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Upper Bounds on the Number of Resonances for Non-compact Riemann Surfaces

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Let X be a Riemannian surface of finite geometric type and with hyperbolic ends. The resolvent $(\Delta^X - s(1-s))^{-1}$, $\operatorname{Re} s > 1$ of the Laplacian on X extends to a meromorphic family of operators on $\mathbf{C}$ and its poles are called resonances. We prove an optimal polynomial bound for their counting function. © 1995 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The purpose of this note is to provide upper bounds on the number of poles of the meromorphic continuation of the resolvent of the Laplacian on non-compact Riemann surfaces $\mathbf{H}^2/\Gamma$ with Γ a discrete isometry group, without torsion and of finite type. In fact we consider the slightly more general class of two dimensional Riemannian manifolds (see Fig. 1) which are obtained from the preceding Riemannian surfaces by compactly supported perturbation of the metric and are similar to the ones studied in [4, 11, 16]. Although the bound is rather far from the precise results in the finite volume case [16, 25], it seems new when the volume is infinite.

The proof is based on Vodev's impressive refinement of the Fredholm determinant method which was used to obtain optimal polynomial bounds, first by Melrose [15] for the obstacle problem and then by Zworski [34] for the Schrödinger equation. It was applied in [30] to give a different proof of the general bounds on the number of poles obtained by Sjöstrand

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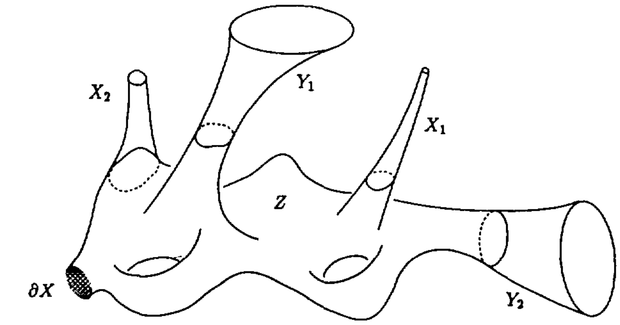

FIG. 1. A Riemannian surface X.

and Zworski [27] in the Euclideans odd dimensional case and to extend them to global bounds in the even dimensions [31, 32]—see [12] for the first results in that direction. The facts about the spectral theory on infinite volume Riemann surfaces come from [9, 11], see also [19], and we remark that the crucial computation of the model scattering matrix was also carried out by Epstein [5].

In higher dimensions ($n > 2$) the meromorphic continuation of the resolvent was studied with great success in [2, 6, 7, 13, 14, 20–22], but our method applies at the moment to a limited class of examples (see Remark 1) yielding, however, the optimal bound $\mathcal{O}(r^n)$.

Let (X, g) (see (Fig. 1) be a complete two dimensional Riemannian manifold (with a compact boundary) with a decomposition

$$X = Z \cup X_1 \cup \cdots \cup X_M \cup Y_1 \cup \cdots \cup Y_N, \tag{1.1}$$

where Z is a compact manifold with boundary, $\partial Z = \partial X \cup \partial X_1 \cup \cdots \cup \partial X_M \cup \partial Y_1 \cup \cdots \cup \partial Y_N$ and each X_i is isometric to

$$X_i \simeq [a_i, \infty)_r \times (\mathbf{R}/h_i \mathbf{Z})_t, \qquad g|_{X_i} \simeq dr^2 + e^{-2r}\, dt^2, a_i > 0, h_i > 0, \tag{1.2}$$

and each Y_j to

$$Y_j \simeq [b_j, \infty)_r \times (\mathbf{R}/l_j \mathbf{Z})_t, \qquad g|_{Y_j} \simeq dr^2 + \cosh^2 r\, dt^2, b_j > 0, l_j > 0. \tag{1.3}$$

We will denote the Laplacian on $L^2(X)$ by Δ with standard boundary conditions (Dirichlet, Robin,...) on the boundary ∂X. The following theorem is essentially well known, with an easy proof obtained by adapting the proof of Theorem 1.1 of [27] to this setting (see Section 5):

THEOREM 1. *The resolvent*

$$(\Delta - s(1-s))^{-1} : L^2(X) \to H^2(X)$$

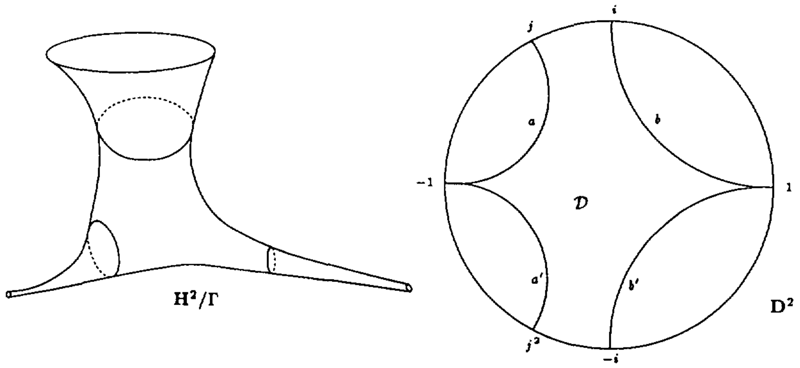

FIG. 2. A Riemann surface $\mathbf{H}^2/\Gamma$. The fundamental domain $\mathscr{D}$ of Γ in the Poincaré Disc $\mathbf{D}^2$ has four geodescics as boundary and the group Γ is freely generated by $\gamma_a z = (2ijz - \sqrt{3})/(\sqrt{3}z + 2ij^2)$ (identifying a and a') and $\gamma_b z = (z(1-i)-1)/(z-i-1)$ (identifying b and b').

defined for $\operatorname{Re} s > \frac{1}{2}$, $s(1-s) \notin \operatorname{spec}_{\mathrm{pp}}(\Delta)$ *extends to a meromorphic family of operators*

$$R(s) : L^2_{\mathrm{comp}}(X) \to H^2_{\mathrm{loc}}(X) \tag{1.4}$$

with poles of finite rank.

By studying the model resolvents (see Sections 2, 3) more carefully we can give a more precise mapping property than (1.4), but we do not need it here. The poles of the operator (1.4) are called the *resonances*. When $N=0$ [16] or $M=0$ [19, 20, 11] the resonances which do not correspond to L^2 embedded eigenvalues (which appear in the case of $N=0$ case only) are identified with the poles of the scattering matrix and, in the constant curvature case (see Fig. 2), with the poles of the meromorphic continuation of the logarithmic derivative of the Selberg zeta function.

THEOREM 2. *If $N(r)$ is the number of resonances counted with their algebraic multiplicity in $\{s : |s| \leqslant r\}$, then, for some constant C,*

$$N(r) \leqslant Cr^2, \qquad r > C. \tag{1.5}$$

Remark 1. The proof of Theorem 2 applies in higher dimension $n \geqslant 3$ as long as analogues of (1.1) hold and the scattering matrices for X_i, Y_j are well understood as meromorphic functions (see Section 3). For example, if $M=0$ and each Y_j is isometric to a neighbourhood of infinity of $\mathbf{H}^n/\langle\gamma\rangle$, $\gamma \in \operatorname{Isom}(\mathbf{H}^n)$ hyperbolic, one obtains $N(r) = \mathcal{O}(r^n)$, which is optimal. This is not the exactly computable case since in the case of non-constant curvature the compact part Z is quite arbitrary.

Remark 2. If $N=0$, then the Selberg trace formula (see [25, p. 668]) in the constant curvature case and the works of Müller [16], Parnovski [18], or Vodev [33] in general, give the asymptotics for $N(r)$

$$N(r) \sim \frac{\operatorname{Vol} X}{2\pi} r^2. \tag{1.6}$$

In the opposite extreme, $M=0$ and $X=\mathbf{H}^2/\Gamma$, it is expected that the Selberg zeta function is of finite order (see [23]). Assuming this, we can apply Theorem 2 above and Corollary 2 of [28] to conclude that

$$N(r) > Cr, \qquad r > C. \tag{1.7}$$

In fact, the meromorphic continuation of the zeta function is provided by [11], the identification of its poles and zeros with resonances by [22], and the order of the canonical Weierstrass product by Theorem 2. The conclusion is sharper than (1.7) by giving a lower bound in any logarithmic neighbourhood of the unitarity axis $\{s : -\operatorname{Re} s < \rho \log\langle s\rangle\}$: for every $d > 0$, a sufficiently small $\varepsilon > 0$, and $\rho > n/(d-\varepsilon^2)$

$$\sum_{|\operatorname{Im} s| \leq r,\ -\operatorname{Re} s < \rho \log\langle s\rangle} e^{-(d-\varepsilon)(1/2 - \operatorname{Re} s)}$$
$$> \left(\frac{1}{2\pi \sinh(d/2)} \sum_{m l(\gamma) = d} l(\gamma) - o_\varepsilon(1)\right) r,$$

where $\gamma \in \Gamma$ is hyperbolic with displacement length $l(\gamma)$.

