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# THE WAVE TRACE FOR RIEMANN SURFACES

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**Abstract**

We present a wave group version of the Selberg trace formula for an arbitrary surface of finite geometry. As an application we give a new lower bound on the number of resonances for hyperbolic surfaces. Motivated by recent results we formulate a conjecture on a lower bound for the counting function of resonances in a strip.

## 1 Introduction

The purpose of this note is to present the Selberg trace formula in terms of the wave group for a general hyperbolic surface, $M$, of finite geometry type. The novelty is in allowing infinite volume surfaces and in considering the consequences of the formula for resonances in that case. The study of the Selberg trace formula in this generality was previously conducted by Patterson [P2] and the first author [G].

In sect. 2 we present the geometric part of the formula and in sect. 3 we recall some facts about the spectrum and resonances which follow from the more general results of our previous work [GZ2]. Sect. 4 is then devoted to a new application to resonance counting.

We recall that as a consequence of the Poisson formula for resonances, the authors [GZ2] gave asymptotic counting estimates in disks

$$\#\big(\mathcal{R}_M \cap \{|\lambda| \leq r\}\big) \asymp r^2 .$$

for the resonance set $\mathcal{R}_M$ (each resonance must be counted according to its multiplicity ; we refer to [Z1] for a small correction to the definition [GZ2] of the multiplicity of the resonance $\lambda = 0$). Here we prove estimates in strips

$$\#\big(\mathcal{R}_M \cap \{|\lambda| \leq r, \operatorname{Im} \lambda < \varepsilon^{-1}\}\big) = \Omega(r^{1-\varepsilon}) .$$

In the discussion of related bounds, we propose a conjecture on the lower bound estimate, related to the entropy of the geodesic flow (which is the key parameter in upper estimates showed by the second author [Z2]). Recently, other results on lower bounds for resonances in situations where the classical dynamics is hyperbolic were obtained by V. Petkov [Pe] in the case of several strictly convex obstacles and by L. Farhy and V. Tsanov [FT] for surfaces.

We start by introducing the geometric and analytic notation which will be used in the paper. We will use the Poincaré half-plane model for the hyperbolic plane $\mathbf{H}^2$ defined as $\mathbf{P}^2 = \{w = u + \mathrm{i}v, v > 0\}$ with metric $v^{-2}(du^2 + dv^2)$. A cusp $C$ is isometric to the limited strip $S_h = \{0 \le u \le h, v \ge 1\}$ with the edges $\{u = 0\}$ and $\{u = h\}$ identified, its boundary being a closed horocycle of length $h$. A funnel $F$ is isometric to a half annulus $A_F = \{u \ge 0, 1 \le u^2 + v^2 \le \mathrm{e}^{2\ell_{\partial F}}\}$ with the edges $\{u^2 + v^2 = 1\}$ and $\{u^2 + v^2 = \mathrm{e}^{2\ell_{\partial F}}\}$ identified, its boundary being totally geodesic of length $\ell_F$; in Fermi coordinates with respect to the geodesic boundary $\mathcal{C} = \partial F$, the funnel is isometric to $\mathbf{R}^+_\delta \times (\mathbf{R}/\ell_{\partial F}\mathbf{Z})_\theta$ with the warped metric $d\delta^2 + \cosh^2 \delta d\theta^2$.

Let $(M, g)$ be a complete surface with a metric $g$ of constant curvature $-1$, of finite geometry type and non-elementary: $M$ is a quotient of the hyperbolic plane $\mathbf{H}^2$ by a Fuchsian group $\Gamma$ which is assumed to be torsion free, non-cyclic and with a finite presentation. The surface $M$ has a finite number of ends of two types: cusps and funnels. More precisely, $M$ is the union of the Nielsen region $N$ and $f$ funnels $F_j$, $j = 1, \ldots, f$, each attached to one of the totally geodesic boundary components $\mathcal{C}_j$ of $N$. The Nielsen region $N$, of finite area $2\pi(2g + c + f - 2)$, is the union of a compact $N_0$ (compact surface of genus $g$ and with $f + c$ boundary components) and $c$ attached cusps $C_i$, $i = 1, \ldots, c$, through their horocycle boundary $\mathcal{H}_i$. Let us denote $\mathcal{D}$ a fundamental domain for the action of $\Gamma$ on $\mathbf{H}^2$ and by $\pi$ the covering projection $\mathbf{H}^2 \to \Gamma \backslash \mathbf{H}^2 = M$.

The Laplacian, $\Delta_M$, is defined on the half-densities by $\Delta_M(u d\mathrm{vol}_M^{1/2}) = (\Delta u) d\mathrm{vol}_M^{1/2}$ where $\Delta$ is the usual Laplace-Beltrami operator acting on functions. The $L^2$-space of sections of the half-density bundle $\Omega^{1/2}M$ will be denoted by $L^2(M; \Omega^{1/2}M)$. We will use the modified spectral parameter $\lambda$, with physical sheet chosen to be $\{\mathrm{Im}\, \lambda < 0\}$: the resolvent $(\Delta_M - 1/4 - \lambda^2)^{-1}$ has finitely many poles there. The modified spectral parameter is related to the true spectral parameter, $\Lambda$, by $\Lambda = \frac{1}{4} + \lambda^2$ and to the usual parameter $s$ of hyperbolic scattering theory by $s = \frac{1}{2} + \mathrm{i}\lambda$ . The choice of $\lambda$ formulates the results similarly to those of euclidean scattering theory.

