MR812282 (87b:35107) 35L70
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Propagation of smoothness for nonlinear second-order strictly hyperbolic differential equations.

Pseudodifferential operators and applications (*Notre Dame, Ind.*, 1984), 21–44, *Proc. Sympos. Pure Math.*, 43, *Amer. Math. Soc., Providence, RI*, 1985.

Let $u \in H^s(\mathcal{O})$, $\mathcal{O} \subset \mathbf{R}^n$, $s > n/2$, be the solution of a semilinear or quasilinear strictly hyperbolic equation with given singular support on an initial hypersurface or in the past. The singular support of u in the future can be larger than that of the solutions to the corresponding linear problem. These "anomalous singularities" that develop have been shown to arise in two ways: if singularity-bearing characteristics for the linear problem cross, then nonlinear singularities can propagate along all forward characteristics issuing from the crossing point (Rauch-Reed, Lascar, Beals); and, for $n > 2$, if $\Gamma = \{x(s)\}$ is a characteristic that is the projection of two null bicharacteristics $\Gamma_\pm = \{x(s), \pm\xi(s)\}$, and if the corresponding linear solution has wave front set containing $\Gamma_+ \cup \Gamma_-$, then nonlinear singularities can "self-spread" from Γ along all possible characteristics (Beals). Singularities of strength approximately $2s - n/2$ appear in a solution to a nonlinear equation of order greater than two.

For second-order equations anomalous singularities of order $2s - n/2$ do not appear; they are at worst of order roughly $3s - n$: this paper provides complete proofs for the general semilinear equations $p_2(x, D)u = f(x, u)$ and $p_2(x, D)u = f(x, u, Du)$, with f smooth, and an outline of the proof in the quasilinear case. The idea behind the proof bears certain similarities to the arguments used in the study of conormal (Bony, Melrose-Ritter) or stratified (Rauch-Reed) solutions. An algebraic property of $p_2(x, \xi)$ allows estimates on $e_{2k}(D)(f(u))$ for certain microlocally elliptic operators e_{2k} in terms of estimates on $(p_2(x, D))^k u$ in the case of singularities due to self-spreading. Since $(p_2(x, D))^k u \in H^{s-k}(\mathcal{O})$ if $p_2(x, D)u = f(u)$ and $k < s - n/2$, it follows that an extra $s - n/2$ derivatives are estimated where e_{2k} is elliptic. A geometric property of the characteristic set of p_2 then shows that, for both self-spread singularities and those due to characteristic crossings, extra derivatives in characteristic directions of order $s - n/2$ are controlled, allowing the conclusion.

Reviewed by *Jean Leray*