The idea of the proof can be described as follows. We treat the manifold X as a compactly supported perturbation of a finite set of hyperbolic half-cylinders and cusps. We also use the standard observation that only the zero mode in the cusp Fourier expansion contributes to the continuous spectrum. Hence, the Laplacian is made to act on a modified Hilbert space, roughly, $\mathscr{H} = \mathscr{H}_{\text{int}} \oplus L^2(\mathbf{R}_+, dr) \oplus L^2(Y_{0l}, d\operatorname{vol}_g)$, where Y_{0l} is the hyperbolic half-cylinder $(\mathbf{R}^+ \times \mathbf{R}/l\mathbf{Z},\ dr^2 + \cosh^2 r\, dt^2)$ and we took $N=M=1$. This is similar to the situation considered in [27, 30] with $\mathbf{R}^+ \sqcup Y_{0l}$ replacing $\mathbf{R}^n \setminus B(0, R)$. Thus, to apply the Fredholm determinant method we need precise information about the scattering theory on Y_{0l}, with, say, a Dirichlet boundary condition. The new difficulty is created by the presence of poles in the resolvent $R^{Y_{0l}}$ of the *free* problem on Y_{0l}. But as the scattering matrix S_{0l} can be computed explicitly (see Lemma A.2) we can control their contribution. More precisely, the estimates for the generalized eigenfunctions E_{0l} (Lemma 4.3) and the resolvent $R^{Y_{0l}}$ (Lemma 4.1) in the *good* half-plane $\operatorname{Re} s > \frac{1}{2}$ are essentially the same as in the Euclidean case. To treat the *bad* half-plane $\operatorname{Re} s < \frac{1}{2}$, we use the scattering matrix S_{0l} to relate the generalized eigenfunctions there to the ones in the *good* half-plane. This gives an expression for the difference of the *free* resolvents kernel $R^{Y_{0l}}$ based on that in [11],

$$(R^{Y_{0l}}(s) - R^{Y_{0l}}(1-s))(y, y')$$
$$= (1-2s)\int_{S^1} S_{0l}(s)\, E_{0l}(1-s, \cdot, y)(\xi)\, E_{0l}(1-s, \xi, y')\, d\xi, \qquad \operatorname{Re} s < \tfrac{1}{2},$$

where all the singularities come from S_{0l}, the scattering matrix. To apply Melrose's method [15] with an improvement obtained by reducing the dimension from two (Y_{0l}) to one (S^1) [34], we use Vodev's argument [29, Lemma 4] (see Proposition 4.1 below). The blow-up in the resulting estimates is then cancelled through a multiplication by an appropriately chosen entire function.

For notational simplicity we will use below the letter C to denote a large, but not necessarily the same, constant. For $s \in \mathbf{C}$ we will denote by $\langle s \rangle$ the shifted modulus $1 + |s|$.

2. The Modified Hilbert Space

To apply the abstract approach to resonance counting [27, 30] we start by modifying the Hilbert space on which the Laplacian acts as an unbounded self-adjoint operator. The need for that is dictated by the presence of cusps $X_1, ..., X_M$ and we proceed as in [4] and [11].

Thus we define

$$\mathscr{H} = \mathscr{H}^{a,b}_{\text{int}} \oplus \bigoplus_{i=1}^{M} L^2(X^a_{0i}, dr) \oplus \bigoplus_{j=1}^{N} L^2(Y^b_j, d\operatorname{vol}_g), \tag{2.1}$$

where with the identifications (1.1)–(1.3)

$$X^a_i \simeq [a, \infty) \times S^1, \quad X^a_i \subset X \text{ if } a > a_i, \quad X^a_{0i} \simeq [a, \infty), \quad i = 1, ..., M,$$
$$Y^b_j \simeq [b, \infty) \times S^1, \quad Y^b_j \subset X \text{ if } b > b_j, \quad j = 1, ..., N,$$
$$\mathscr{H}^{a,b}_{\text{int}} = L^2(Z) \oplus \bigoplus_{i=1}^{M} [L^2(X_i \backslash X^a_i, d\operatorname{vol}_g) \oplus {}_0L^2(X^a_i, d\operatorname{vol}_g)]$$
$$\oplus \bigoplus_{j=1}^{N} L^2(Y_j \backslash Y^b_j, d\operatorname{vol}_g).$$

The component of f in the ith term in (2.1) is the 0th Fourier coefficient $\widehat{f^i_0}(r) = h_i^{-1} \int_0^{l_i} f(r, t)\, dt$, $r > a$ along the cusp X^a_i. The space ${}_0L^2(X^a_i, d\operatorname{vol}_g)$

is the space of functions f in $L^2(X_i^a, d\,\mathrm{vol}_g)$ with zero horocyclic integrals $\widehat{f_0^i}(r)$, $r > a$.

We define the corresponding pseudo-Laplacian $\Delta^{a,b}$ (see [4]): it is the Friedrich extension of the symmetric Laplacian $(\Delta^X, \mathscr{C}_0^\infty(X) \cap \mathscr{H}_{\mathrm{int}}^{a,b})$.

The proof of the lemma below is a modification of the proof in the case $N=0$ (see [4]):

LEMMA 2.1. *The operator $\Delta^{a,b}$ is self-adjoint with compact resolvent. The eigenvalues satisfy*

$$\#\{\mu : \mu^2 \in \mathrm{spec}(\Delta^{a,b}),\ |\mu| \leqslant r\} = \frac{\mathrm{Vol}_b(X)}{2\pi} r^2 + o(r^2)$$

with

$$\mathrm{Vol}_b(X) = \mathrm{Vol}\left(X \Big\backslash \bigcup_{j=1}^{N} Y_j^b\right).$$

We have two types of model problems. For the cusp ends, we take the Euclidean half-line $X_0^0 \simeq \mathbf{R}^+$ and the shifted Dirichlet Laplacian

$$\Delta_0^0 = D_r^2 + \tfrac{1}{4} \tag{2.3}$$

on $L^2(X_0^0)$ with the domain $H^2(X_0^0) \cap H_0^1(X_0^0)$. Let $\Delta_0^{X_i}$ denote the corresponding differential operator on the line $X_{0i}^0 \simeq \mathbf{R}^+$ corresponding in (2.1) to the cusp X_i.

For the cylindrical ends Y_j, the model depends on the length l_j of the closed geodesic in the cylinder, and we take the half-cylinder $Y_{0l} \simeq \mathbf{R}^+ \times \mathbf{R}/l\mathbf{Z}$. Again it is convenient to use the Dirichlet problem realization Δ_{0l} of

$$\Delta^{Y_{0l}} = D_r^2 - i \tanh r D_r + \cosh^{-2} r D_t^2 \tag{2.4}$$

on $L^2(Y_{0l}, d\,\mathrm{vol}_g)$ with the domain $H^2(Y_{0l}, d\,\mathrm{vol}_g) \cap H_0^1(Y_{0l}, d\,\mathrm{vol}_g)$. Let $\Delta^{Y_j^0}$ denote the corresponding differential operator on the half-cylinder $Y_j^0 \simeq Y_{0l_j}$.

Let χ be a smooth function on $\mathbf{R}$ with support in $(-\infty, \frac{2}{3}]$, $\chi(r) = 1$ for $r < \frac{1}{3}$ and let χ_A be the translate $\chi(\cdot - A)$. Using the identification (1.1) we define the following linear cutoff operators acting on $f \in \mathscr{H}$:

$$\chi_{a,b}^{X_i} f = \begin{cases} \chi_a f & \text{if } f \in L^2(X_{0i}^a, dr), \\ f & \text{if } f \in \mathscr{H}_{\mathrm{int}}^{a,b} \oplus \bigoplus_{k \neq i} L^2(X_{0k}^a, dr) \oplus \bigoplus_{j=1}^{N} L^2(Y_j^b, d\,\mathrm{vol}_g), \end{cases}$$

$$\chi_{a,b}^{Y_j} f = \begin{cases} \chi_b f & \text{if } f \in L^2(Y_j^b, d\,\mathrm{vol}_g), \\ f & \text{if } f \in \mathscr{H}_{\mathrm{int}}^{a,b} \oplus \bigoplus_{i=1}^{M} L^2(X_{0i}^a, dr) \oplus \bigoplus_{k \neq j} L^2(Y_k^b, d\,\mathrm{vol}_g), \end{cases}$$

and

$$\chi_{a,b}^{Z} = \begin{cases} f & \text{if } f \in \mathscr{H}_{\mathrm{int}}^{a,b}, \\ \chi_a f & \text{if } f \in \bigoplus_{i=1}^{M} L^2(X_{0i}^a, dr), \\ \chi_b f & \text{if } f \in \bigoplus_{j=1}^{N} L^2(Y_j^b, d\,\mathrm{vol}_g). \end{cases}$$

We then immediately have

$$\Delta = \chi_{a+1,b+1}^{Z} \Delta^{a+2,b+2} \chi_{a,b}^{Z} + \sum_{i=1}^{M} (1 - \chi_{a-1,b-1}^{X_i}) \Delta_0^{X_i} (1 - \chi_{a,b}^{X_i})$$
$$+ \sum_{j=1}^{N} (1 - \chi_{a-1,b-1}^{Y_j}) \Delta^{Y_j^0} (1 - \chi_{a,b}^{Y_j}). \tag{2.5}$$

This representation, combined with Lemma 2.1 and detailed information about the model resolvents $(\Delta_0^0 - s(1-s))^{-1}$ (Lemma 2.2), $(\Delta_{0l} - s(1-s))^{-1}$ (Section 3), give the meromorphy of $(\Delta - s(1-s))^{-1}$ (see Section 5) and then the estimates on the poles (see Section 6).