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## 2 The Wave Selberg Trace Formula

To formulate the trace formula in terms of the wave group of $M$, we introduce the 0-integral by a regularization defined in geometric terms. We recall that the finite part, $\mathrm{FP}_{\varepsilon\to 0}\, h(\varepsilon)$, is defined as $h_0$ when $h(\varepsilon) = h_0 + \sum_j h_j \varepsilon^{-\lambda_j} \log^{m_j} \varepsilon + o(1)$, $\mathrm{Re}\,\lambda_j \geq 0$, $m_j \in \mathbf{N}$ ($h_0$ is unique – see for instance [H, Lemma 3.2.1]).

Definition 1. *The regularized integral* ${}^0\!\!\int \omega$ *of a* $n$*-density* $\omega$ *on* $M$ *is defined, if it exists, as the finite part*

$$ {}^0\!\!\int_M \omega = \mathop{\mathrm{FP}}_{\varepsilon\to 0} \int_{M(\varepsilon)} \omega $$

*where* $M(\varepsilon)$ *is defined by*

$$ M(\varepsilon) = N_0 \cup \bigcup_{i=1}^{c} \left\{ m\in C_i, d(m,\mathcal{H}_i) \leq -\log\varepsilon \right\} \cup \bigcup_{j=1}^{f} \left\{ m\in F_j, d(m,\mathcal{C}_j) \leq -\log\varepsilon \right\}. $$

This was introduced in [GZ2] by analogy with the $b$-integral of Melrose [Me] which arises in the study of pseudodifferential calculus on manifolds with cylindrical ends, the $b$-calculus. The underlying pseudodifferential calculus in the infinite volume case is the 0-calculus of Mazzeo and Melrose [MMe] and hence the notation. The regularization in the cusps is very closely related to the $b$-integral. We remark that in the definition above we chose specific defining functions of the boundary at infinity. In general, the 0-integral in $n$ dimensions depends on the $n$-jet of the defining function at the boundary while the $b$-integral depends only on the 1-jet.

Let us remark that the 0-volume of a funnel $F$ is zero, as indicated by a simple calculation in Fermi coordinates:

$$ \mathrm{Vol}\left(F \cap \{d(m,\partial F) \leq -\log\varepsilon\}\right) = \ell_{\partial F} \int_0^{-\log\varepsilon} \cosh\delta\, d\delta = \ell_{\partial F} \frac{\varepsilon^{-1}-\varepsilon}{2}. $$

Hence the 0-volume of $M$ is the volume of its Nielsen region $N$.

For a smoothing operator $K$ with a half-density kernel $K$, its 0-trace is defined as the 0-integral of $\kappa = \Delta^* K$, where $\Delta$ is the diagonal embedding $M \to M \times M$. If $K$ is trace-class in its action on $L^2(M;\Omega^{1/2}M)$, its 0-trace and $L^2$-trace coincide.

Using the 0-trace we can give a wave equation version of the Selberg trace formula. It is essentially well known in the finite volume case.

**Theorem 1.** *Let $M = \Gamma\backslash\mathbf{H}^2$ be a Riemann surface of finite type with $c$ cusps, each with boundary length $h_i$, $i = 1, \ldots, c$. Let $N$ denote its Nielsen region, $\mathcal{P}_M$ the set of its primitive oriented closed geodesics $\mathcal{C}$ with length $\ell_{\mathcal{C}}$ and Poincaré map $P_{\mathcal{C}}$. Then, in the distributional sense in $t \in \mathbf{R}$,*

$$\begin{aligned} &0\text{-tr} \cos t\sqrt{\Delta_M - \tfrac{1}{4}} \\ &\quad = -\frac{\mathrm{Vol}(N)}{8\pi} \frac{\cosh(t/2)}{\sinh^2(t/2)} + \frac{1}{2} \sum_{\mathcal{C} \in \mathcal{P}_M} \sum_{k \in \mathbf{N}} \frac{\ell_{\mathcal{C}}}{|1 - P_{\mathcal{C}}^k|^{1/2}} \delta(|t| - k\ell_{\mathcal{C}}) \\ &\qquad + \frac{c}{4} \coth(|t|/2) + \left[ c(\gamma - \log 2) - \sum_{i=1}^{c} \log h_i \right] \delta(t)\,, \qquad (2.1) \end{aligned}$$

*where* $\cosh t \sinh^{-2} t$ (resp. $\coth |t|$) *is to be understood as the distributional derivative of* $-\sinh^{-1} t$ *defined as* a principal value (resp. $\mathrm{sgn}(t) \log|\sinh t|$).

*Proof.* The kernel density

$$\begin{aligned} &W_{\mathbf{H}^2}(t)(w, w') \\ &\quad = \mathrm{sgn}(t) \tfrac{1}{4\pi} \left[ \sinh^2(t/2) - \sinh^2(d(w,w')/2) \right]_+^{-1/2} d\mathrm{vol}_{\mathbf{H}^2}^{1/2}(w) d\mathrm{vol}_{\mathbf{H}^2}^{1/2}(w') \end{aligned}$$

of the wave operator $\frac{\sin t\sqrt{\Delta_{\mathbf{H}^2} - \frac{1}{4}}}{\sqrt{\Delta_{\mathbf{H}^2} - \frac{1}{4}}}$ induces the kernel $W_M(t)$ of $\frac{\sin t\sqrt{\Delta_M - \frac{1}{4}}}{\sqrt{\Delta_M - \frac{1}{4}}}$ by taking the mean on the group $\Gamma$

