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On smoothing property of Schrödinger propagators.
Functional-analytic methods for partial differential equations (*Tokyo*, 1989), 20–35, *Lecture Notes in Math.*, 1450, *Springer, Berlin*, 1990.

The Schrödinger equation under consideration is of the form $i\partial_t u = \frac{1}{2}\sum_j [i\partial_j + A_j(t,x)]^2 u + V(t,x)$, $j = 1,\cdots,n$, $t \in \mathbf{R}$, $x \in \mathbf{R}^n$, where ∂_j is the component of the gradient ∂_x and both the vector-potential $A = (A_1,\cdots,A_n)$ and the scalar potential V are real-valued. Let $B_{jk} = \partial_j A_k - \partial_k A_j$ and assume that: $|\partial_x^\alpha A(t,x)| + |\partial_x^\alpha \partial_t A(t,x)| \le C_\alpha$ for any multi-index α; $|\partial_x^\alpha B_{jk}(t,x)| \le C_\alpha (1+|x|)^{-1-\varepsilon}$ for some $\varepsilon > 0$; $\partial_x^\alpha V$ is continuous for any α, and $|\partial_x^\alpha V(t,x)| \le C_\alpha$ for $|\alpha| \ge 2$. Define

$$\|f\|^2_{\Sigma(2)} = \sum_{|\alpha+\beta|\le 2} \|x^\alpha \partial_x^\beta f\|^2,$$

where $\|\cdot\|$ is the norm in $L^2(\mathbf{R}^n)$, and $\Sigma(2) = \{f \in L^2(\mathbf{R}^n)\colon \|f\|_{\Sigma(2)} < \infty\}$. A preceding paper proved the existence in $L^2(\mathbf{R}^n)$ of a unitary operator $U(t,s)$, depending on t and $s \in \mathbf{R}$, having the following property: Let $u_0 \in C^1(\mathbf{R}, L^2(\mathbf{R}^n)) \cap C^0(\mathbf{R}, \Sigma(2))$; then the Schrödinger equation has a unique solution $u(\cdot) = U(\cdot,s)u_0$ satisfying the initial condition $u(s) = u_0$.

Now let $\mathcal{S}(\mathbf{R}^n)$ be the space of rapidly decreasing functions and $\langle x\rangle = (1+x^2)^{1/2}$, $\langle D\rangle = (1-\Delta)^{1/2}$. The author proves the following theorems. Theorem 1: Let $T > 0$ be small, $\mu > 1/2$, $\rho \ge 0$. Then there exists a constant $C_{\rho,\mu} > 0$ such that for $s \in \mathbf{R}$, $\int_{s-T}^{s+T} \|\langle x\rangle^{-\mu-\rho}\langle D\rangle^\rho U(t,s)f\|^2\,dt \le C_{\rho\mu}\|\langle D\rangle^{\rho-1/2}f\|^2$, $f \in \mathcal{S}(\mathbf{R}^n)$. Theorem 2: Let $T > 0$ be small, $p \ge 2$, $0 \le 2/\theta = 2\sigma + n(\frac{1}{2} - 1/p) < 1$ and $\rho \in \mathbf{R}$. Then there exists a constant $C_{p\rho\sigma} > 0$ such that

$$\Big[\int_{s-T}^{s+T} \Big\{\int_{\mathbf{R}^n} |\langle x\rangle^{-2\sigma-|\rho|}\langle D\rangle^{\rho+\sigma}U(t,s)f(x)|^p\,dx\Big\}^{\theta/p} dt\Big]^{1/\theta} \le C_{p\rho\sigma}\|\langle D\rangle^\rho f\|,$$

for $s \in \mathbf{R}$, $f \in \mathcal{S}(\mathbf{R}^n)$. A maximal inequality and a summability theorem follow from Theorem 1.

Reviewed by *Jean Leray*