We conclude this section with

LEMMA 2.2. *The resolvent in* $\mathrm{Re}\, s > \frac{1}{2}$

$$(\Delta_0^0 - s(1-s))^{-1} : L^2(X_0^0) \to H^2(X_0^0)$$

extends to an entire family of operators

$$R_0^0(s) : L^2_{\mathrm{comp}}(X_0^0) \to H^2_{\mathrm{loc}}(X_0^0)$$

satsisfying the estimate

$$\|\chi R_0^0(s) \chi\|_{L^2 \to H^2} \leqslant C e^{-c \,\mathrm{Re}\, s} + C, \qquad s \in \mathbf{C},\ c > 0,$$

for any $\chi \in \mathscr{C}_0^\infty(\mathbf{R}^+)$.

Proof. We have an explicit expression

$$R_0^0(s)(r, r') = \frac{2ie^{(1/2)(r-r')}}{2s-1} \left(e^{(1/2-s)|r-r'|} - e^{(1/2-s)|r+r'|}\right),$$

from which the conclusion is immediate. ∎

3. Scattering Theory for the Hyperbolic Cylinder

The purpose of this section is to present the relevant scattering theory for the hyperbolic half-cylinder Y_{0l}. We will use both the general approach from [11] and [19] and a more direct one based on rotational symmetry and a reduction to one dimensional problems.

The resolvent

$$R^{Y_{0l}}(s) = (\Delta_{0l} - s(1-s))^{-1} : L^2(Y_{0l}) \to H^2(Y_{0l}) \cap H_0^1(Y_{0l}), \qquad \operatorname{Re} s > \tfrac{1}{2},$$

admits a meromorphic continuation (see Lemma 1.2 in [11])

$$R^{Y_{0l}}(s) : L^2_{\text{comp}}(Y_{0l}) \to H^2_{\text{loc}}(Y_{0l}), \qquad s \in \mathbf{C}.$$

We will denote by $Y_{0l}(\infty)$ the boundary at infinity for Y_{0l}: through the Fermi coordinates based at the boundary ∂Y_{0l}, $Y_{0l}(\infty)$ is identified to $\mathbf{R}/l\mathbf{Z}$. If $d(y, \partial Y_{0l})$ denotes the distance to the neck ∂Y_{0l}, then the generalized eigenfunctions or Eisenstein functions $E_{0l}(s, \xi, y)$, $\xi \in Y_{0l}(\infty)$, $y \in Y_{0l}$, are defined for $\operatorname{Re} s > \frac{1}{2}$ as regularized boundary values of the resolvent kernel

$$E_{0l}(s, \xi, y) = \lim_{y' \to \xi} e^{sd(y', \partial Y_{0l})} (\Delta_{0l} - s(1-s))^{-1} (y, y'), \qquad \operatorname{Re} s > \tfrac{1}{2}. \tag{3.1}$$

The Eisenstein functions, as well as the resolvent kernel, admit a meromorphic extension to $\mathbf{C}$ and, by applying the Green formula at infinity, it is proved in Proposition 2.1 of [11] that

$$\begin{aligned} R^{Y_{0l}}(s, y, y') &- R^{Y_{0l}}(1-s, y, y') \\ &= (1-2s) \int_{Y_{0l}(\infty)} E_{0l}(s, \xi, y)\, E_{0l}(1-s, \xi, y')\, d\xi. \end{aligned} \tag{3.2}$$

The scattering matrix $S_{0l}(s)$ is now defined to describe the behaviour at infinity of the Eisenstein functions:

$$\begin{aligned} &E_M(s, m_\infty, (r, n_\infty)) \\ &\quad = \frac{e^{(s-1)r}}{2s-1} \delta(m_\infty - n_\infty) - \frac{e^{-sr}}{2s-1} S_{0l}(s)(m_\infty, n_\infty) + o(e^r) \end{aligned}$$

for $\operatorname{Re} s \in (0, 1)$, s not a resonance, and with convergence $(o(e^r))$ in the sense of distributions. The scattering matrix S_{0l} extends to $\mathbf{C}$ as a meromorphic function with polar parts of finite rank. We also have the functional equation (see Corollary 2.6 in [11])

$$E_{0l}(s, \xi, y) = S_{0l}(s)\, E_{0l}(1-s, \cdot, y)(\xi), \qquad s \in \mathbf{C}. \tag{3.3}$$

In Section 4 we will need estimates given in the next two lemmas:

LEMMA 3.1. *Let K be a compact subset of Y_{0l} and $\varepsilon > 0$. Then there exists a constant C, depending only on K and ε such that*

$$|\partial_\xi^k E_{0l}(s, \xi, y)| \leqslant k!\, C^k e^{C\langle s\rangle}, \qquad \xi \in Y_{0l}(\infty),\ y \in K,\ \operatorname{Re} s > \varepsilon,\ k \in \mathbf{N}.$$

Proof. Let t_∞ be the coordinate on $Y_{0l}(\infty) \simeq \mathbf{R}/l\mathbf{Z}$ induced by the Fermi coordinates (r, t) on the cylinder $C_l = \mathbf{H}^2/\langle \gamma_l \rangle$. If E_l are the Eisenstein functions for C_l defined similarly to (3.1) and τ is the symmetry with respect to the collar geodesic on C_l, by the method of images we have

$$E_{0l}(s, \xi, y) = E_l(s, \xi, y) - E_l(s, \xi, \tau y),$$

where, in terms of Fermi coordinates and following the definition (3.1) (see [11] and the formula (3.8) below),

$$E_l(s, \xi, y) = \frac{\Gamma(s)}{\Gamma(s+1/2)} \frac{\cosh^{-s} r}{2\sqrt{\pi}} \sum_{n \in \mathbf{Z}} [e^{t - t_\infty + nl} - 2 \tanh r + e^{t_\infty - t - nl}]^{-s}. \tag{3.5}$$

If $r_K = \sup_{y \in K} |r|$, there exists a neighbourhood $\{|\operatorname{Im} z| \leqslant \delta\}$ of $\mathbf{R}$ (depending on r_K) where the series

$$\sum_{n \in \mathbf{Z}} [e^{z + nl} - 2 \tanh r + e^{-z - nl}]^{-s}$$

defines an analytic function, periodic with period l for $|r| \leqslant r_K$ and $\operatorname{Re} s > \varepsilon$ and with bound

$$\left| \sum_{n \in \mathbf{Z}} [e^{z + nl} - 2 \tanh r + e^{-z - nl}]^{-s} \right|$$
$$\leqslant \frac{2e^{(l+\delta)|s|}}{1 - e^{-l\varepsilon}} \sup |[1 - 2 \tanh r e^{-(z + nl)} + e^{-2(z + nl)}]^{-s}|,$$

where the sup is on the set $\{n \geqslant 0,\ |r| \leqslant r_K,\ |\operatorname{Re} z| \leqslant l,\ |\operatorname{Im} z| \leqslant \delta\}$.

Then, by the Cauchy formula

$$|\partial_{t_\infty}^k E_l(s, t_\infty, y)| \leqslant k!\, \delta^{-k} \sup_{|\operatorname{Im} z| \leqslant \delta} |E_l(s, z, y)|,$$

which gives the estimate (3.4) for E_l. The estimate for E_{0l} follows. ∎

LEMMA 3.2. *Let $Q_i \in \mathrm{Diff}^{p_i}(Y_{0l})$ have coefficients of compact support in Y^0_{0l} and assume that the union of the supports of coefficients of Q_1 is disjoint from the union of the supports of coefficients of Q_2. Then*

$$\|Q_1 R^{Y_{0l}}(s)\, Q_2\| \leqslant C_{p_1, p_2, \varepsilon}\langle s\rangle^{p_1+p_2-\tau}, \qquad \operatorname{Re} s > \frac{\tau}{2} + \varepsilon,\ \varepsilon > 0,\ \tau = 0, 1. \tag{3.6}$$

Proof. When $\operatorname{Re} s > \frac{1}{2} + \varepsilon$, we have the standard resolvent estimate

$$\|(\Delta_{0l} - s(1-s))^{-1}\|_{L^2(Y_{0l}) \to L^2(Y_{0l})} \leqslant \frac{1}{\varepsilon\langle s\rangle}. \tag{3.7}$$

Since for $\chi_1, \chi_2 \in \mathscr{C}^\infty(Y_{0l}) \cap L^\infty(Y_{0l})$ with $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \varnothing$

$$\begin{aligned} &\Delta^{Y_{0l}}\chi_1(\Delta_{0l} - s(1-s))^{-1}\chi_2 \\ &\quad = [\Delta^{Y_{0l}}, \chi_1](\Delta_{0l} - s(1-s))^{-1}\chi_2 + s(1-s)\,\chi_1(\Delta_{0l} - s(1-s))^{-1}\chi_2, \end{aligned}$$

the estimate (3.6) follows from (3.7) by iteration and interpolation—see for instance the proof of Lemma 3 in [34].