$$W_M(t) = \sum_{\gamma \in \Gamma} (\gamma^* \otimes \mathrm{Id}) W_{\mathbf{H}^2}(t)$$

with 0-trace $w_M(t) = 0\text{-tr}\, W_M(t)$ such that

$$\begin{aligned} w_M(t) &= \frac{\mathrm{sgn}(t)}{4\pi} \sum_{\gamma \in \Gamma} {}^0\!\!\int_{\mathcal{D}} \left[ \sinh^2(t/2) - \sinh^2(d(\gamma w, w)/2) \right]_+^{-\frac{1}{2}} d\mathrm{vol}_{\mathbf{H}^2}(w) \\ &= \frac{\mathrm{sgn}(t)}{4\pi} \operatorname*{FP}_{\varepsilon \to 0} \sum_{\gamma \in \Gamma} \int_{\mathcal{D}(\varepsilon)} [\sinh^2(t/2) - \sinh^2(d(\gamma w, w)/2)]_+^{-\frac{1}{2}} d\mathrm{vol}_{\mathbf{H}^2}(w)\,, \qquad (2.2) \end{aligned}$$

where $\mathcal{D}(\varepsilon)$ is the lift in $\mathcal{D}$ of $M(\varepsilon)$.

Let us consider $t$ non-negative. To evaluate $w_M(t)$, as usual in Selberg trace formula calculations, the different terms are grouped in partial sums along conjugacy class, $w_{[\gamma]}(\varepsilon, t)$ corresponding in (2.2) to the conjugacy class $[\gamma] \simeq \Gamma/Z(\gamma)$ where $Z(\gamma)$ is the centralizer of $\gamma$ in $\Gamma$. Using the invariance of the kernel $W_{\mathbf{H}^2}(t)$, we have

$$w_{[\gamma]}(\varepsilon,t) = \frac{1}{4\pi}\sum_{\tilde\gamma\in[\gamma]}\int_{\mathcal{D}(\varepsilon)}\left[\sinh^2(t/2)-\sinh^2(d(\tilde\gamma w,w)/2)\right]_+^{-\frac12} d\mathrm{vol}_{\mathbf{H}^2}(w) \tag{2.3}$$

$$= \frac{1}{4\pi}\int_{\mathcal{M}_\gamma(\varepsilon)}\left[\sinh^2(t/2)-\sinh^2(d(\gamma w,w)/2)\right]_+^{-\frac12} d\mathrm{vol}_{\mathbf{H}^2}(w)\,. \tag{2.4}$$

The set $\mathcal{M}_\gamma(\varepsilon)$ converges, for $\varepsilon\to0$, to a fundamental domain $\mathcal{D}(Z(\gamma))$ for the action of $Z(\gamma)$ on the hyperbolic plane. The contribution of the trivial element in (2.2) is $\mathrm{Vol}(N)/(4\pi\sinh(t/2))$, since $\mathrm{FP}_{\varepsilon\to0}\mathrm{Vol}(M(\varepsilon)) = \mathrm{Vol}(N)$.

The sum $w_{[\gamma_0^k]}(\varepsilon,t)$ with $\gamma_0^k$ the $k$-th power of the primitive hyperbolic $\gamma_0$ (conjugate to the isometry $w\to \mathrm{e}^{\ell_{\gamma_0}}w$) gives an integrand summable on $\mathcal{D}(Z(\gamma))$ with the sum

$$\operatorname*{FP}_{\varepsilon\to0} w_{[\gamma_0^k]}(\varepsilon,t) = w_{[\gamma_0^k]}(0,t) = \frac{\ell_{\gamma_0}}{4\sinh(|k|\ell/2)}\left[|t|-|k|\ell_{\gamma_0}\right]_+^0 .$$

Through the correspondence between the primitive conjugacy classes $[\gamma_0]$ of hyperbolic elements and the primitive closed oriented geodesics $\mathcal{C}$ (of length $\ell_\mathcal{C}=\ell_{\gamma_0}$ and Poincaré map $P_\mathcal{C}$ with eigenvalues $e^{\pm\ell_\mathcal{C}}$, so that the determinant $|1-P_\mathcal{C}^k|$ is equal to $4\sinh^2(k\ell_\mathcal{C}/2)$), the contributions of hyperbolic elements to $w_M(t)$ can be written as

$$\frac12\sum_{\mathcal{C}\in\mathcal{P}_M}\sum_{k\in\mathbf{N}}\frac{\ell_\mathcal{C}}{|1-P_\mathcal{C}^k|^{1/2}}\left[|t|-k\ell_\mathcal{C}\right]_+^0 .$$

For a primitive parabolic element $\gamma_i$ associated to the cusp $C_i$, the terms corresponding to the conjugates of $\gamma_i^k, k\in\mathbf{Z}\setminus\{0\}$, give the sum

$$I_i\big(\varepsilon,\sinh(t/2)\big) = \sum_{k\in\mathbf{Z}\setminus\{0\}} w_{[\gamma_i^k]}(\varepsilon,t)$$

$$= \frac{1}{2\pi}\int_{\mathcal{M}_{\gamma_i}(\varepsilon)}\sum_{k\in\mathbf{N}}\left[\sinh^2(t/2)-\left(\frac{kh_i}{v}\right)^2\right]_+^{-1/2} d\mathrm{vol}_{\mathbf{H}^2}(w)\,. \tag{2.5}$$

The sets $\mathcal{M}_{\gamma_i}(\varepsilon)$ and $\{0\le u\le h_i,\, v\le\varepsilon^{-1}\}$ differ, modulo $\Gamma$, by a set where the integrand in (2.5) is summable, with sum vanishing when $\varepsilon\to0$. Hence $I_i(\varepsilon,s)$ has the same finite part as

$$J_i(\varepsilon,s) = \frac{h_i}{2\pi}\int_0^{\varepsilon^{-1}}\sum_{k\in\mathbf{N}}\left[s^2-\left(\frac{kh_i}{v}\right)^2\right]_+^{-1/2}\frac{dv}{v^2} \tag{2.6}$$

which is

$$\operatorname*{FP}_{\varepsilon\to 0} J_i(\varepsilon,s) = \frac{\log|s|}{2} + \frac{\gamma-\log h_i}{2} - \frac{\log 2}{2}\,,$$

with $\gamma$ the Euler constant.

