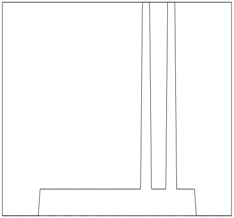
Numerical example 1



Attenuation map a, (128×128)



Emission activity f, (128×128)



Emission profile

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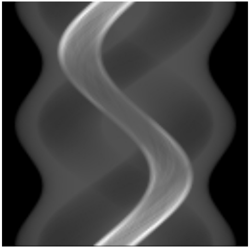
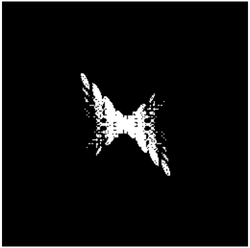
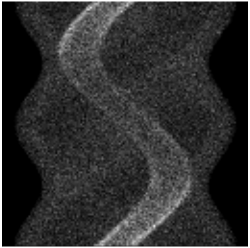
Noiseless and noisy projections



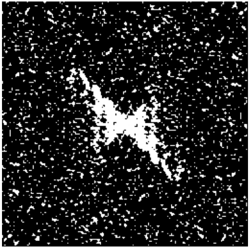
Projections g (128×128)



Spectrum $|\hat{g}|$



Projections $\|p - g\|_2 / \|g\|_2 = 30\%$



Spectrum $|\hat{p}|$

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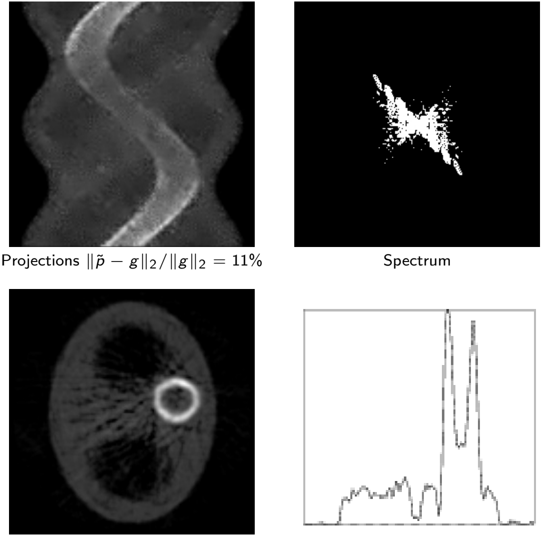
Space-variant Wiener filter $A_{8,8}^{sym}$ and O.A. reconstruction



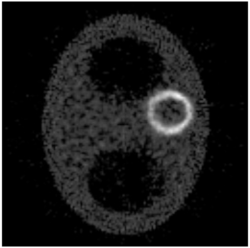
Projections $\|\tilde{p} - g\|_2 / \|g\|_2 = 11\%$

Spectrum

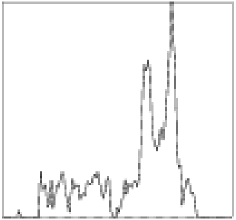
OAR : $\|r - r_0\|_2 / \|r_0\|_2 = 36\%$

Profile

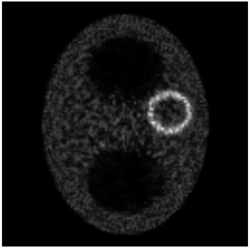
Iterative reconstructions (60 It)



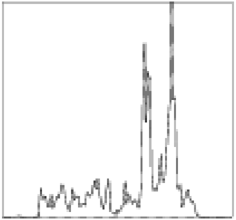
Gradient : $\|r - r_0\|_2 / \|r_0\|_2 = 43\%$



Profile

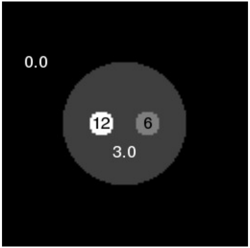


Em : $\|r - r_0\|_2 / \|r_0\|_2 = 42\%$

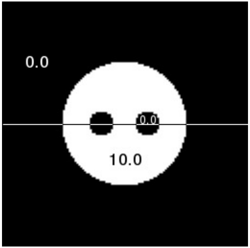


Profile

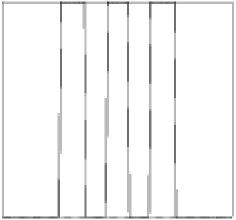
Numerical example 2



Attenuation map a, (128×128)

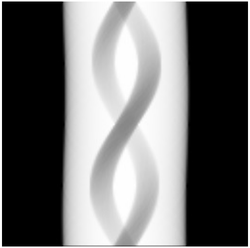


Emission activity f, (128×128)

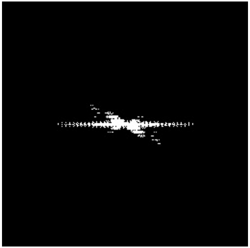


Emission profile

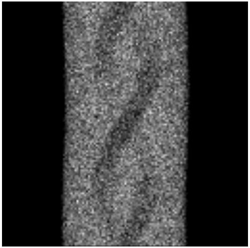
Noiseless and noisy projections



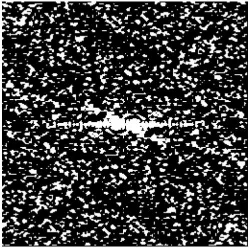
Projections g (128×128)



Spectrum $|\hat{g}|$

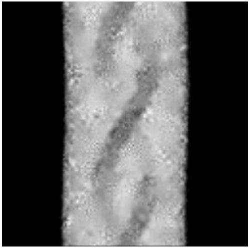


Projections $\|p - g\|_2 / \|g\|_2 = 30\%$

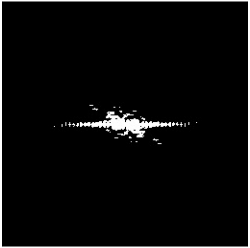


Spectrum $|\hat{p}|$

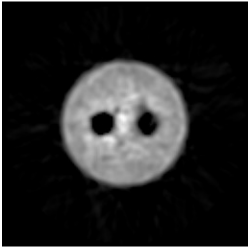
Space-variant Wiener filter $A^{sym}_{8,8}$ and O.A. reconstruction



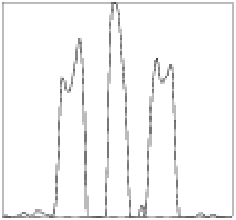
Projections $\|\tilde{p} - g\|_2 / \|g\|_2 = 10\%$



Spectrum

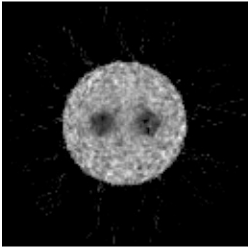


OAR : $\|r - r_0\|_2 / \|r_0\|_2 = 22\%$

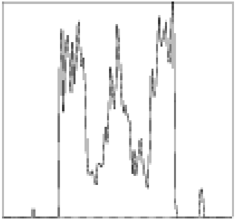


Profile

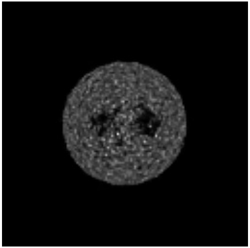
Iterative reconstructions (60 it)



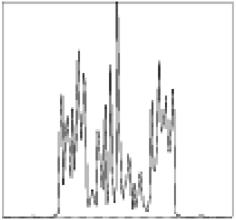
Gradient : $\|r - r_0\|_2 / \|r_0\|_2 = 24\%$



Profile



Em : $\|r - r_0\|_2 / \|r_0\|_2 = 52\%$



Profile

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