The case $\tau = 0$ is more complicated: Let $Q(\partial_{m_1}, \partial_{m_2})$ be a differential operator with support in a compact subset of $C_l \times C_l \setminus \Delta$, of total order q, $Q(\partial_{z_1}, \partial_{z_2})$ be the covering operator on $\mathbf{H}^2 \times \mathbf{H}^2$. If the projection of (z_1, z_2) is in the support of Q, $\sigma(z_1, z_2) = \sinh^2 d(z_1, z_2) \asymp e^{2d(z_1, z_2)}$. Then, for $t \in [0, 1]$

$$\begin{aligned} &|Q(\partial_{z_1}, \partial_{z_2})(t + \sigma(z_1, z_2))^{-s}| \\ &\quad \leqslant C_1^{\operatorname{Re} s}\langle s\rangle^q (t + \sigma(z_1, z_2))^{-\operatorname{Re} s} \leqslant C_2^{\operatorname{Re} s} e^{-2 \operatorname{Re} s\, d(z_1, z_2)}. \end{aligned}$$

As the kernel of $(\Delta^{C_l} - s(1-s))^{-1}$ is

$$\begin{aligned} &(\Delta^{C_l} - s(1-s))^{-1}(m_1, m_2) \\ &\quad = \sum_{n \in \mathbf{Z}} \frac{1}{4\pi} \int_0^1 (t(1-t))^{s-1}(t + \sigma(z_1, \gamma_l^n z_2))^{-s}\, dt \end{aligned} \tag{3.8}$$

we have, for $\operatorname{Re} s \geqslant \varepsilon$,

$$\begin{aligned} &|Q(\partial_{m_1}, \partial_{m_2})(\Delta^{C_l} - s(1-s))^{-1}(m_1, m_2)| \\ &\quad \leqslant C_2^{\operatorname{Re} s} \sum_{n \in \mathbf{Z}} e^{-2 \operatorname{Re} s\, d(z_1, \gamma_l^n z_2)} \leqslant \frac{C C_2^{\operatorname{Re} s}}{1 - e^{-\varepsilon l}} \leqslant \tilde{C}_1, \qquad \varepsilon < \operatorname{Re} s < C'. \end{aligned}$$

Applying the Laplace operator as in the case of $\tau = 1$ gives (3.6) with $\tau = 0$. ∎

To compute $S_{0l}(s)$ and to find the resonances of Y_{0l} we use the rotational symmetry and thus reduce the problem to one dimension. Through the conjugation by $\cosh^{1/2} r$, the Laplacian

$$\Delta^{Y_{0l}} = D_r^2 + i \tanh r D_r + \frac{\Delta^{\mathbf{R}/l\mathbf{Z}}}{\cosh^2 r}$$

is equivalent to the differential operator $D_r^2 + (\Delta^{\mathbf{R}/l\mathbf{Z}} + \frac{1}{4}) \cosh^2 r + \frac{1}{4}$ on $L^2(\mathbf{R}^+ \times \mathbf{R}/l\mathbf{Z}, dr\, dt)$. Taking a Fourier expansion in the t variable, we get

$$\Delta^{Y_{0l}} \simeq \bigoplus_{m \in \mathbf{Z}} D_r^2 + \frac{\left(\frac{2\pi m}{l}\right)^2 + \frac{1}{4}}{\cosh^2 r} + \frac{1}{4} \tag{3.9}$$

on $L^2(Y_{0l}, d\,\mathrm{vol}_{Y_{0l}}) \simeq l^2(\mathbf{Z}, L^2(\mathbf{R}^+, dr))$. The Dirichlet Laplacian Δ_{0l} is then unitarily equivalent to a direct sum of one dimensional Schrödinger operators $\bar{H}_{0,\nu}$, $\nu = -\frac{1}{2} - i2\pi m/l$, $m \in \mathbf{Z}$ defined in the Appendix. Hence

$$R^{Y_{0l}}(s) = \bigoplus_{m \in \mathbf{Z}} (\bar{H}_{0,-1/2-2i\pi m/l} - k^2)^{-1}, \qquad s = \tfrac{1}{2} - ik,\ \mathrm{Re}\, s > \tfrac{1}{2}.$$

Since the Eisenstein functions can also be decomposed into Fourier series we obtain that

$$S_{0l}(s) = \bigoplus_{m \in \mathbf{Z}} s(\bar{H}_{0,-1/2-2i\pi m/l})(k),\ s = \tfrac{1}{2} - ik,$$

where $s(\bar{H}_{0,\nu})(k)$ is the scattering matrix for the Pöschl–Teller potential $V_{0,\nu}$. We note that the poles of $s(\bar{H}_{0,\nu})(k)$ in $\mathrm{Im}\, k < 0$ correspond to the poles of $(\bar{H}_{0,\nu} - k^2)^{-1}$ but $s(\bar{H}_{0,\nu})(k)$ has *non-physical* poles in the upper half-plane, which is a well known phenomenon for potentials with exact exponential decay (see [17, p. 420]).

Combining this with Lemma A.2 we obtain

LEMMA 3.3. *For the Dirichlet Laplacian on the hyperbolic half-cylinder* Y_{0l},

$$S_{0l}(s) = \bigoplus_{m \in \mathbf{Z}} s_{lm}(s),$$

where

$$s_{lm}(s) = -2^{2s-1} \frac{\Gamma\left(\frac{1}{2} - s\right) \Gamma\left(\left(1 + s - i\frac{2\pi m}{l}\right)\Big/2\right) \Gamma\left(\left(1 + s + i\frac{2\pi m}{l}\right)\Big/2\right)}{\Gamma\left(s - \frac{1}{2}\right) \Gamma\left(\left(2 - s - i\frac{2\pi m}{l}\right)\Big/2\right) \Gamma\left(\left(2 - s + i\frac{2\pi m}{l}\right)\Big/2\right)}.$$

This immediately allows us to find the exact resonance set for the model problem:

LEMMA 3.4. *The resonance set of the Dirichlet Laplacian on the hyperbolic half-cylinder Y_{0l} is given by the half-lattice $\mathscr{L}_l$:*

$$\mathscr{L}_l = \left\{ im\frac{2\pi}{l} - n : m \in \mathbf{Z}, n \in 2\mathbf{N} - 1 \right\}.$$

4. ESTIMATES ON THE CHARACTERISTIC VALUES

If A is a trace class operator with eigenvalues $\{\lambda_j(A)\}_{j \geq 1}$, $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)| \to 0$, the Fredholm determinant is defined by $\det(1 + A) = \prod_{j=1}^{\infty} (1 + \lambda_j(A))$. For a compact operator A, its characteristic values $\mu_1(A) \geq \cdots \geq \mu_n(A) \to 0$ are defined as the eigenvalues of the self-adjoint operator $|A| = \sqrt{A^*A}$. Weyl's classical inequality (see [8, p. 35])

$$\prod_{j=1}^{N} (1 + |\lambda_j(A)|) \leq \prod_{j=1}^{N} (1 + \mu_j(A)) \tag{4.1}$$

applied to the determinant

$$|\det[1 + A]| \leq \det[1 + |A|] \tag{4.2}$$

has been crucial in the pole counting estimates [15, 34, 30] and, using a more local approach developed by Sjöstrand [26], in [27].

To study the determinants arising here, we need to estimate the characteristic values of the cutoff free resolvent. We start with the *good* half-plane, $\operatorname{Re} s > \varepsilon$, where the situation is essentially the same as in the Euclidean case (see [34, Lemma 3; 30, inequality (2.5)]).