The theorem results from taking the distribution derivative of the trace formula,

$$\text{0-tr}\,\frac{\sin t\sqrt{\Delta_M-\frac14}}{\sqrt{\Delta_M-\frac14}} = \frac{\operatorname{Vol}(N)}{4\pi\sinh(t/2)} + \frac{\operatorname{sgn}(t)}{2}\sum_{\mathcal{C}\in\mathcal{P}_M}\sum_{k\in\mathbf{N}}\frac{\ell_{\mathcal{C}}}{|1-P_{\mathcal{C}}^k|^{1/2}}\big[|t|-k\ell_{\mathcal{C}}\big]_+^0$$
$$+\frac{c}{2}\operatorname{sgn}(t)\log\big|\sinh(t/2)\big| + \frac{\operatorname{sgn}(t)}{2}\Big[c(\gamma-\log 2)-\sum_{i=1}^{c}\log h_i\Big]. \quad\square$$

## 3 Spectral Theory and Resonances

The Laplacian $\Delta_M$ has $[1/4,\infty)$ as continuous spectrum, each cusp contributing with multiplicity 1 and each funnel with infinite multiplicity. If there are funnels, its discrete spectrum, $\sigma_{\mathrm{pp}}(\Delta_M)$, is finite and included in $(0,1/4)$. If there are funnels and cusps, there are eigenvalues and the least one is of the form $\delta(1-\delta)$ with $\delta$ the Poincaré exponent of the group $\Gamma$ or the entropy of the geodesic flow.

The finer analysis of the spectrum is carried out by considering the resolvent acting on spaces of distributions. In term of the parameter $\lambda$ related to the actual spectral parameter $\Lambda$ by $\Lambda=\frac14+\lambda^2$, the meromorphic $L^2(M;\Omega^{1/2}M)$-bounded operator valued function defined on the *physical sheet* $\{\operatorname{Im}\lambda<0\}$

$$R_M(\lambda)=\big(\Delta_M-\tfrac14-\lambda^2\big)^{-1}: L^2(M;\Omega^{\frac12}M)\to L^2(M;\Omega^{\frac12}M)$$

admits a meromorphic extension [GZ1] to $\mathbf{C}$ when the values of the function $R_M(\lambda)$ are taken as operators between the space of smooth compactly supported half-densities $\mathcal{C}_0^\infty(M;\Omega^{1/2}M)$ to the space of distributional sections $\mathcal{C}^{-\infty}(M;\Omega^{1/2}M)$.

The singularities in $\mathbf{C}$ of the function $R_M$ define the resonance set $\mathcal{R}_M$ of the surface $M$: it is proved in [GZ2] that the polar parts at any resonance $\lambda_0$ is built from operators with finite rank, so the multiplicity $m_{\lambda_0}$ is defined as the dimension of the sum of the ranges of the $A_k$ where $R_M(\lambda)=\sum_{k=1}^N A_k(\lambda-\lambda_0)^{-k}+H_{\lambda_0}(\lambda)$ with $H_{\lambda_0}$ holomorphic in a neighbourhood of $\lambda_0\neq 0$. When $\lambda_0=0$ that dimension has to be taken twice – see [Z1].

The counting function $N_M$ of the resonance set $\mathcal{R}_M$ is defined by

$$N_M(r) = \sum_{\lambda \in \mathcal{R}_M \cap \{|\lambda| \leq r\}} m_\lambda = \#\big(\mathcal{R}_M \cap \{|\lambda| \leq r\}\big)\,,$$

where the elements in $\mathcal{R}_M$ appear in the last member according to their multiplicities (which is the usual abuse of notation also observed here).

In [GZ2] we obtained the Poisson formula for resonances:

$$0\text{-tr}\cos t\sqrt{\Delta_M - \tfrac{1}{4}} = \tfrac{1}{2}\sum_{\lambda \in \mathcal{R}_M} \mathrm{e}^{\mathrm{i}\lambda|t|}\,, \quad t \neq 0\,, \tag{3.1}$$

which combined with Theorem 1 gives a trace formula relating resonances spectrum and closed geodesic length spectrum.

To sketch the proof of the Poisson trace formula (3.1) we need to recall some definitions related to the continuous spectrum. The Eisenstein eigenfunctions $E_M(\lambda)$ are, for a hyperbolic surface $M$ and up to simple factors, meromorphic extensions of Eisenstein series defined on the physical sheet. Let $P$ denote the Poisson kernel defined on $\partial_\infty \mathbf{H^2} \times \mathbf{H}^2$, explicitly in the Poincaré model by

$$P(x,w) = \frac{\operatorname{Im} w}{(\operatorname{Re} w - x)^2 + (\operatorname{Im} w)^2}\,, \quad (x,w) \in \mathbf{R} \times \mathbf{P}^2\,.$$

The Eisenstein eigenfunctions (in fact sections of the half-density bundle $\Omega^{1/2}M$) are defined in [GZ2] as renormalized boundary values of the resolvent kernel $R_M(\lambda)(m,m')$ as $m'$ go to infinity: they are parametrized by the set of points at infinity of $M$, which is build by one point (at infinity) for each cusp and the funnel boundary $\partial_F M = \sqcup_{j=1}^f \partial_\infty F_j$ where $\partial_\infty F$ is the boundary at infinity of the funnel $F$.

The Eisenstein function corresponding to the cusp $C$ is given by

$$\begin{aligned} E_M(\lambda, C, m) &= \big[\mathrm{e}^{-(\frac{1}{2}+\mathrm{i}\lambda)d(m',\partial C)} R_M(\lambda)(m,m')\mathrm{dvol}_M^{-\frac{1}{2}}(m')\big]_{m' \in C, d(m',\partial C) \to \infty} \\ &= \tfrac{1}{2\mathrm{i}\lambda}\Big(\sum_{\gamma \in \Gamma/\Gamma_{\xi_i}} P^{\frac{1}{2}+\mathrm{i}\lambda}(\xi, \gamma w)\Big)\ \mathrm{dvol}_M^{1/2}(m)\,, \end{aligned}$$

if $\xi \in \overline{\mathcal{D}}$, with stabilizer $\Gamma_\xi$ in $\Gamma$, goes down via the projection $\pi$ on the point at infinity of the cusp $C$ of $M$ and $w \in \mathbf{P}^2$ projects through $\pi$ on $m \in M$.

To describe the Eisenstein eigenfunctions corresponding to the boundary $\partial_\infty F$ of a funnel $F$, let us recall that the funnel is a conformal compact manifold with boundary: in the coordinates $(\rho, \theta) \in (0, \pi/2)_\rho \times (\mathbf{R}/\ell_{\partial F}\mathbf{Z})_\theta$

with $\sin\rho = \cosh^{-1}\delta$, the hyperbolic metric is $\sin^{-2}\rho(d\rho^2 + d\theta^2)$. Let us denote $\widetilde{F}(\varepsilon) = \{m \in F, \rho(m) \geq \varepsilon\}$ and $|N^*\partial\widetilde{F}(\varepsilon)|^\sigma$ the bundle of $\sigma$-conormal densities on $\partial\widetilde{F}(\varepsilon)$, which is trivialized by the section $|d\rho|^\sigma$. The Eisenstein eigenfunction $E_M(\lambda, F, m)$, section of the bundle $\Omega^{1/2}_{\partial_\infty F} \otimes |N^*\partial_\infty F|^{\mathrm{i}\lambda}$ with base $\partial_\infty F$, is given by

$$E_M(\lambda,\xi,m) = \left[R_M(\lambda)(m,m')\left|\frac{d\rho(m')}{\rho(m')}\right|^{-\frac{1}{2}+\mathrm{i}\lambda}\right]_{m'\in F, d(m',\partial F)\to\infty}$$
$$= \frac{1}{2\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\mathrm{i}\lambda)}{\Gamma(1+\mathrm{i}\lambda)}\mathrm{e}^{\theta(\frac{1}{2}+\mathrm{i}\lambda)}\Big(\sum_{\gamma\in\Gamma}P^{\frac{1}{2}+\mathrm{i}\lambda}(\mathrm{e}^\theta, w)\Big)d\mathrm{vol}_M^{\frac{1}{2}}(m)d\theta^{\frac{1}{2}}|d\rho|^{\mathrm{i}\lambda},$$

where $w \in \mathbf{P}^2$ projects by $\pi$ on $m$ and $\xi \in \partial_\infty F \simeq \mathbf{R}/\ell_{\partial F}\mathbf{Z}$ has coordinate $\theta$: here it is assumed that the fundamental domain $\mathcal{D}$ (represented in the Poincaré plane) covers the funnel $F$ by the half-annulus $A_F$, with the boundary $\partial_\infty F$ corresponding to the action of the isometry $w \to \mathrm{e}^{\ell_F}w$ of $\Gamma$ on $\{u > 0\}$.

The Eisenstein eigenfunction $E_M(\lambda)$ is the set of these eigenfunctions

$$E_M(\lambda; m) = \big((E_M(\lambda, C_i, m))_{i=1}^c, (E_M(\lambda, \xi_j, m))_{j=1}^f)\big)$$
$$\in \mathbf{C}^c \oplus \mathcal{C}^\infty\big(\partial_\infty F; \Omega^{1/2}_{\partial_\infty F} \otimes |N^*\partial_F M|^{\mathrm{i}\lambda}\big)$$

for which the scattering matrix is an intertwining operator for the opposite values of the spectral parameter $\lambda$ (see Sect. 2 of [GZ2] for more details):

$$S_M(\lambda)E_M(-\lambda; m) = E_M(\lambda; m)\,.$$

We define the "free" scattering matrix, which is the identity on $\mathbf{C}^c$ and the scattering matrix on each of the free funnels, $F_j$ (with Dirichlet boundary conditions on the neck $\partial F_j$):

$$S_{c,F}(\lambda) = \mathrm{Id}_{\mathbf{C}^c} \oplus \bigoplus_{j=1}^f S_{F_j^0}(\lambda)\,.$$