LEMMA 4.1. *Under the assumption of Lemma 3.2 and for any $m \in \mathbf{N}_0$*

$$\begin{gathered} \mu_j(Q_1 R^{Y_{0l}}(s)\, Q_2) \leq C_{m,\varepsilon}\, j^{-m} \langle s \rangle^{2m + p_1 + p_2 - \tau}, \\ \operatorname{Re} s > \frac{\tau}{2} + \varepsilon,\ \varepsilon > 0,\ \tau = 0, 1. \end{gathered} \tag{4.3}$$

Proof. We follow the simple argument from [15]: let $\Omega \subset Y_{0l}$ be an open set with $\bar{\Omega}$ compact and $\partial\Omega$ smooth. If the coefficients of Q_1 are supported in Ω then, denoting by Δ_Ω the Dirichlet realization of the Laplacian on Ω,

$$Q_1 R^{Y_{0l}}(s)\, Q_2 = \Delta_\Omega^{-m}\, \Delta_\Omega^m Q_1 R^{Y_{0l}}(s)\, Q_2$$

and hence

$$\mu_j(Q_1 R^{Y_{0l}}(s) Q_2) \leqslant \mu_j(\Delta_\Omega^{-m}) \|\Delta_\Omega^m Q_1 R^{Y_{0l}}(s) Q_2\|$$
$$\leqslant C_{m,\varepsilon} j^{-m} \langle s \rangle^{2m+p_1+p_2-\tau},$$

where we used Lemma 3.2 with Q_1 replaced by $\Delta_\Omega^m Q_1$ and the standard Weyl estimates on the eigenvalues of the Dirichlet Laplacian. ∎

As indicated already in the Introduction, the estimates in the *bad* half-plane $\mathrm{Re}\, s < \frac{1}{2}$ are based on the representation of the spectral measure in terms of the resolvent. The new phenomenon comes from the presence of poles already in the *free* case and for sufficiently precise estimates we will need the detailed information from Section 3.

LEMMA 4.2. *If* $\mathrm{Re}\, s < \frac{1}{2} - \varepsilon$, *then*

$$\mu_j(S_{0l}(s)) \leqslant \begin{cases} d^{-2}(s, \mathscr{L}_l) e^{C\langle s \rangle \log \langle s \rangle}, & j = 1, 2, \\ \exp\left[C\langle s \rangle + 2\,\mathrm{Re}\left(\frac{1}{2} - s\right) \log\left(\frac{\langle s \rangle}{j}\right)\right], & j > 2. \end{cases} \tag{4.4}$$

Proof. By Lemma 3.3, the characteristic values of $S_{0l}(s)$ are

$$|s_{lm}(s)| = \left| \frac{\pi}{4}\left(s - \frac{1}{2}\right) \sin \pi \left(\frac{1}{2} - s\right) \Gamma\left(\frac{1}{2} - s\right)^2 \right.$$
$$\times \left[\sin \frac{\pi}{2}(s + im\omega + 1) \sin \frac{\pi}{2}(s - im\omega + 1) \right.$$
$$\left.\left. \times \Gamma(-s + im\omega + 1) \Gamma(-s - im\omega + 1) \right]^{-1} \right|, \qquad m \in \mathbf{Z}, \tag{4.5}$$

where we introduced $\omega = 2\pi/l$ and we used the complement formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

and the duplication formula

$$\Gamma(2z) = \frac{4^z}{2\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$

We start with the following estimate: for $\mathrm{Re}\, z > \varepsilon$, $|\mathrm{Im}\, z| \leqslant \omega/2$, $k \in \mathbf{Z}^*$

$$\left| \sin \frac{\pi}{2}(z - ik\omega) \Gamma(z - ik\omega) \right|^{-1}$$
$$\leqslant C \exp\left[C\,\mathrm{Re}\, z - \frac{1}{2}\left(\mathrm{Re}\, z - \frac{1}{2}\right) \log((\mathrm{Re}\, z)^2 + \omega^2 k^2)\right]. \tag{4.6}$$

In fact, if $A \asymp B$ means that there exists C independent of z and k such that $A/C \leqslant B \leqslant CA$, we have from Stirling's formula

$$|\Gamma(z-i\omega k)| = |\sqrt{2\pi}\, e^{-z+i\omega k}(z-i\omega k)^{(z-i\omega k-1/2)}| \left(1+\mathcal{O}_\varepsilon\left(\frac{1}{|z|+k}\right)\right)$$

$$\asymp e^{-\operatorname{Re} z}\, e^{(\operatorname{Re} z - 1/2)\log(|z+i\omega k|)}\, e^{\omega k \operatorname{Arg}(z-i\omega k)},$$

where $|\operatorname{Arg}(z-i\omega k)| < \pi$. Since for $k \in \mathbf{Z}^*$

$$\left|\sin\frac{\pi}{2}(z-i\omega k)\right|^{-1} \leqslant 2e^{\pi\omega/4}\, e^{-(\pi/2)\omega|k|}(1-e^{-\omega\pi(|k|-1/2)})^{-1}$$

and

$$\operatorname{Arg}(z-ik\omega) = -\operatorname{sgn}(k)\frac{\pi}{2} + \mathcal{O}\left(\frac{|z|}{|k|^2}\right),$$

(4.6) follows.

To estimate (4.5) we consider two cases, $|\operatorname{Im} s| > \omega/2$ and $|\operatorname{Im} s| \leqslant \omega/2$. For $x \in \mathbf{R}$ let $[\![x]\!]$ denote the integer closest to x when $x \notin \mathbf{Z}/2$ and $\operatorname{sgn}(x)[\,|x|\,]$ if $x \in \mathbf{Z}/2$. With $z = 1 - s + i\omega[\![\operatorname{Im} s/\omega]\!]$ and $k = m + [\![\operatorname{Im} s/\omega]\!]$,

$$\sin\frac{\pi}{2}(s+im\omega+1)\,\Gamma(-s-im\omega+1) = \sin\frac{\pi}{2}(z-ik\omega)\,\Gamma(z-ik\omega)$$

and the assumptions of (4.6) are satisfied.

Hence for $m \neq -[\![\operatorname{Im} s/\omega]\!]$ we obtain from (4.6) that

$$\left|\sin\frac{\pi}{2}(s+im\omega+1)\,\Gamma(-s-im\omega+1)\right|^{-1}$$

$$\leqslant C\exp\left[C\operatorname{Re}(1-s) - \frac{1}{2}\operatorname{Re}\left(\frac{1}{2}-s\right)\right.$$

$$\left.\times\log\left(\left|\operatorname{Re}\left(\frac{1}{2}-s\right)\right|^2 + \omega^2\left(m+\left[\!\!\left[\frac{\operatorname{Im} s}{\omega}\right]\!\!\right]\right)^2\right)\right].$$

Also,

$$|(s-\tfrac{1}{2})\sin\pi(\tfrac{1}{2}-s)\,\Gamma(\tfrac{1}{2}-s)^2| \leqslant e^{C\langle s\rangle}\, e^{2\operatorname{Re}(1/2-s)\log\langle s\rangle}.$$

Thus when $|\operatorname{Im} s| > \omega/2$, that is $[\![\operatorname{Im} s/\omega]\!] \neq 0$,

$$
\begin{aligned}
|s_{lm}(s)| \leqslant e^{C\langle s\rangle} \exp\Bigg[\operatorname{Re}\left(\frac{1}{2}-s\right)\Bigg[\log\langle s\rangle^2 - \frac{1}{2}\log\left(\left|\operatorname{Re}\left(\frac{1}{2}-s\right)\right|^2\right. \\
\left. + \omega^2\left(m + \left[\!\!\left[\frac{\operatorname{Im} s}{\omega}\right]\!\!\right]\right)^2\right) - \frac{1}{2}\log\left(\left|\operatorname{Re}\left(\frac{1}{2}-s\right)\right|^2\right. \\
\left. + \omega^2\left(m - \left[\!\!\left[\frac{\operatorname{Im} s}{\omega}\right]\!\!\right]\right)^2\right)\Bigg]\Bigg], \qquad m \neq \pm\left[\!\!\left[\frac{\operatorname{Im} s}{\omega}\right]\!\!\right].
\end{aligned} \tag{4.7}
$$

When $|\operatorname{Im} s| > \omega/2$ and $m = \pm[\![\operatorname{Im} s/\omega]\!]$, then

$$
\begin{aligned}
|s_{lm}(s)| \leqslant d^{-1}(s, \mathscr{L}_l)\, e^{C\langle s\rangle} \exp\Bigg[\operatorname{Re}\left(\frac{1}{2}-s\right)\Bigg[\log\langle s\rangle^2 \\
- \operatorname{Re}\left(\frac{1}{2}-s\right)\left[\log\left|\operatorname{Re}\left(\frac{1}{2}-s\right)\right|\right. \\
- \frac{1}{4}\log\left(\left|\operatorname{Re}\left(\frac{1}{2}-s\right)\right|^2 + 4\omega^2\left[\!\!\left[\frac{\operatorname{Im} s}{\omega}\right]\!\!\right]^2\right)\Bigg]\Bigg].
\end{aligned} \tag{4.8}
$$