The relative scattering matrix is then given by

$$\widetilde{S}_M(\lambda) = S_{c,F}(\lambda)^{-1}S_M(\lambda)\,.$$

It is of the form $I + A_M(\lambda)$ where $A_M(\lambda)$ is a trace class operator. Hence we can define the determinant and the scattering phase:

$$\tau_M(\lambda) = \det \widetilde{S}_M(\lambda)\,,$$
$$\sigma_M(\lambda) = \tfrac{\mathrm{i}}{2\pi}\log\tau_M(\lambda)\,.$$

Then the Birman-Krein type formula (see Proposition 4.5 of [GZ2]) is

$$0\text{-tr}\cos t\sqrt{\Delta_M - \tfrac{1}{4}} = \tfrac{1}{2}\widehat{\tfrac{d\sigma_M}{d\lambda}}(t) + \sum_{\frac{1}{4}\neq\frac{1}{4}+\lambda^2\in\sigma_{\mathrm{pp}}(\Delta_M)} \cos(t\lambda) + \tfrac{1}{2}m_0\,,$$

where we normalize $\lambda$'s in the summation by demanding that $\operatorname{Im}\lambda \geq 0$, $\operatorname{Re}\lambda \geq 0$ and where $m_0$ is the multiplicity at 0. The Poisson formula follows, once some control on the growth of the scattering determinant is obtained.

We remark that in the above formula the need for the relative scattering matrix arises only when $f \neq 0$ and that in that case only finitely many eigenvalues, all with $\operatorname{Re}\lambda = 0$, can appear.

## 4 Lower Bounds on the Number of Resonances

We would like to use the formulæ of sect. 2 and 3 to establish some new lower bounds on the number of resonances for convex co-compact surfaces. In applying ideas originating in the work of Ikawa [I] and of Sjöstrand and the second author [SZ] we take advantage of the exactness of the dynamical side of the Poisson formula.

We first recall the existing lower bounds. A general Tauberian argument of [SZ] applied as in that paper, or more naturally using the formula presented here, shows that for every $d > 0$, a sufficiently small $\varepsilon > 0$ and $\rho > 2/(d-\varepsilon^2)$ we have

$$\sum_{\substack{\lambda\in\mathcal{R}_M\cap\{|\operatorname{Re}\lambda|\leq r\}\\ |\operatorname{Im}\lambda|\leq\rho\log|\lambda|}} \exp\big(-(d-\varepsilon)\operatorname{Im}\lambda\big) > \big(B - o_{d,\varepsilon}(1)\big)r\,,\ \ B = \frac{1}{\pi}\sum_{k\ell_{\mathcal{C}}=d}\frac{\ell_{\mathcal{C}}}{|1-P_{\mathcal{C}}^k|^{1/2}}.$$

This is valid for any Riemann surface of finite geometry type (hence the statement is slightly more general than that in [SZ]). If we can control singularities in the trace then a similar result remains valid for more general surfaces of the type considered in [GZ2]. In [GZ1] and [GZ2] we provided global bounds

$$r^2/C \leq N_M(r) \leq Cr^2\,,\ r > C\,, \tag{4.1}$$

with $N_M(r) \stackrel{\text{def}}{=} \#(\mathcal{R}_M \cap \{|\lambda| \leq r\})$. To formulate the new result we recall the Hardy-Littlewood notation: $f(r) = \Omega(g(r))$ if and only if there does *not* exist a constant $C$ for which $|f(r)| \leq C|g(r)|$.

**Theorem 2.** *Let $M$ be a hyperbolic surface of finite geometry. For any* $\varepsilon \in (0,1/2)$

$$\#\big(\mathcal{R}_M \cap \{|\lambda| \leq r, \operatorname{Im}\lambda < \varepsilon^{-1}\}\big) = \Omega(r^{1-\varepsilon})\,. \tag{4.2}$$

*Proof.* Let us put $\varphi_{\gamma,d}(t) = \varphi(\gamma^{-1}(t-d))$ for $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$ satisfying

$$\varphi(0) > 0\,, \quad \varphi \geq 0\,, \quad \operatorname{supp}\varphi \subset [-1,1]\,.$$

To apply the Poisson formula (3.1), we take $d \geq 1$ and $\gamma \leq 1$ (so that $\varphi_{\gamma,d}$ has support in $\mathbf{R}^+$), $d$ will be taken in the length spectrum of $M$ and $\gamma$ chosen small. The Poisson formula gives

$$0\text{-tr} \int_0^\infty \varphi_{\gamma,d}(t) \cos t\sqrt{\Delta_M - \tfrac{1}{4}}dt = \tfrac{1}{2} \sum_{\lambda \in \mathcal{R}_M} \widehat{\varphi}_{\gamma,d}(-\lambda)\,. \tag{4.3}$$

The left-hand side is analysed using Theorem 1 which gives

$$\begin{aligned} 0\text{-tr} &\int_0^\infty \varphi_{\gamma,d}(t) \cos t\sqrt{\Delta_M - \tfrac{1}{4}}dt \\ &= \sum_{|k\ell_{\mathcal{C}} - d| < \gamma} \frac{\ell_{\mathcal{C}}}{|1 - P_{\mathcal{C}}^k|^{\frac{1}{2}}} \varphi_{\gamma,d}(k\ell_{\mathcal{C}}) - \operatorname{Vol}(N) \int_0^\infty \frac{\cosh(t/2)}{8\pi\sinh^2(t/2)} \varphi_{\gamma,d}(t)dt \\ &\qquad + \frac{c}{4} \int_0^\infty \frac{\varphi_{\gamma,d}(t)}{\tanh(t/2)}dt\,. \end{aligned}$$

The unique negative term in the sum is equal to

$$\gamma \operatorname{Vol}(N) \int_{-1}^1 \varphi(s) \left( \frac{\cosh(t/2)}{8\pi\sinh^2(t/2)} \right)_{|t = d + s\gamma} ds = \mathcal{O}(\gamma) \mathrm{e}^{-\frac{d}{2}}\,.$$

Forgetting all the non-negative terms, except the one corresponding to the length $d$, we obtain the lower bound

$$\begin{aligned} 0\text{-tr} \int_0^\infty \varphi_{\gamma,d}(t) \cos t\sqrt{\Delta_M - \tfrac{1}{4}} &\geq C^{-1} \mathrm{e}^{-\frac{d}{2}}(1 + \mathcal{O}(\gamma)) - \mathcal{O}(\gamma)\mathrm{e}^{-\frac{d}{2}} \\ &\geq C^{-1}\mathrm{e}^{-\frac{d}{2}}(1 - \mathcal{O}(\gamma))\,. \end{aligned} \tag{4.4}$$

By $C$ we will denote below a large, but not necessarily the same, constant.

On the other hand, noting that

$$\big|\widehat{\varphi}_{\gamma,d}(\zeta)\big| = \big|\gamma\widehat{\varphi}(\gamma\zeta)\mathrm{e}^{-\mathrm{i}d\zeta}\big| \leq C_M \gamma \mathrm{e}^{(d \pm \gamma)\operatorname{Im}\zeta} \big(1 + |\gamma\zeta|\big)^{-M}\,, \quad \pm \operatorname{Im}\zeta \geq 0\,,$$

for all $M$, we see that

$$\Bigg| \sum_{\lambda \in \mathcal{R}_M \cap \{\operatorname{Im}\lambda \geq \alpha\}} \widehat{\varphi}_{\gamma,d}(-\lambda) \Bigg| \leq C_M \gamma \mathrm{e}^{-\alpha(d-\gamma)} \int_0^\infty (1 + \gamma r)^{-M} dN_M(r)\,.$$

By using the global upper bound (4.1) we obtain

$$\Big|\sum_{\lambda\in\mathcal{R}_M\cap\{\mathrm{Im}\,\lambda\geq\alpha\}}\widehat{\varphi}_{\gamma,d}(-\lambda)\Big|\leq C\gamma^{-1}\mathrm{e}^{-\alpha d}\,. \tag{4.5}$$

Assume now that

$$P(\alpha,r)=\#\big(\mathcal{R}_M\cap\{|\lambda|\leq r,\mathrm{Im}\,\lambda<\alpha\}\big)\leq P_\varepsilon(\alpha)r^{1-\varepsilon}$$

so that

$$\Big|\sum_{\lambda\in\mathcal{R}_M\cap\{\mathrm{Im}\,\lambda<\alpha\}}\widehat{\varphi}_{\gamma,d}(-\lambda)\Big|\leq C\gamma\int_0^\infty(1+\gamma r)^{-M}P(\alpha,dr)$$
$$+\sum_{\lambda\in\mathcal{R}_M\cap\{\mathrm{Im}\,\lambda\leq 0\}}\big|\widehat{\varphi}_{\gamma,d}(-\lambda)\big|\leq C\gamma^\varepsilon+C\gamma\mathrm{e}^{\frac{d+\gamma}{2}}\,. \tag{4.6}$$

Here we used the fact that there are only a finite number of resonances $\lambda$ with $\mathrm{Im}\,\lambda\leq 0$, in which case $-\mathrm{Im}\,\lambda\leq\sup(\delta-\frac{1}{2},0)\leq\frac{1}{2}$. Combining (4.4) with (4.5), (4.6) in (4.3) we obtain

$$C^{-1}\mathrm{e}^{-d/2}\big(1-\mathcal{O}(\gamma)\big)\leq C\gamma^{-1}\mathrm{e}^{-\alpha d}+C\gamma^\varepsilon+C\gamma\mathrm{e}^{d/2}\,.$$

If we choose $\gamma=\mathrm{e}^{-\beta d}$, then

$$C^{-1}\mathrm{e}^{-\frac{d}{2}}\big(1-\mathcal{O}(\gamma)\big)\leq C\mathrm{e}^{-(\alpha-\beta)d}+C\mathrm{e}^{-\varepsilon\beta d}+C\mathrm{e}^{-d(\beta-\frac{1}{2})}$$

which gives a contradiction if $\varepsilon\beta>1/2$, $\alpha-\beta>1/2$ and $\beta>1$ (which is implied by $\varepsilon\beta>\frac{1}{2}$ as $\varepsilon<\frac{1}{2}$). Hence we can take any $\alpha>\frac{1}{2}(1+\varepsilon^{-1})$. □

Remark. The lower bound (4.2) is based on the existence of only one closed geodesic (and its iterates). For the hyperbolic plane, a non-trapping example, we only have a finite number of resonances in every strip: $\mathcal{R}_{\mathbf{H}^2}=\mathrm{i}\big(\frac{1}{2}+\mathbf{N}_0\big)$ ($\mathrm{i}\big(\frac{1}{2}+k\big)$ with multiplicity $2k+1$): by explicit calculations the exact trace formula

$$0\text{-tr}\cos t\sqrt{\Delta_{\mathbf{H}^2}-\tfrac{1}{4}}=-\,0\text{-Vol}(\mathbf{H}^2)\frac{\cosh(t/2)}{8\pi\sinh^2(t/2)}=\frac{1}{2}\sum_{\lambda\in\mathcal{R}_{\mathbf{H}^2}}\mathrm{e}^{\mathrm{i}\lambda|t|}\,,\quad t\neq 0$$

is valid, where $0\text{-Vol}(\mathbf{H}^2)=-2\pi$ is defined, for any base-point $m_0$, by

$$0\text{-Vol}(\mathbf{H}^2)=\operatorname*{FP}_{\varepsilon\to 0}\mathrm{Vol}\left(\{d_{\mathbf{H}^2}(m_0,m)\leq-\log\varepsilon\}\right).$$