When $|\operatorname{Im} s| \leqslant \omega/2$, that is $[\![\operatorname{Im} s/\omega]\!] = 0$, we easily see that (4.7) still holds. The estimate (4.8), however, needs to be modified to

$$
|s_{l0}(s)| \leqslant d^{-2}(s, -2\mathbf{N}+1)\, e^{C\langle s\rangle}.
$$

It is now convenient to write $S_{0l} = S_{0l}^{+} + S_{0l}^{-}$, $S_{0l}^{+} \simeq \bigoplus_{m\geqslant 0} s_{lm}$, $S_{0l}^{-} \simeq \bigoplus_{m<0} s_{lm}$ so that

$$
\mu_{m+n-1}(S_{0l}) \leqslant \mu_m(S_{0l}^{+}(s)) + \mu_n(S_{0l}^{-}(s))
$$

and thus it suffices to estimate $\mu_j(S_{0l}^{\pm}(s))$. Since $\mu_j(S_{0l}^{\pm}(s)) \leqslant \|S_{0l}^{\pm}(s)\| \leqslant \max_{\pm m\geqslant 0}|s_{ml}(s)|$, (4.4), $j = 1, 2$ follows. To obtain the case $j > 2$ we make the following elementary observation: if $a_1 \geqslant \cdots \geqslant a_j \geqslant \ldots$, $b_1 \geqslant \cdots \geqslant b_k \geqslant \ldots$, are sequences of positive numbers and if for some bijection of $k \to j_k$ of $\mathbf{N}$, $a_{j_k} \leqslant b_k$, then $a_j \leqslant b_j$. Assume now that $\operatorname{Im} s \geqslant 0$ and that $0 \leqslant m \neq [\![\operatorname{Im} s/\omega]\!]$. We obtain from (4.7) that

$$
|s_{lm}(s)| \leqslant e^{C\langle s\rangle} \exp\left[\operatorname{Re}\left(\frac{1}{2}-s\right)\log\left[\frac{\langle s\rangle^2}{(m+[\![\operatorname{Im} s/\omega]\!])(m-[\![\operatorname{Im} s/\omega]\!])}\right]\right].
$$

Reordering the sequence on the right hand side according to size we obtain a decreasing sequence k_j satisfying

$$
k_j \leqslant e^{C\langle s\rangle} \exp\left[\operatorname{Re}\left(\frac{1}{2}-s\right)\log\left[\frac{\langle s\rangle^2}{j^2}\right]\right]
$$

and that gives the desired estimate on characteristic values. The other cases, $\operatorname{Im} s < 0$ or $m < 0$ are considered similarly. ∎

To estimate the characteristic values of the difference of the resolvents it is instructive to estimate the characteristic values of operators with Eisenstein functions as kernels:

LEMMA 4.3. *If* $\mathbf{E}^{\chi}_{0l}(s) : L^2(Y_{0l}, d\,\mathrm{vol}_g) \to L^2(Y_{0l}(\infty), d\xi)$ *is defined by*

$$\mathbf{E}^{\chi}_{0l}(s)\, u(\xi) = \int_{Y_{0l}} E_{0l}(s, \xi, y)\, \chi(y)\, u(y)\, d\,\mathrm{vol}_g, \qquad \chi \in \mathscr{C}^{\infty}_0(Y_{0l}),$$

then for $\operatorname{Re} s > \varepsilon$

$$\mu_j(\mathbf{E}^{\chi}_{0l}(s)) \leqslant \exp[C\langle s\rangle - j/C].$$

Proof. In view of the analytic estimates in Lemma 3.1, we can apply the method of Melrose as in the proof of Proposition 2 in [34]. For the convenience of the reader we briefly recall the argument:

$$\begin{aligned}\mu_j(\mathbf{E}^{\chi}_{0l}(s)) &= \mu_j((D^2_\xi + 1)^{-k}(D^2_\xi + 1)^k\, \mathbf{E}^{\chi}_{0l}(s))\\ &\leqslant \mu_j((D^2_\xi + 1)^{-k})\; \|(D^2_\xi + 1)^k\, \mathbf{E}^{\chi}_{0l}(s)\|_{L^2(Y_{0l}) \to L^2(Y_{0l}(\infty))}.\end{aligned}$$

By Lemma 3.1 and using $\mu_j((D^2_\xi + 1)^{-k}) \leqslant C^k j^{-2k}$, this is bounded by $C^k j^{-2k}(2k)!\, e^{C\langle s\rangle}$. The lemma follows from optimization in k. ∎

We recall that the gain in the estimates above comes from reducing the dimension from 2 in Y_{0l} to 1 in $Y_{0l}(\infty)$. To combine Lemmas 4.2 and 4.3 in order to estimate the free resolvent we now use an argument similar to that of Vodev:

PROPOSITION 4.1. *If* $\operatorname{Re} s < \frac{1}{2} - \varepsilon$ *and* $\chi \in \mathscr{C}^{\infty}_0(Y_{0l})$ *then*

$$\mu_j(\chi(R^{Y_{0l}}(s) - R^{Y_{0l}}(1-s))\chi)$$
$$\leqslant \begin{cases} d^{-2}(s, \mathscr{L}_l)\exp[C\langle s\rangle \log\langle s\rangle], & j \leqslant 2,\\ \exp[C\langle s\rangle + 2\operatorname{Re} s(\frac{1}{2} - s)\log\langle s\rangle/j], & 2 < j \leqslant C\langle s\rangle,\\ \exp[-j/C], & j > C\langle s\rangle.\end{cases}$$

Proof. We start with formula (3.2) which we rewrite using (3.3)

$$\chi(y)(R^{Y_{0l}}(s)(y, y') - R^{Y_{0l}}(1-s)(y, y'))\,\chi(y')$$

$$= (1-2s)\int_{Y_{0l}} S_{0l}(s)\, E_{0l}(1-s, \cdot, y)(\xi)\, \chi(y)\, E_{0l}(1-s, \xi, y')\, \chi(y')\, d\xi$$

so that

$$\chi(R^{Y_{0l}}(s) - R^{Y_{0l}}(1-s))\chi = (1-2s)^t \mathbf{E}^{\chi}_{0l}(1-s)^t S_{0l}(s)\, \mathbf{E}^{\chi}_{0l}(1-s)$$

and consequently

$$\begin{aligned} &\mu_j(\chi(R^{Y_{0l}}(s) - R^{Y_{0l}}(1-s))\chi) \\ &\quad \leqslant |1-2s|\, \mu_{j_1}(\mathbf{E}^{\chi}_{0l}(1-s))\, \mu_{j_2}(S_{0l}(s))\, \mu_{j_3}(\mathbf{E}^{\chi}_{0l}(1-s)), \\ &\quad\quad j = j_1 + j_2 + j_3 - 2, \end{aligned}$$

where we use the standard property of characteristic values (see [8, p. 29])

$$\mu_{m+n-1}(A_1 A_2) \leqslant \mu_m(A_1)\, \mu_n(A_2). \tag{4.9}$$

Hence as in the proof of Lemma 4.3 we have for all k and $j > C\langle s\rangle$

$$\mu_j(\chi(R^{Y_{0l}}(s) - R^{Y_{0l}}(1-s))\chi) \leqslant C^k j^{-2k}(2k)!\, e^{C\langle s\rangle},$$

where we applied Lemma 4.2. The conclusion for large j follows as in the proof of Lemma 4.3 while for small j from Lemma 4.2. ∎

We note that the smoothness of E_{0l} in y implies that the same estimates are true for $Q\chi(R^{Y_{0l}}(s) - R^{Y_{0l}}(1-s))\chi$, where Q is a differential operator.

We conclude this section by including estimates on the characteristic values for the cusp end (see Lemma 2.2). Although stronger results are easily available we content ourselves with the following consequence of the proofs of Lemma 4.1 above and of Lemma 4 of [34]:

PROPOSITION 4.2. *Let P_1 and P_2 be differential operators on X^0_0, of orders p_1 and p_2, respectively, and with coefficients of compact support. Then*

$$\mu_j(P_1(R^0_0(s) - R^0_0(1-s))P_2) \leqslant \exp[C\langle s\rangle - j/C].$$

If the support of the coefficients of P_1 are disjoint from the supports of coefficients of P_2 and Re $s > -C$ *then*

$$\mu_j(P_1 R^0_0(s) P_2) \leqslant C_m j^{-2m}\langle s\rangle^{2m+p_1+p_2}.$$

5. MEROMORPHIC CONTINUATION

Since the resolvents $R^{X_i}_0(s) = (\Delta^{X_i}_0 - s(1-s))^{-1}$ and $R^{Y^0_j}(s) = (\Delta^{Y^0_j} - s(1-s))^{-1}$ are holomorphic and meromorphic in $s \in \mathbf{C}$, respectively, the meromorphic continuation of

$$R(s) = (\Delta - s(1-s))^{-1} : L^2_{\mathrm{comp}}(X) \to H^2_{\mathrm{loc}}(X)$$

follows as in the proof in [27] which we will now briefly recall.

For $s_0 \in \mathbf{C}$ with $\operatorname{Re} s_0 > \frac{1}{2}$ to be chosen later we define, with the notation of (2.5) and a, b large,

$$Q_0(s_0) = \chi^Z_{a+2,b+2} R(s_0)\, \chi^Z_{a+1,b+1},$$

$$Q(s) = \sum_{i=1}^{M} (1-\chi^{X_i}_{a,b})\, R^{X_i}_0(s)(1-\chi^{X_i}_{a+1,b+1})$$

$$+ \sum_{j=1}^{N} (1-\chi^{Y_j}_{a,b})\, R^{Y^0_j}(s)(1-\chi^{Y_j}_{a+1,b+1}).$$

Since

$$\chi^Z_{a,b} + \sum_{i=1}^{M} (1-\chi^{X_i}_{a,b}) + \sum_{j=1}^{N} (1-\chi^{Y_j}_{a,b}) = I,$$

we obtain

$$(\Delta - s(1-s))(Q_0(s_0) + Q(s)) = I + L(s_0, s),$$

where

$$L(s_0, s) = [\Delta, \chi^Z_{a+2,b+2}]\, R(s_0)\, \chi^Z_{a+1,b+1} + (s_0(1-s_0) - s(1-s))\, Q_0(s_0)$$

$$- \sum_{i=1}^{M} [\Delta^{X_i}_0, \chi^{X_i}_{a,b}]\, R^{X_i}_0(s)(1-\chi^{X_i}_{a+1,b+1})$$

$$- \sum_{j=1}^{N} [\Delta^{Y^0_j}, \chi^{Y_j}_{a,b}]\, R^{Y^0_j}(s)(1-\chi^{Y_j}_{a+1,b+1}). \qquad (5.1)$$

If $\operatorname{Re} s > \frac{1}{2}$ and $(1-s)s \notin \operatorname{spec}(\Delta)$ we conclude that

$$Q_0(s_0) + Q(s) = R(s)(I + L(s_0, s))$$

and consequently that

$$(Q_0(s_0) + Q(s))\, \chi^Z_{a+3,b+3} = R(s)\, \chi^Z_{a+3,b+3}(I + L(s_0, s)\, \chi^Z_{a+3,b+3}).$$

The meromorphy of $R(s)\, \chi^Z_{a+3,b+3}$ for $s \in \mathbf{C}$ now follows from the meromorphy of $(I + L(s_0, s)\, \chi^Z_{a+3,b+3})^{-1}$. As $L(s_0, s)\, \chi^Z_{a+3,b+3}$ is a compact operator, meromorphic in s with poles of finite rank, the analytic Fredholm theory applies as long as $I + L(s_0, s)\, \chi^Z_{a+3,b+3}$ is invertible for some $s \in \mathbf{C}$. By taking $s = s_0$ and by choosing $\operatorname{Re} s_0$ large enough so that the norm of (5.1) is small we conclude the proof of the meromorphic continuation of $R(s)$ and thus obtain Theorem 1 of Section 1 above.

We conclude this section with the observation

LEMMA 5.1. *The set of poles of $R(s)$ with multiplicities is included in the union of $\bigcup_{j=1}^{M} \mathscr{L}_{l_j}$ and the set of zeroes of $D(s)=\det(1+K(s_0, s)^3)$, where $K(s_0, s)=L(s_0, s)\,\chi^Z_{a+3,b+3}$.*

Proof. Let us recall that, for an n-dimensional manifold M^n, an operator in $\Psi^{-1}_{\text{comp}}(M^n)$ is in the Schatten class $\mathfrak{S}_{n+1}$ (see [8, p. 91] for the definitions). Since for $s \notin \bigcup_{j=1}^{N} \mathscr{L}_{l_j}$, $K(s_0, s)$ is built with such operators on X and on the real line, $K(s_0, s)^3$ is a trace class operator and hence we can define the determinant $D(s)$ which is a meromorphic function in $\mathbf{C}$. The agreement of multiplicities follows as in the Appendix of [31]. Strictly speaking we characterized only the poles of $R(s)\,\chi^Z_{a+3,b+3}$, but the independence of the cutoff function can be seen as in the proof of Proposition 3.6 of [27] or in Section 3 of [30]. ∎

6. Proof of the Main Estimate

The argument of this section is based on Lemma 5.1 above. In comparison with the previous determinant estimates [15, 34, 30–32] we now need to control the contributions from the poles of $K(s_0, s)$. To remove these poles we introduce an entire function g_l given as a Weierstrass product

$$g_l(s)=\prod_{\lambda \in \tilde{\mathscr{L}}_l} E\left(\frac{s}{\lambda}, 2\right), \tag{6.1}$$

where as usual $E(z, 2)=(1-z)\exp(z+z^2/2)$ and the set $\tilde{\mathscr{L}}_l$ is the minimal subset of $\mathbf{C}$ containing $\mathscr{L}_l$ and invariant under the multiplication by the square roots of ± 1.

Since the number of elements of $\tilde{\mathscr{L}}_l$ in a disk of radius r grows like r^2 and since $\sum_{\epsilon \lambda \in \tilde{\mathscr{L}}_l,\, |\lambda| \leqslant r} \lambda^{-2}=0$ we conclude from Lindelöf's theorem (see [3, Thm. 2.10.1]) that

$$|g_l(s)| \leqslant e^{C\langle s\rangle^2}. \tag{6.2}$$

We shall need some inequalities on determinants

LEMMA 6.1. *Let A, B be compact operators in the Schatten class $\mathfrak{S}_p$ and S, T trace class operators. Then*

$$\begin{aligned} |\det(1+(A+B)^p)| &\leqslant \det(1+|A+B|^p)^p \\ &\leqslant \det(1+2^{p-1}|A|^p)^{2p} \det(1+2^{p-1}|B|^p)^{2p}, \end{aligned} \tag{6.3}$$

$$|\det(1+ST)| \leqslant \det(1+|S|)^2 \det(1+|T|)^2.$$

Proof. By (4.9) we have, for $k \geqslant 1$, $j \geqslant 1$,

$$\mu_{p(j-1)+k}(A^p) \leqslant \mu_j(A)^{p-1} \mu_{j+k-1}(A) \leqslant \mu_j(A)^p.$$

Then, according to (4.2) and the additive analogue of (4.9) we have

$$\begin{aligned}
&|\det(1+(A+B)^p)| \\
&\leqslant \prod_{k=1}^{p} \prod_{j=1}^{\infty} 1+\mu_{p(j-1)+k}((A+B)^p) \\
&\leqslant \left[\prod_{j=1}^{\infty} 1+\mu_j(A+B)^p\right]^p \\
&\leqslant \left[\prod_{j=1}^{\infty} 1+(\mu_j(A)+\mu_{j+1}(B))^p \prod_{j=1}^{\infty} 1+(\mu_j(A)+\mu_j(B))^p\right]^p \\
&\leqslant \left[\prod_{j=1}^{\infty} (1+2^{p-1}\mu_j(A)^p)(1+2^{p-1}\mu_j(B)^p)\right]^{2p} \\
&= \det(1+2^{p-1}\,|A|^p)^{2p} \det(1+2^{p-1}\,|B|^p)^{2p}.
\end{aligned}$$

The proof of the other inequality is similar. ∎

We now have the crucial

LEMMA 6.2. *For* $|\mathrm{Re}\, s - \frac{1}{2}| > \varepsilon$, *some* $P \in \mathbf{N}$, *and a constant* C_ε

$$\left|\prod_{j=1}^{N} g_{l_j}(s)^P D(s)\right| \leqslant e^{C_\varepsilon \langle s\rangle^2}.$$

Proof. Let us introduce the operator $K_2(s)$ defined by

$$\begin{aligned}
K(s_0, s) = {} & [\Delta, \chi^Z_{a+2,b+2}]\, R(s_0)\, \chi^Z_{a+1,b+1} \\
& + (s_0(1-s_0) - s(1-s))\, Q_0(s_0) + K_2(s).
\end{aligned}$$

Since, for each compact K, the characteristic values μ_j of the imbedding of $H^k(K)$ in $L^2(X)$ are such that $\mu_j = \mathcal{O}(j^{-k/2})$, we have

$$\begin{aligned}
\mu_j([\Delta, \chi^Z_{a+2,b+2}]\, R(s_0)\, \chi^Z_{a+1,b+1}) &= \mathcal{O}(j^{-1/2}); \\
\mu_j(Q_0(s_0)) &= \mathcal{O}(j^{-1}).
\end{aligned}$$

From this it easily follows that

$$\det(1+4\,|s(1-s)-s_0(1-s_0)|^3\,|Q_0(s_0)|^3) \leqslant e^{C\langle s\rangle^2}. \tag{6.6}$$

$$|D(s)| \leqslant e^{C\langle s\rangle^2}, \qquad \operatorname{Re} s > \tfrac{1}{2}+\varepsilon.$$