We note that this exact trace formula allows an extension of the validity of (3.1) to surfaces whose hyperbolic ends are compact perturbations of cusps, funnels or the hyperbolic plane (in [GZ2] only cusps and funnels were considered).

The result presented in Theorem 2 is very weak. We expect the following stronger result:

CONJECTURE. *There exists* $\alpha > 0$ *such that*

$$\#\big(\mathcal{R}_M \cap \{|\lambda| \leq r, \operatorname{Im}\lambda < \alpha\}\big) \geq \tfrac{r^{1+\delta}}{C}\,, \quad r > C\,, \tag{4.7}$$

*where* $\delta = \dim \Lambda(\Gamma)$ *with* $\Lambda(\Gamma)$ *the limit set of the group* $\Gamma$.

In [Z2], the second author proved an upper bound $Cr^{1+\delta}$ for the left hand side of (4.7) when $\Gamma$ has only hyperbolic elements. The only example at the moment showing that this is optimal is that of a hyperbolic cylinder where $\delta = 0$ and the bound certainly holds (see [GZ1]). We recall that following the work of Patterson [P1] and Sullivan [Su] $M$ can be thought of, from the dynamical point of view, as a hyperbolic manifold of dimension $1+\delta$, which is half of the dimension of its trapped set in the cotangent bundle. If we think of resonances in a strip as the analogues of eigenvalues then the exponent $1+\delta$ should appear naturally in their Weyl law.

We conclude with a simple example which indicates that neglecting finer aspects of the distribution of the length spectrum (length of closed geodesics) and of the coefficients in the trace formula will not give the conjecture, at least when $\delta < 1/2$:

REMARK. Let $\{\ell_{\mathcal{C}}\} = \mathbf{N}$ considered as a hypothetic length spectrum where each length $n$ appears with (pseudo-)multiplicity $\mathrm{e}^{\delta n}$ so that

$$\tfrac{\mathrm{e}^{\delta r}}{\delta} \leq \#\{\ell_{\mathcal{C}} \leq r\} \leq \tfrac{\mathrm{e}^{\delta(r+1)}}{\delta}\,. \tag{4.8}$$

Then by (the usual) Poisson formula

$$\sum_{\ell_{\mathcal{C}}} \mathrm{e}^{-\ell_{\mathcal{C}}/2}\delta(t-\ell_{\mathcal{C}}) = \sum_{j\in\mathbf{N}} \mathrm{e}^{\mathrm{i}\lambda_j t}\,, \quad t > 0\,.$$

where $\lambda_j = 2\pi j + \mathrm{i}(\frac{1}{2}-\delta)$. Hence the $\lambda_j$'s have a linear growth in a fixed strip while the left hand side has the basic qualitative properties (e.g. (4.8)) of the geometric side of the trace formula.

## References


[FT] L. FARHY, V. TSANOV, Strips with infinite number of zeros for a class of Selberg zeta functions, preprint, 1998.

[G] L. GUILLOPÉ, Fonctions zêta de Selberg et surfaces de géométrie finie, Adv. Stud. Pure Math. 21 (1992), 33–70.

[GZ1] L. GUILLOPÉ, M. ZWORSKI, Upper bounds on the number of resonances for non-compact Riemann surfaces, J. Funct. Anal. 129 (1995), 364–389.

[GZ2] L. Guillopé, M. Zworski, Scattering asymptotics for Riemann surfaces, Ann. of Math. 145 (1997), 597–660.

[H] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Springer Verlag, Berlin, 1983.

[I] M. Ikawa, On the existence of poles of the scattering for several convex bodies, Proc. Japan Acad. 64 (1988), 91–93.

[MMe] R. Mazzeo, R.B. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, J. Funct. Anal. 75 (1987), 260–310.

[Me] R.B. Melrose, The Atiyah-Patodi-Singer Index Theorem, A K Peters, Wellesley, 1994.

[P1] S.J. Patterson, The limit set of a Fuchsian group, Acta Math. 136 (1976), 241–273.

[P2] S.J. Patterson, The Selberg zeta function of a Kleinian group, in "Number Theory, Trace Formulas and Discrete Group," Academic Press, Boston (1989), 409–441,

[Pe] V. Petkov, Distribution of scattering poles for several strictly convex obstacles, preprint, 1997.

[SZ] J. Sjöstrand, M. Zworski, Lower bounds on the number of scattering poles, Comm. Partial Differential Equations 18 (1993), 847–857.

[Su] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, Publ. Math. Inst. Hautes Étud. Sci. 50 (1979), 171–202.

[Z1] M. Zworski, Poisson formulæ for resonances, Séminaire É.D.P., École Polytechnique, XIII-1–12, 1996-1997.

[Z2] M. Zworski, Dimension of the limit set and the density of resonances for convex co-compact hyperbolic quotients, Inventiones Math. 136 (1999), 353–409.



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