To estimate $\det[1+2\,|K_2(s)|]$ in the *bad* half-plane $\operatorname{Re} s < \frac{1}{2}-\varepsilon$, it suffices, by (6.3) with $p=1$, to estimate the determinants coming from each term of the sum

$$\begin{aligned} K_2(s) = K_2(1-s) &+ \sum_{i=1}^{M} [\Delta_0^{X_i}, \chi_{a,b}^{X_i}](R_0^{X_i}(s) \\ &- R_0^{X_i}(1-s))(\chi_{a+3,b+3}^{Z} - \chi_{a+1,b+1}^{X_i}) \\ &+ \sum_{j=1}^{N} [\Delta^{Y_j^0}, \chi_{a,b}^{Y_j}](R^{Y_j^0}(s) - R^{Y_j^0}(1-s))(\chi_{a+3,b+3}^{Z} - \chi_{a+1,b+1}^{Y_j}). \end{aligned} \tag{6.8}$$

Since $\operatorname{Re}(1-s) > \frac{1}{2}+\varepsilon$ the first one is bounded by $\exp C\langle s\rangle^2$. Let us introduce

$$G_i(s) = \det[1+4\,|[\Delta_0^{X_i}, \chi_{a,b}^{X_i}](R_0^{X_i}(s) - R_0^{X_i}(1-s))(\chi_{a+3,b+3}^{Z} - \chi_{a+1,b+1}^{X_i})|],$$
$$H_j(s) = \det[1+4\,|[\Delta^{Y_j^0}, \chi_{a,b}^{Y_j}](R^{Y_j^0}(s) - R^{Y_j^0}(1-s))(\chi_{a+3,b+3}^{Z} - \chi_{a+1,b+1}^{Y_j})|].$$

Proposition 4.2 easily gives $G_i(s) \leqslant e^{C\langle s\rangle^2}$. The function H_j is singular, but for $s \notin \mathscr{L}_{l_j}$ we have by Proposition 4.1 (or rather by the remark following the proof)

$$\begin{aligned} H_j(s) &\leqslant d^{-2}(s, \mathscr{L}_{l_j})\, e^{C\langle s\rangle \log\langle s\rangle} \prod_{j \leqslant C\langle s\rangle} e^{C_1\langle s\rangle + C\langle s\rangle \log(\langle s\rangle/j)} \\ &\times \prod_{j > C\langle s\rangle} (1+e^{-j/C}) \leqslant d^{-2}(s, \mathscr{L}_{l_j})\, e^{C\langle s\rangle^2}. \end{aligned}$$

Since $|g_{l_j}(s)| \leqslant d(s, \mathscr{L}_{l_j})\, e^{C\langle s\rangle^2}$, the lemma follows for P big enough. ∎

The second part of Lemma 4.1 applied with a fixed $m>0$ gives

LEMMA 6.3. *For* $\operatorname{Re} s > \varepsilon$

$$|D(s)| \leqslant e^{C_\varepsilon\langle s\rangle^3}. \tag{6.9}$$

Proof. This is immediate from (4.3) with $\tau = 0$ and the *good* half-plane part of the proof of Lemma 6.2. ∎

The proof of Theorem 2 is now easily completed: the function $\prod_{j=1}^{N} g_{l_j}(s)^P D(s)$ is entire. Lemma 6.2 and 6.3 together with the Phragmén–Lindelöf principle show that it is bounded by $\exp C\langle s\rangle^2$ from which the conclusion follows from Jensen's inequality.

APPENDIX: Hyperbolic Laplacians and Pöschl–Teller Potentials

We recall the following definition coming essentially from [24]: the Pöschl–Teller potential $V_{\mu,\nu}$ is defined on $\mathbf{R}$ by

$$V_{\mu,\nu}(r) = \mu(\mu+1)\sinh^{-2} r - \nu(\nu+1)\cosh^{-2} r, \qquad r \in \mathbf{R},$$

and for a real potential $V_{\mu,\nu}$ the parameters μ, ν are taken in $-\frac{1}{2} + i\mathbf{R}^+ \cup [-\frac{1}{2}, +\infty)$.

The motivation is provided by the following lemma, the proof of which we omit:

LEMMA A.1. (i) *The Laplacian on $L^2(\mathbf{H}^n, d\,\mathrm{vol}_{\mathbf{H}^n})$ is unitarily equivalent to*

$$D_r^2 + \frac{\frac{(n-1)(n-3)}{4} + \Delta^{S^{n-1}}}{\sinh^2 r} + \frac{(n-1)^2}{4}$$

on $L^2(\mathbf{R}^+, L^2(S^{n-1}, d\,\mathrm{vol}_{S^{n-1}}), dr)$.

(ii) *Let X be the cylinder $\mathbf{H}^n/\langle\gamma\rangle$, where γ is the hyperbolic isometry with displacement length l and acting trivially on the orthogonal of its axis. Then the Laplacian on $L^2(X, d\,\mathrm{vol}_X)$ is unitarily equivalent to*

$$D_r^2 + \frac{\frac{(n-2)(n-4)}{4} + \Delta^{S^{n-2}}}{\sinh^2 r} + \frac{\frac{1}{4} + \Delta^{\mathbf{R}/l\mathbf{Z}}}{\cosh^2 r} + \frac{(n-1)^2}{4}$$

on $L^2(\mathbf{R}^+, L^2(S^{n-2} \times \mathbf{R}/l\mathbf{Z}, d\,\mathrm{vol}_{S^{n-2}\times\mathbf{R}/l\mathbf{Z}}), dr)$.

The properties of the Pöschl–Teller Hamiltonians useful to describe the resonances of the hyperbolic spaces of the preceding lemma are stated in the following assertion

LEMMA A.2. *Let $V_{\mu,\nu}$ be a real Pöschl–Teller potential with $\mu \geqslant -\frac{1}{2}$. Let $H_{\mu,\nu}$ (resp. L_ν) be the symmetric Hamiltonian $D_r^2 + V_{\mu,\nu}$ (resp. $D_r^2 + V_{0,\nu}$) with domain $\mathscr{C}_0^\infty(\mathbf{R}^+)$ (resp. $\mathscr{C}_0^\infty(\mathbf{R})$ and $\bar{H}_{\mu,\nu}$, $\bar{L}_\nu$ their Friedrich extension.*

The Hamiltonian $\bar{H}_{\mu,\nu}$ (resp. $\bar{L}_\nu$) has $\mathbf{R}^+$ as continuous spectrum which is purely absolutely continuous of multiplicity one (resp. two). The determinant of the scattering matrix is given for $\bar{H}_{\mu,\nu}$ by the reflection coefficient

$$s(\bar{H}_{\mu,\nu})(k) = -\frac{\Gamma(ik)\,\Gamma((\mu+\nu-ik)/2+1)\,\Gamma(\mu-\nu-ik+1)/2)\,2^{-ik}}{\Gamma(-ik)\,\Gamma((\mu+\nu+ik)/2+1)\,\Gamma((\mu-\nu+ik+1)/2)2^{ik}}, \tag{A1}$$

and for $\bar{L}_\nu$ by

$$s(\bar{L}_\nu)(k) = \frac{\Gamma(ik)^2\Gamma(\nu-ik+1)\,\Gamma(-\nu-ik)}{\Gamma(-ik)^2\,\Gamma(\nu+ik+1)\,\Gamma(-\nu+ik)}.$$

The Hamiltonian $\bar{H}_{\mu,\nu}$ (resp. $\bar{L}_\nu$) has non-empty discrete spectrum if and only if $\nu-\mu>1$ (resp. $\nu>0$). When it is non-empty the discrete spectrum is given by

$$\sigma_{\mathrm{d}}(\bar{H}_{\mu,\nu}) = \{-(\nu-\mu-1-2n)^2,\ n\in\mathbf{N},\ 2n<\nu-\mu-1\},$$
$$\sigma_{\mathrm{d}}(\bar{L}_\nu) = \{-(\nu-n)^2,\ n\in\mathbf{N},\ n<\nu\}.$$

Proof. Through a conjugation by $\sinh^{\mu+1} r \cosh^{\nu+1} r$ and the change of variable $u = -\sinh^2 r$, the Schrödinger equation

$$D_r^2\psi + V_{\mu,\nu}\psi - k^2\psi = 0 \tag{A2}$$

is reduced to the hypergeometric equation

$$u(1-u)\,F''(u) + [(\mu+3/2)-(\mu+\nu+3)u]\,F'(u) - [(\mu+\nu+2)/2)^2 + (k/2)^2]\,F = 0.$$

The Schrödinger equation (A2) has the following independent solutions ([1, 15.5.1]) (if $\mu \neq -\frac{1}{2}$):

$$E_{\mu,\nu}(k)(r) = \sin^{1+\mu} r \cosh^{1+\nu} r \times {}_2F_1((\mu+\nu-ik+2)/2,\ (\mu+\nu+ik+2)/2,\ \mu+\tfrac{3}{2};\ -\sinh^2 r), \tag{A3}$$

$$F_{\mu,\nu}(k)(r) = \sinh^{-\mu} r \cosh^{1+\nu} r \times {}_2F_1((-\mu+\nu-ik+1)/2,\ (-\mu+\nu+ik+1)/2,\ \tfrac{1}{2}-\mu;\ -\sinh^2 r). \tag{A4}$$

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